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GRADED QUANTUM GROUPS AND QUASITRIANGULAR HOPF GROUP-COALGEBRAS#

Alexis Virelizier
Max Planck Institute for Mathematics, Bonn, Germany

Starting from a Hopf algebra endowed with an action of a group π by Hopf automorphisms, we construct (by a "twisted" double method) a quasitriangular Hopf π-coalgebra. This method allows us to obtain non-trivial examples of quasitriangular Hopf π-coalgebras for any finite group π and for infinite groups π such as $GL_n(\Bbbk)$. In particular, we define the graded quantum groups, which are Hopf π-coalgebras for $\pi = \mathbb{C}[[h]]^l$ and generalize the Drinfeld-Jimbo quantum enveloping algebras.

Key Words: Drinfeld double; Graded quantum groups; Hopf algebra automorphisms; Quasi-triangular Hopf group-coalgebras.

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INTRODUCTION

Let π be a group. Turaev (2000) introduced the notion of a *braided π category* and showed that such a category gives rise to a 3-dimensional homotopy quantum field theory (the target being a $K(\pi, 1)$ space). Moreover braided π-categories, also called *π-equivariant categories*, provide a suitable mathematical formalism for the description of orbifold models that arise in the study of conformal field theories in which π is the group of automorphisms of the vertex operator algebra, see Kirillov (2004).

The algebraic structure whose category of representations is a braided π-category is that of a *quasitriangular Hopf π-coalgebra*, see Turaev (2000), Virelizier (2002). The aim of the present article is to construct examples of quasitriangular Hopf π-coalgebras. Note that quasitriangular Hopf π-coalgebras are also used in Virelizier (2001) to construct HKR-type invariants of flat π-bundles over link complements and over 3-manifolds.

Following Turaev (2000), a Hopf π-coalgebra is a family $H = \{H_\alpha\}_{\alpha\in\pi}$ of algebras (over a field $\Bbbk$) endowed with a comultiplication $\Delta = \{\Delta_{\alpha,\beta} : H_{\alpha\beta} \to H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$, a counit $\varepsilon : H_1 \to \Bbbk$, and an antipode $S = \{S_\alpha : H_\alpha \to H_{\alpha^{-1}}\}_{\alpha\in\pi}$ which verify some compatibility conditions. A crossing for H is a family of algebra

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Address correspondence to Alexis Virelizier, Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany; Fax: +49-228-402277; E-mail: virelizi@mpim-bonn.mpg.de

isomorphisms $\varphi = \{\varphi_\beta : H_\alpha \to H_{\beta\alpha\beta^{-1}}\}_{\alpha,\beta\in\pi}$, which preserves the comultiplication and the counit, and which yields an action of π in the sense that $\varphi_\beta\varphi_{\beta'} = \varphi_{\beta\beta'}$. A crossed Hopf π-coalgebra H is quasitriangular when it is endowed with an R-matrix $R = \{R_{\alpha,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ verifying some axioms (involving the crossing φ) which generalize the classical ones given in Drinfeld (1987). Note that the case $\pi = 1$ is the standard setting of Hopf algebras.

Starting from a crossed Hopf π-coalgebra $H = \{H_\alpha\}_{\alpha\in\pi}$, Zunino (2004) constructed a double $Z(H) = \{Z(H)_\alpha\}_{\alpha\in\pi}$ of H, which is a quasitriangular Hopf π-coalgebra in which H is embedded. One has that $Z(H)_\alpha = H_\alpha \otimes \left(\bigoplus_{\beta\in\pi} H^*_\beta\right)$ as a vector space. Unfortunately, each component $Z(H)_\alpha$ is infinite-dimensional (unless $H_\beta = 0$ for all but a finite number of $\beta \in \pi$).

To obtain non-trivial examples of quasitriangular Hopf π-coalgebras with finite-dimensional components, we restrict ourselves to a less general situation: our initial datum is not any crossed Hopf π-coalgebra but a Hopf algebra endowed with an action of π by Hopf algebra automorphisms. Remark indeed that the component H_1 of a Hopf π-coalgebra $H = \{H_\alpha\}_{\alpha\in\pi}$ is a Hopf algebra and that a crossing for H induces an action of π on H_1 by Hopf automorphisms.

In this article, starting from a Hopf algebra A endowed with an action $\phi : \pi \to \mathrm{Aut}_{\mathrm{Hopf}}(A)$ of a group π by Hopf automorphisms, we construct a quasitriangular Hopf π-coalgebra $D(A, \phi) = \{D(A, \phi_\alpha)\}_{\alpha\in\pi}$. The algebra $D(A, \phi_\alpha)$ is constructed in a manner similar to the Drinfeld double (in particular $D(A, \phi_\alpha) = A \otimes A^*$ as a vector space) except that its product is "twisted" by the Hopf automorphism $\phi_\alpha : A \to A$. The algebra $D(A, \mathrm{id}_A)$ is the usual Drinfeld double. Note that the algebras $D(A, \phi_\alpha)$ and $D(A, \phi_\beta)$ are in general not isomorphic when $\alpha \neq \beta$.

This method allows us to define non-trivial examples of quasitriangular Hopf π-coalgebras for any finite group π and for infinite groups π such as $\mathrm{GL}_n(\mathbb{k})$. In particular, given a complex simple Lie algebra $\mathfrak{g}$ of rank l, we define the *graded quantum groups* $\{U^\alpha_h(\mathfrak{g})\}_{\alpha\in(\mathbb{C}^*)^l}$ and $\{U^\alpha_h(\mathfrak{g})\}_{\alpha\in\mathbb{C}[[h]]^l}$, which are crossed Hopf group-coalgebras. They are obtained as quotients of $D(U_q(\mathfrak{b}_+), \phi)$ and $D(U_h(b_+), \phi')$, where $\mathfrak{b}_+$ denotes the Borel subalgebra of $\mathfrak{g}$, ϕ is an action of $(\mathbb{C}^*)^l$ by Hopf automorphisms of $U_q(\mathfrak{b}_+)$, and ϕ' is an action of $\mathbb{C}[[h]]^l$ by Hopf automorphisms of $U_h(\mathfrak{b}_+)$. Furthermore, the crossed Hopf $\mathbb{C}[[h]]^l$-coalgebra $\{U^\alpha_h(\mathfrak{g})\}_{\alpha\in\mathbb{C}[[h]]^l}$ is quasitriangular.

The article is organized as follows. In Section 1, we review the basic definitions and properties of Hopf π-coalgebras. In Section 2, we define the twisted double of a Hopf algebra A endowed with an action of a group π by Hopf automorphisms. In Section 3, we explore the case $A = \mathbb{k}[G]$, where G is a finite group. In Section 4, we give an example of a quasitriangular Hopf $\mathrm{GL}_n(\mathbb{k})$-coalgebra. Finally, we define the graded quantum groups in Sections 5 and 6.

Throughout this article, π is a group (with neutral element 1) and $\mathbb{k}$ is a field. Unless otherwise specified, the tensor product $\otimes = \bigotimes_{\mathbb{k}}$ is assumed to be over $\mathbb{k}$.

1. HOPF GROUP-COALGEBRAS

In this section, we review some definitions and properties concerning Hopf group-coalgebras. For a detailed treatment of the theory of Hopf group-coalgebras, we refer to Virelizier (2002).

1.1. Hopf π-Coalgebras

A *Hopf π-coalgebra* (over $\Bbbk$) is a family $H = \{H_\alpha\}_{\alpha\in\pi}$ of $\Bbbk$-algebras endowed with a family $\Delta = \{\Delta_{\alpha,\beta} : H_{\alpha\beta} \to H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ of algebra homomorphisms (the *comultiplication*) and an algebra homomorphism $\varepsilon : H_1 \to \Bbbk$ (the *counit*) such that, for all $\alpha, \beta, \gamma \in \pi$,

$$\big(\Delta_{\alpha,\beta} \otimes \mathrm{id}_{H_\gamma}\big)\Delta_{\alpha\beta,\gamma} = \big(\mathrm{id}_{H_\alpha} \otimes \Delta_{\beta,\gamma}\big)\Delta_{\alpha,\beta\gamma}, \tag{1.1}$$

$$\big(\mathrm{id}_{H_\alpha} \otimes \varepsilon\big)\Delta_{\alpha,1} = \mathrm{id}_{H_\alpha} = \big(\varepsilon \otimes \mathrm{id}_{H_\alpha}\big)\Delta_{1,\alpha}, \tag{1.2}$$

and with a family $S = \{S_\alpha : H_\alpha \to H_{\alpha^{-1}}\}_{\alpha\in\pi}$ of $\Bbbk$-linear maps (the *antipode*) which verifies that, for all $\alpha \in \pi$,

$$m_\alpha(S_{\alpha^{-1}} \otimes \mathrm{id}_{H_\alpha})\Delta_{\alpha^{-1},\alpha} = \varepsilon 1_\alpha = m_\alpha(\mathrm{id}_{H_\alpha} \otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}}, \tag{1.3}$$

where $m_\alpha : H_\alpha \otimes H_\alpha \to H_\alpha$ and $1_\alpha \in H_\alpha$ denote, respectively, the multiplication and unit element of H_α.

When $\pi = 1$, one recovers the usual notion of a Hopf algebra. In particular $(H_1, m_1, 1_1, \Delta_{1,1}, \varepsilon, S_1)$ is a Hopf algebra.

Remark that the notion of a Hopf π-coalgebra is not self-dual and that if $H = \{H_\alpha\}_{\alpha\in\pi}$ is a Hopf π-coalgebra, then $\{\alpha \in \pi \mid H_\alpha \neq 0\}$ is a subgroup of π.

A Hopf π-coalgebra $H = \{H_\alpha\}_{\alpha\in\pi}$ is said to be of *finite type* if, for all $\alpha \in \pi$, H_α is finite-dimensional (over $\Bbbk$). Note that it does not mean that $\bigoplus_{\alpha\in\pi} H_\alpha$ is finite-dimensional (unless $H_\alpha = 0$ for all but a finite number of $\alpha \in \pi$).

The antipode of a Hopf π-coalgebra $H = \{H_\alpha\}_{\alpha\in\pi}$ is anti-multiplicative: each $S_\alpha : H_\alpha \to H_{\alpha^{-1}}$ is an anti-homomorphism of algebras, and anti-comultiplicative: $\varepsilon S_1 = \varepsilon$ and $\Delta_{\beta^{-1},\alpha^{-1}} S_{\alpha\beta} = \tau_{H_{\alpha^{-1}},H_{\beta^{-1}}}(S_\alpha \otimes S_\beta)\Delta_{\alpha,\beta}$ for any $\alpha, \beta \in \pi$, see Virelizier (2002, Lemma 1.1).

The antipode $S = \{S_\alpha\}_{\alpha\in\pi}$ of $H = \{H_\alpha\}_{\alpha\in\pi}$ is said to be *bijective* if each S_α is bijective. As for Hopf algebras, the antipode of a finite type Hopf π-coalgebra is always bijective, see Virelizier (2002, Corollary 3.7(a))).

1.2. Crossed Hopf π-Coalgebras

A Hopf π-coalgebra $H = \{H_\alpha\}_{\alpha\in\pi}$ is said to be *crossed* if it is endowed with a family $\varphi = \big\{\varphi_\beta : H_\alpha \to H_{\beta\alpha\beta^{-1}}\big\}_{\alpha,\beta\in\pi}$ of algebra isomorphisms (the *crossing*) such that, for all $\alpha, \beta, \gamma \in \pi$,

$$(\varphi_\beta \otimes \varphi_\beta)\Delta_{\alpha,\gamma} = \Delta_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}}\varphi_\beta, \tag{1.4}$$

$$\varepsilon\varphi_\beta = \varepsilon, \tag{1.5}$$

$$\varphi_\alpha\varphi_\beta = \varphi_{\alpha\beta}. \tag{1.6}$$

It is easy to check that $\varphi_1|_{H_\alpha} = \mathrm{id}_{H_\alpha}$ and $\varphi_\beta S_\alpha = S_{\beta\alpha\beta^{-1}}\varphi_\beta$ for all $\alpha, \beta \in \pi$.

1.3. Quasitriangular Hopf π-Coalgebras

A crossed Hopf π-coalgebra $H = \{H_\alpha\}_{\alpha\in\pi}$ is said to be *quasitriangular* if it is endowed with a family $R = \{R_{\alpha,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ of invertible elements (the *R-matrix*) such that, for all $\alpha, \beta, \gamma \in \pi$ and $x \in H_{\alpha\beta}$,

$$R_{\alpha,\beta} \cdot \Delta_{\alpha,\beta}(x) = \tau_{\beta,\alpha}(\varphi_{\alpha^{-1}} \otimes \mathrm{id}_{H_\alpha})\Delta_{\alpha\beta\alpha^{-1},\alpha}(x) \cdot R_{\alpha,\beta}, \tag{1.7}$$

$$(\mathrm{id}_{H_\alpha} \otimes \Delta_{\beta,\gamma})(R_{\alpha,\beta\gamma}) = (R_{\alpha,\gamma})_{1\beta 3} \cdot (R_{\alpha,\beta})_{12\gamma}, \tag{1.8}$$

$$(\Delta_{\alpha,\beta} \otimes \mathrm{id}_{H_\gamma})(R_{\alpha\beta,\gamma}) = [(\mathrm{id}_{H_\alpha} \otimes \varphi_{\beta^{-1}})(R_{\alpha,\beta\gamma\beta^{-1}})]_{1\beta 3} \cdot (R_{\beta,\gamma})_{\alpha 23}, \tag{1.9}$$

$$(\varphi_\beta \otimes \varphi_\beta)(R_{\alpha,\gamma}) = R_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}}, \tag{1.10}$$

where $\tau_{\beta,\alpha}$ denotes the flip map $H_\beta \otimes H_\alpha \to H_\alpha \otimes H_\beta$ and, for $\Bbbk$-spaces P, Q and $r = \sum_j p_j \otimes q_j \in P \otimes Q$, we set $r_{12\gamma} = r \otimes 1_\gamma \in P \otimes Q \otimes H_\gamma$, $r_{\alpha 23} = 1_\alpha \otimes r \in H_\alpha \otimes P \otimes Q$, and $r_{1\beta 3} = \sum_j p_j \otimes 1_\beta \otimes q_j \in P \otimes H_\beta \otimes Q$.

Note that $R_{1,1}$ is a (classical) R-matrix for the Hopf algebra H_1.

When π is abelian and φ is *trivial* (that is, $\varphi_\beta|_{H_\alpha} = \mathrm{id}_{H_\alpha}$ for all $\alpha, \beta \in \pi$), one recovers the definition of a quasitriangular π-colored Hopf algebra given in Ohtsuki (1993).

The R-matrix always verifies (see Virelizier, 2002, Lemma 6.4) that, for any $\alpha, \beta, \gamma \in \pi$,

$$(\varepsilon \otimes \mathrm{id}_{H_\alpha})(R_{1,\alpha}) = 1_\alpha = (\mathrm{id}_{H_\alpha} \otimes \varepsilon)(R_{\alpha,1}), \tag{1.11}$$

$$(S_{\alpha^{-1}}\varphi_\alpha \otimes \mathrm{id}_{H_\beta})(R_{\alpha^{-1},\beta}) = R^{-1}_{\alpha,\beta} \quad \text{and} \quad (\mathrm{id}_{H_\alpha} \otimes S_\beta)(R^{-1}_{\alpha,\beta}) = R_{\alpha,\beta^{-1}}, \tag{1.12}$$

$$(S_\alpha \otimes S_\beta)(R_{\alpha,\beta}) = (\varphi_\alpha \otimes \mathrm{id}_{H_{\beta^{-1}}})(R_{\alpha^{-1},\beta^{-1}}), \tag{1.13}$$

and provides a solution of the π-colored Yang-Baxter equation:

$$(R_{\beta,\gamma})_{\alpha 23} \cdot (R_{\alpha,\gamma})_{1\beta 3} \cdot (R_{\alpha,\beta})_{12\gamma} = (R_{\alpha,\beta})_{12\gamma} \cdot [(\mathrm{id}_{H_\alpha} \otimes \varphi_{\beta^{-1}})(R_{\alpha,\beta\gamma\beta^{-1}})]_{1\beta 3} \cdot (R_{\beta,\gamma})_{\alpha 23}. \tag{1.14}$$

1.4. Ribbon Hopf π-Coalgebras

A quasitriangular Hopf π-coalgebra $H = \{H_\alpha\}_{\alpha\in\pi}$ is said to be *ribbon* if it is endowed with a family $\theta = \{\theta_\alpha \in H_\alpha\}_{\alpha\in\pi}$ of invertible elements (the *twist*) such that, for any $\alpha, \beta \in \pi$,

$$\varphi_\alpha(x) = \theta_\alpha^{-1} x \theta_\alpha \qquad \text{for all } x \in H_\alpha, \tag{1.15}$$

$$S_\alpha(\theta_\alpha) = \theta_{\alpha^{-1}}, \tag{1.16}$$

$$\varphi_\beta(\theta_\alpha) = \theta_{\beta\alpha\beta^{-1}}, \tag{1.17}$$

$$\Delta_{\alpha,\beta}(\theta_{\alpha\beta}) = (\theta_\alpha \otimes \theta_\beta) \cdot \tau_{\beta,\alpha}((\varphi_{\alpha^{-1}} \otimes \mathrm{id}_{H_\alpha})(R_{\alpha\beta\alpha^{-1},\alpha})) \cdot R_{\alpha,\beta}. \tag{1.18}$$

Note that θ_1 is a (classical) twist of the quasitriangular Hopf algebra H_1.

1.5. Hopf π-Coideals

Let $H = \{H_\alpha\}_{\alpha \in \pi}$ be a Hopf π-coalgebra. A *Hopf π-coideal* of H is a family $I = \{I_\alpha\}_{\alpha \in \pi}$, where each I_α is an ideal of H_α, such that, for any $\alpha, \beta \in \pi$,

$$\Delta_{\alpha,\beta}(I_{\alpha\beta}) \subset I_\alpha \otimes H_\beta + H_\alpha \otimes I_\beta, \tag{1.19}$$

$$\varepsilon(I_1) = 0, \tag{1.20}$$

$$S_\alpha(I_\alpha) \subset I_{\alpha^{-1}}. \tag{1.21}$$

The quotient $\overline{H} = \{\overline{H}_\alpha = H_\alpha / I_\alpha\}_{\alpha \in \pi}$, endowed with the induced structure maps, is then a Hopf π-coalgebra. If H is furthermore crossed, with a crossing φ such that, for any $\alpha, \beta \in \pi$,

$$\varphi_\beta(I_\alpha) \subset I_{\beta\alpha\beta^{-1}}, \tag{1.22}$$

then so is $\overline{H}$ (for the induced crossing).

2. TWISTED DOUBLE OF HOPF ALGEBRAS

In this section, we give a method (the *twisted double*) for defining a quasitriangular Hopf π-coalgebra from a Hopf algebra endowed with an action of a group π by Hopf automorphisms.

2.1. Hopf Pairings

Recall that a *Hopf pairing* between two Hopf algebras A and B (over $\Bbbk$) is a bilinear pairing $\sigma : A \times B \to \Bbbk$ such that, for all $a, a' \in A$ and $b, b' \in B$,

$$\sigma(a, bb') = \sigma(a_{(1)}, b)\sigma(a_{(2)}, b'), \tag{2.1}$$

$$\sigma(aa', b) = \sigma(a, b_{(2)})\sigma(a', b_{(1)}), \tag{2.2}$$

$$\sigma(a, 1) = \varepsilon(a) \qquad \text{and} \qquad \sigma(1, b) = \varepsilon(b). \tag{2.3}$$

Note that such a pairing always verifies that, for any $a \in A$ and $b \in B$,

$$\sigma(S(a), S(b)) = \sigma(a, b), \tag{2.4}$$

since both σ and $\sigma(S \times S)$ are the inverse of $\sigma(\mathrm{id} \times S)$ in the algebra $\mathrm{Hom}_\Bbbk(A \times B, \Bbbk)$ endowed with the convolution product.

Let $\sigma : A \times B \to \Bbbk$ be a Hopf pairing. Its annihilator ideals are $I_A = \{a \in A \mid \sigma(a, b) = 0 \text{ for all } b \in B\}$ and $I_B = \{b \in B \mid \sigma(a, b) = 0 \text{ for all } a \in A\}$. It is easy to check that I_A and I_B are Hopf ideals of A and B, respectively. Recall that σ is said to be *non-degenerate* if I_A and I_B are both reduced to 0. A degenerate Hopf pairing $\sigma : A \times B \to \Bbbk$ induces (by passing to the quotients) a Hopf pairing $\bar{\sigma} : A/I_A \times B/I_B \to \Bbbk$, which is non-degenerate.

Most of Hopf algebras we shall consider in the sequel will be defined by generators and relations. The following provides us with a method of constructing Hopf pairings, see Van Daele (1993), Kassel et al. (1997).

Let $\widetilde{A}$ (resp. $\widetilde{B}$) be a free algebra generated by elements $a_1, \dots, a_p$ (resp. $b_1, \dots, b_q$) over $\Bbbk$. Suppose that $\widetilde{A}$ and $\widetilde{B}$ have Hopf algebra structures such that each $\Delta(a_i)$ for $1 \leq i \leq p$ (resp. $\Delta(b_j)$ for $1 \leq i \leq q$) is a linear combination of tensors $a_r \otimes a_s$ (resp. $b_r \otimes b_s$). Given pq scalars $\lambda_{i,j} \in \Bbbk$ with $1 \leq i \leq p$ and $1 \leq j \leq q$, there is a unique Hopf pairing $\sigma : \widetilde{A} \times \widetilde{B} \to \Bbbk$ such that $\sigma(a_i, b_j) = \lambda_{i,j}$.

Suppose now that A (resp. B) is the algebra obtained as the quotient of $\widetilde{A}$ (resp. $\widetilde{B}$) by the ideal generated by elements $r_1, \dots, r_m \in \widetilde{A}$ (resp. $s_1, \dots, s_n \in \widetilde{B}$). Suppose also that the Hopf algebra structure in $\widetilde{A}$ (resp. $\widetilde{B}$) induces a Hopf algebra structure in A (resp. B). Then a Hopf pairing $\sigma : \widetilde{A} \times \widetilde{B} \to \Bbbk$ induces a Hopf pairing $A \times B \to \Bbbk$ if and only if $\sigma(r_i, b_j) = 0$ for all $1 \leq i \leq m$ and $1 \leq j \leq q$, and $\sigma(a_i, s_j) = 0$ for all $1 \leq i \leq p$ and $1 \leq j \leq n$.

2.2. The Twisted Double Construction

Definition-Lemma 2.1. *Let $\sigma : A \times B \to \Bbbk$ be a Hopf pairing between two Hopf algebras A and B. Let $\phi : A \to A$ be a Hopf algebra endomorphism of A. Set $D(A, B; \sigma, \phi) = A \otimes B$ as a $\Bbbk$-space. Then $D(A, B; \sigma, \phi)$ has a structure of an associative and unitary algebra given, for any $a, a' \in A$ and $b, b' \in B$, by*

$$(a \otimes b) \cdot (a' \otimes b') = \sigma(\phi(a'_{(1)}), S(b_{(1)}))\sigma(a'_{(3)}, b_{(3)}) a a'_{(2)} \otimes b_{(2)} b', \tag{2.5}$$

$$1_{D(A,B;\sigma,\phi)} = 1_A \otimes 1_B. \tag{2.6}$$

Moreover, the linear embeddings $A \hookrightarrow D(A, B; \sigma, \phi)$ and $B \hookrightarrow D(A, B; \sigma, \phi)$ defined by $a \mapsto a \otimes 1_B$ and $b \mapsto 1_A \otimes b$, respectively, are algebra morphisms.

Remark 2.2. (a) Note that $D(A, B; \sigma, \mathrm{id}_A)$ is the underlying algebra of the usual quantum double of A and B (obtained by using the Hopf pairing σ).

(b) If ϕ and ϕ' are different Hopf algebra endomorphisms of A, then the algebras $D(A, B; \sigma, \phi)$ and $D(A, B; \sigma, \phi')$ are not in general isomorphic, see Remark 4.2.

Proof. Let $a, a', a'' \in A$ and $b, b', b'' \in B$. Using the fact that σ is a Hopf pairing and ϕ is a Hopf algebra endomorphism, we have that

$$\begin{aligned}
&\big((a \otimes b) \cdot (a' \otimes b')\big) \cdot (a'' \otimes b'') \\
&= \sigma(\phi(a'_{(1)}), S(b_{(1)}))\sigma(a'_{(3)}, b_{(5)})\sigma(\phi(a''_{(1)}), S(b_{(2)} b'_{(1)})) \\
&\quad \times \sigma(a''_{(3)}, b_{(4)} b'_{(3)}) a a'_{(2)} a''_{(2)} \otimes b_{(3)} b'_{(2)} b'' \\
&= \sigma(\phi(a'_{(1)}), S(b_{(1)}))\sigma(a'_{(3)}, b_{(5)})\sigma(\phi(a''_{(1)}), S(b'_{(1)}))\sigma(\phi(a''_{(2)}), S(b_{(2)})) \\
&\quad \times \sigma(a''_{(4)}, b_{(4)})\sigma(a''_{(5)}, b'_{(3)}) a a'_{(2)} a''_{(3)} \otimes b_{(3)} b'_{(2)} b'',
\end{aligned}$$

and

$$\begin{aligned}
&(a \otimes b) \cdot \big((a' \otimes b') \cdot (a'' \otimes b'')\big) \\
&= \sigma(\phi(a''_{(1)}), S(b'_{(1)}))\sigma(a''_{(5)}, b'_{(3)})\sigma(\phi(a'_{(1)} a''_{(2)}), S(b_{(1)})) \\
&\quad \times \sigma(a'_{(3)} a''_{(4)}, b_{(3)}) a a'_{(2)} a''_{(3)} \otimes b_{(2)} b'_{(2)} b''
\end{aligned}$$

$$= \sigma(\phi(a''_{(1)}), S(b'_{(1)}))\sigma(a''_{(5)}, b'_{(3)})\sigma(\phi(a'_{(1)}), S(b_{(1)}))\sigma(\phi(a''_{(2)}), S(b_{(2)}))$$
$$\times \sigma(a'_{(3)}, b_{(5)})\sigma(a''_{(4)}, b_{(4)})aa'_{(2)}a''_{(3)} \otimes b_{(3)}b'_{(2)}b''.$$

Hence the product is associative. Moreover $1_A \otimes 1_B$ is the unit element since

$$\begin{aligned}(a \otimes b) \cdot (1 \otimes 1) &= \sigma(\phi(1), S(b_{(1)}))\sigma(1, b_{(3)})a \otimes b_{(2)}\\ &= \varepsilon(S(b_{(1)}))\varepsilon(b_{(3)})a \otimes b_{(2)} = a \otimes b,\end{aligned}$$

and

$$\begin{aligned}(1 \otimes 1) \cdot (a \otimes b) &= \sigma(\phi(a_{(1)}), S(1))\sigma(a_{(3)}, 1)a_{(2)} \otimes b\\ &= \varepsilon(\phi(a_{(1)}))\varepsilon(a_{(3)})a_{(2)} \otimes b = a \otimes b.\end{aligned}$$

Finally, for any $a, a' \in A$ and $b, b' \in B$, we have that

$$\begin{aligned}(a \otimes 1) \cdot (a' \otimes 1) &= \sigma(\phi(a'_{(1)}), S(1))\sigma(a'_{(3)}, 1)aa'_{(2)} \otimes 1\\ &= \varepsilon(\phi(a'_{(1)}))\varepsilon(a'_{(3)})aa'_{(2)} \otimes 1\\ &= aa' \otimes 1,\end{aligned}$$

and

$$\begin{aligned}(1 \otimes b) \cdot (1 \otimes b') &= \sigma(\phi(1), S(b_{(1)}))\sigma(1, b_{(3)})1 \otimes b_{(2)}b'\\ &= \varepsilon(S(b_{(1)}))\varepsilon(b_{(3)})1 \otimes b_{(2)}b'\\ &= 1 \otimes bb'.\end{aligned}$$

Therefore $A \hookrightarrow D(A, B; \sigma, \phi)$ and $B \hookrightarrow D(A, B; \sigma, \phi)$ are algebra morphisms. □

In the sequel, the group of Hopf automorphisms of a Hopf algebra A will be denoted by $\mathrm{Aut}_{\mathrm{Hopf}}(A)$.

Theorem 2.3. *Let $\sigma : A \times B \to \Bbbk$ be a Hopf pairing between two Hopf algebras A and B, and $\phi : \pi \to \mathrm{Aut}_{\mathrm{Hopf}}(A)$ be group homomorphism (that is, an action of π on A by Hopf automorphisms). Then the family of algebras $D(A, B; \sigma, \phi) = \{D(A, B; \sigma, \phi_\alpha)\}_{\alpha\in\pi}$ (see Definition 2.1) has a structure of a Hopf π-coalgebra given, for any $a \in A$, $b \in B$, and $\alpha, \beta \in \pi$, by:*

$$\Delta_{\alpha,\beta}(a \otimes b) = (\phi_\beta(a_{(1)}) \otimes b_{(1)}) \otimes (a_{(2)} \otimes b_{(2)}), \tag{2.7}$$

$$\varepsilon(a \otimes b) = \varepsilon_A(a)\varepsilon_B(b), \tag{2.8}$$

$$S_\alpha(a \otimes b) = \sigma(\phi_\alpha(a_{(1)}), b_{(1)})\sigma(a_{(3)}, S(b_{(3)}))\phi_\alpha S(a_{(2)}) \otimes S(b_{(2)}). \tag{2.9}$$

Proof. The coassociativity (1.1) follows directly from the coassociativity of the coproducts of A and B and the fact that $\phi_{\beta\gamma} = \phi_\beta\phi_\gamma$. Axiom (1.2) is a direct consequence of $\varepsilon_A\phi_\alpha = \varepsilon_A$. Since $\phi_1 = \mathrm{id}_A$ and $D(A, B; \sigma, \mathrm{id}_A)$ is underlying algebra of the usual quantum double of A and B, the counit ε is multiplicative. Let us verify that $\Delta_{\alpha,\beta}$ is multiplicative. Let $a, a' \in A$ and $b, b' \in B$. On one hand we have:

$$\begin{aligned}
&\Delta_{\alpha,\beta}((a \otimes b) \cdot (a' \otimes b')) \\
&\quad = \sigma(\phi_{\alpha\beta}(a'_{(1)}), S(b_{(1)}))\sigma(a'_{(3)}, b_{(3)})\Delta_{\alpha,\beta}(aa'_{(2)} \otimes b_{(2)}b') \\
&\quad = \sigma(\phi_{\alpha\beta}(a'_{(1)}), S(b_{(1)}))\sigma(a'_{(4)}, b_{(4)})\phi_\beta(a_{(1)}a'_{(2)}) \otimes b_{(2)}b'_{(1)} \otimes a_{(2)}a'_{(3)} \otimes b_{(3)}b'_{(2)}.
\end{aligned}$$

One the other hand,

$$\begin{aligned}
&\Delta_{\alpha,\beta}(a \otimes b) \cdot \Delta_{\alpha,\beta}(a' \otimes b') \\
&\quad = (\phi_\beta(a_{(1)}) \otimes b_{(1)} \otimes a_{(2)} \otimes b_{(2)}) \cdot (\phi_\beta(a'_{(1)}) \otimes b'_{(1)} \otimes a'_{(2)} \otimes b'_{(2)}) \\
&\quad = \sigma(\phi_\alpha\phi_\beta(a'_{(1)}), S(b_{(1)}))\sigma(\phi_\beta(a'_{(3)}), b_{(3)})\sigma(\phi_\beta(a'_{(4)}), S(b_{(4)}))\sigma(a'_{(6)}, b_{(6)}) \\
&\qquad \times \phi_\beta(a_{(1)})\phi_\beta(a'_{(2)}) \otimes b_{(2)}b'_{(1)} \otimes a_{(2)}a'_{(5)} \otimes b_{(5)}b'_{(2)} \\
&\quad = \sigma(\phi_{\alpha\beta}(a'_{(1)}), S(b_{(1)}))\sigma(\phi_\beta(a'_{(3)}), b_{(3)}S(b_{(4)}))\sigma(a'_{(5)}, b_{(6)}) \\
&\qquad \times \phi_\beta(a_{(1)}a'_{(2)}) \otimes b_{(2)}b'_{(1)} \otimes a_{(2)}a'_{(4)} \otimes b_{(5)}b'_{(2)} \\
&\quad = \sigma(\phi_{\alpha\beta}(a'_{(1)}), S(b_{(1)}))\sigma(a'_{(4)}, b_{(4)})\phi_\beta(a_{(1)}a'_{(2)}) \otimes b_{(2)}b'_{(1)} \otimes a_{(2)}a'_{(3)} \otimes b_{(3)}b'_{(2)}.
\end{aligned}$$

Let us verify the first equality of (1.3). Let $a \in A$, $b \in B$, and $\alpha \in \pi$. Denote the multiplication in $D(A, B; \sigma, \phi_\alpha)$ by m_α. We have

$$\begin{aligned}
&m_\alpha(S_{\alpha^{-1}} \otimes \mathrm{id}_{D(A,B;\sigma,\phi_\alpha)})\Delta_{\alpha^{-1},\alpha}(a \otimes b) \\
&\quad = \sigma(a_{(1)}, b_{(1)})\,\sigma(\phi_\alpha(a_{(3)}), S(b_{(5)}))\sigma(\phi_\alpha(a_{(4)}), S^2(b_{(4)})) \\
&\qquad \times \sigma(a_{(6)}, S(b_{(2)}))S(a_{(2)})a_{(5)} \otimes S(b_{(3)})b_{(6)} \\
&\quad = \sigma(a_{(1)}, b_{(1)})\sigma(\phi_\alpha(a_{(3)}), S(b_{(5)})S^2(b_{(4)})) \\
&\qquad \times \sigma(a_{(5)}, S(b_{(2)}))S(a_{(2)})a_{(4)} \otimes S(b_{(3)})b_{(6)} \\
&\quad = \sigma(a_{(1)}, b_{(1)})\sigma(a_{(4)}, S(b_{(2)}))S(a_{(2)})a_{(3)} \otimes S(b_{(3)})b_{(4)} \\
&\quad = \sigma(a_{(1)}, b_{(1)})\sigma(a_{(2)}, S(b_{(2)}))1 \otimes 1 \\
&\quad = \sigma(a, b_{(1)}S(b_{(2)}))1 \otimes 1 = \varepsilon(a)\varepsilon(b)1 \otimes 1.
\end{aligned}$$

The second equality of (1.3) can be verified similarly. □

Let $\sigma : A \times B \to \Bbbk$ be a Hopf pairing between two Hopf algebras A and B, and $\phi : \pi \to \mathrm{Aut}_{\mathrm{Hopf}}(A)$ be an action of π on A by Hopf automorphisms. An action $\psi : \pi \to \mathrm{Aut}_{\mathrm{Hopf}}(B)$ of π on B by Hopf automorphisms is said to be (σ, ϕ)-*compatible* if, for all $a \in A$, $b \in B$ and $\beta \in \pi$,

$$\sigma(\phi_\beta(a), \psi_\beta(b)) = \sigma(a, b). \tag{2.10}$$

Lemma 2.4. *Let $\sigma : A \times B \to \Bbbk$ be a Hopf pairing between two Hopf algebras A and B. Let $\phi : \pi \to \mathrm{Aut}_{\mathrm{Hopf}}(A)$ and $\psi : \pi \to \mathrm{Aut}_{\mathrm{Hopf}}(B)$ be two actions of π by*

Hopf automorphisms. Suppose that ψ is (σ, ϕ)-compatible. Then the Hopf π-coalgebra $D(A, B; \sigma, \phi) = \{D(A, B; \sigma, \phi_\alpha)\}_{\alpha\in\pi}$ (see Theorem 2.3) admits a crossing φ given, for any $a \in A$, $b \in B$ and $\beta \in \pi$, by

$$\varphi_\beta(a \otimes b) = \phi_\beta(a) \otimes \psi_\beta(b). \tag{2.11}$$

Proof. Let $\alpha, \beta \in \pi$. We have that $\varphi_\beta(1_A \otimes 1_B) = \phi_\beta(1_A) \otimes \psi_\beta(1_B) = 1_A \otimes 1_B$ and, for any $a, a' \in A$ and $b, b' \in B$,

$$\begin{aligned}
&\varphi_\beta(a \otimes b) \cdot \varphi_\beta(a' \otimes b') \\
&\quad = \sigma(\phi_{\beta\alpha\beta^{-1}}(\phi_\beta(a')_{(1)}), S(\psi_\beta(b)_{(1)}))\sigma(\phi_\beta(a')_{(3)}, \psi_\beta(b)_{(3)}) \\
&\qquad \times \phi_\beta(a)\phi_\beta(a')_{(2)} \otimes \psi_\beta(b)_{(2)}\psi_\beta(b') \\
&\quad = \sigma(\phi_\beta\phi_\alpha(a'_{(1)})), \psi_\beta S(b_{(1)}))\sigma(\phi_\beta(a'_{(3)}), \psi_\beta(b_{(3)}))\phi_\beta(a)\phi_\beta(a'_{(2)}) \otimes \psi_\beta(b_{(2)})\psi_\beta(b') \\
&\quad = \sigma(\phi_\alpha(a'_{(1)})), S(b_{(1)}))\sigma(a'_{(3)}, b_{(3)})\phi_\beta(aa'_{(2)}) \otimes \psi_\beta(b_{(2)}b') \\
&\quad = \varphi_\beta((a \otimes b) \cdot (a' \otimes b')).
\end{aligned}$$

Moreover ϕ_β and ψ_β are bijective and so is φ_β. Therefore $\varphi_\beta : D(A, B; \sigma, \phi_\alpha) \to D(A, B; \sigma, \phi_{\beta\alpha\beta^{-1}})$ is an algebra isomorphism.

Finally, for any $a \in A$, $b \in B$ and $\alpha, \beta, \gamma \in \pi$, we have that:

$$\begin{aligned}
\Delta_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}}(\varphi_\beta(a \otimes b)) &= \phi_{\beta\gamma\beta^{-1}}(\phi_\beta(a)_{(1)}) \otimes \psi_\beta(b)_{(1)} \otimes \phi_\beta(a)_{(2)} \otimes \psi_\beta(b)_{(2)} \\
&= \phi_{\beta\gamma\beta^{-1}}\phi_\beta(a_{(1)}) \otimes \psi_\beta(b_{(1)}) \otimes \phi_\beta(a_{(2)}) \otimes \psi_\beta(b_{(2)}) \\
&= \phi_\beta\phi_\gamma(a_{(1)}) \otimes \psi_\beta(b_{(1)}) \otimes \phi_\beta(a_{(2)}) \otimes \psi_\beta(b_{(2)}) \\
&= (\varphi_\beta \otimes \varphi_\beta)\Delta_{\alpha,\gamma}(a \otimes b), \\
\varepsilon\varphi_\beta(a \otimes b) &= \varepsilon(\phi_\beta(a))\varepsilon(\psi_\beta(b)) = \varepsilon(a)\varepsilon(b) = \varepsilon(a \otimes b),
\end{aligned}$$

and

$$\varphi_\alpha\varphi_\beta(a \otimes b) = \phi_\alpha\phi_\beta(a) \otimes \psi_\alpha\psi_\beta(b) = \phi_{\alpha\beta}(a) \otimes \psi_{\alpha\beta}(b) = \varphi_{\alpha\beta}(a \otimes b).$$

Hence φ satisfies Axioms (1.4), (1.5) and (1.6). □

Corollary 2.5. *Let $\sigma : A \times B \to \Bbbk$ be a Hopf pairing and $\phi : \pi \to \mathrm{Aut}_{\mathrm{Hopf}}(A)$ be an action of π on A by Hopf automorphisms. Suppose that σ is non-degenerate and that A (and so B) is finite dimensional. Then there exists a unique action $\phi^* : \pi \to \mathrm{Aut}_{\mathrm{Hopf}}(B)$ which is (σ, ϕ)-compatible. It is characterized, for any $a \in A$, $b \in B$ and $\beta \in \pi$, by*

$$\sigma(a, \phi^*_\beta(b)) = \sigma(\phi_{\beta^{-1}}(a), b). \tag{2.12}$$

Consequently the Hopf π-coalgebra $D(A, B; \sigma, \phi) = \{D(A, B; \sigma, \phi_\alpha)\}_{\alpha\in\pi}$ (see Theorem 2.3) is crossed with crossing defined by $\varphi_\beta = \phi_\beta \otimes \phi^*_\beta$ for any $\beta \in \pi$.

Proof. Let $\beta \in \pi$. Since σ is non-degenerate and A and B are finite dimensional, the map $b \in B \mapsto \sigma(\cdot, b) \in A^*$ is a linear isomorphism, and so (2.12) does uniquely define a linear map $\phi^*_\beta : B \to B$. Since σ is a Hopf pairing and $\phi_{\beta^{-1}}$ is a Hopf algebra isomorphism of A, the map ϕ^*_β is a Hopf algebra isomorphism of B. Moreover ϕ^* is an action since $\phi^*_1 = \mathrm{id}_B$ (because $\phi_1 = \mathrm{id}_A$) and $\sigma(a, \phi^*_{\alpha\beta}(b)) = \sigma(\phi_{\beta^{-1}\alpha^{-1}}(a), b) = \sigma(\phi_{\beta^{-1}}\phi_{\alpha^{-1}}(a), b) = \sigma(\phi_{\alpha^{-1}}(a), \phi^*_\beta(b)) = \sigma(a, \phi^*_\alpha\phi^*_\beta(b))$ for any $a \in A$, $b \in B$ and $\alpha, \beta \in \pi$. Finally (2.12) says exactly that ϕ^* is (σ, ϕ)-compatible. □

Theorem 2.6. *Let $\sigma : A \times B \to \Bbbk$ be a Hopf pairing between two Hopf algebras A and B, and $\phi : \pi \to \mathrm{Aut}_{\mathrm{Hopf}}(A)$ be an action of π on A by Hopf automorphisms. Suppose that σ is non-degenerate and that A (and so B) is finite dimensional. Then the crossed Hopf π-coalgebra $D(A, B; \sigma, \phi) = \{D(A, B; \sigma, \phi_\alpha)\}_{\alpha\in\pi}$ (see Corollary 2.5) is quasitriangular with R-matrix given, for all $\alpha, \beta \in \pi$, by*

$$R_{\alpha,\beta} = \sum_i (e_i \otimes 1_B) \otimes (1_A \otimes f_i), \tag{2.13}$$

where $(e_i)_i$ and $(f_i)_i$ are basis of A and B, respectively, such that $\sigma(e_i, f_j) = \delta_{i,j}$.

Remark 2.7. (a) The element $\sum_i (e_i \otimes 1_B) \otimes (1_A \otimes f_i) \in A \otimes B \otimes A \otimes B$ is canonical, i.e., independent of the choices of the basis $(e_i)_i$ of A and $(f_i)_i$ of B such that $\sigma(e_i, f_j) = \delta_{i,j}$.

(b) Note that the hypothesis A is finite dimensional ensures that the sum $\sum_i (e_i \otimes 1_B) \otimes (1_A \otimes f_i)$ lies in $A \otimes B \otimes A \otimes B$. More generally, assume that A and B are graded Hopf algebras with finite dimensional homogeneous components and that σ is compatible with the gradings. Then the quotient Hopf algebras A/I_A and B/I_B are also graded and can be identified via σ with the duals of each other. Suppose also that the action ϕ respects the grading so does the quotient $\bar\phi : \pi \to \mathrm{Aut}_{\mathrm{Hopf}}(A/I_A)$. In this case, there exists a unique action $\pi \to \mathrm{Aut}_{\mathrm{Hopf}}(B/I_B)$ which is $(\bar\sigma, \bar\phi)$-compatible, where $\bar\sigma : A/I_A \times B/I_B \to \Bbbk$ is the induced Hopf pairing. Then the Hopf π-coalgebra $D(A/I_A, B/I_B; \bar\sigma, \bar\phi)$ is quasitriangular by the same construction as in Theorem 2.6.

Proof. Fix basis (e_i) of A and (f_i) of B such that $\sigma(e_i, f_j) = \delta_{i,j}$ (such basis always exist since σ is non-degenerate). Note that $x = \sum_i \sigma(x, f_i)e_i$ and $y = \sum_i \sigma(e_i, y)f_i$ for any $x \in A$ and $y \in B$.

Recall that, since $\sum_i e_i \otimes 1_B \otimes 1_A \otimes f_i$ is the R-matrix of the usual quantum double $D(A, B, \sigma, \mathrm{id}_A)$, we have

$$\sum_{i,j} S(e_i)e_j \otimes f_i f_j = 1_A \otimes 1_B, \tag{2.14}$$

$$\sum_i e_i \otimes f_{i(1)} \otimes f_{i(2)} = \sum_{i,j} e_i e_j \otimes f_j \otimes f_i, \tag{2.15}$$

$$\sum_i e_{i(1)} \otimes e_{i(2)} \otimes f_i = \sum_{i,j} e_i \otimes e_j \otimes f_i f_j. \tag{2.16}$$

Let $\alpha, \beta \in \pi$. From (2.14) and since A (resp. B) can be viewed as a subalgebra of $D(A, B; \sigma, \phi_\alpha)$ (resp. $D(A, B; \sigma, \phi_\beta)$) via $a \mapsto a \otimes 1_B$ (resp. $b \mapsto 1_A \otimes b$), we get that $R_{\alpha,\beta}$ is invertible in $D(A, B; \sigma, \phi_\alpha) \otimes D(A, B; \sigma, \phi_\beta)$ with inverse

$$R_{\alpha,\beta}^{-1} = \sum_i S(e_i) \otimes 1_B \otimes 1_A \otimes f_i.$$

Let $a \in A$, $b \in B$ and $\alpha, \beta \in \pi$. For all $x \in A$, we have that:

$$\begin{aligned}
&(\mathrm{id}_{A\otimes B\otimes A} \otimes \sigma(x, \cdot))(R_{\alpha,\beta} \cdot \Delta_{\alpha,\beta}(a \otimes b)) \\
&\quad = \sum_i \sigma(\phi_\beta(a_{(2)}), S(f_{i(1)}))\sigma(a_{(4)}, f_{i(3)})\sigma(x, f_{i(2)}b_{(2)})e_i\phi_\beta(a_{(1)}) \otimes b_{(1)} \otimes a_{(3)} \\
&\quad = \sum_i \sigma(\phi_\beta S^{-1}(a_{(2)}), f_{i(1)})\sigma(a_{(4)}, f_{i(3)})\sigma(x_{(1)}, f_{i(2)})\sigma(x_{(2)}, b_{(2)})e_i\phi_\beta(a_{(1)}) \otimes b_{(1)} \otimes a_{(3)} \\
&\quad = \sum_i \sigma(a_{(4)}x_{(1)}\phi_\beta S^{-1}(a_{(2)}), f_i)\sigma(x_{(2)}, b_{(2)})e_i\phi_\beta(a_{(1)}) \otimes b_{(1)} \otimes a_{(3)} \\
&\quad = \sigma(x_{(2)}, b_{(2)})a_{(4)}x_{(1)}\phi_\beta(S^{-1}(a_{(2)})a_{(1)}) \otimes b_{(1)} \otimes a_{(3)} \\
&\quad = \sigma(x_{(2)}, b_{(2)})a_{(2)}x_{(1)} \otimes b_{(1)} \otimes a_{(1)},
\end{aligned}$$

and, since $x_{(1)} \otimes x_{(2)} \otimes x_{(3)} \otimes x_{(4)} = \sum_i \sigma(x_{(2)}, f_i)x_{(1)} \otimes e_{i(1)} \otimes e_{i(2)} \otimes e_{i(3)}$,

$$\begin{aligned}
&(\mathrm{id}_{A\otimes B\otimes A} \otimes \sigma(x, \cdot))(\tau_{\beta,\alpha}(\varphi_{\alpha^{-1}} \otimes \mathrm{id}_{H_\alpha})\Delta_{\alpha\beta\alpha^{-1},\alpha}(a \otimes b) \cdot R_{\alpha,\beta}) \\
&\quad = \sum_i \sigma(\phi_\alpha(e_{i(1)}), S(b_{(2)}))\, \sigma(e_{i(3)}, b_{(4)})\sigma(x, \phi^*_{\alpha^{-1}}(b_{(1)})f_i)a_{(2)}e_{i(2)} \otimes b_{(3)} \otimes a_{(1)} \\
&\quad = \sum_i \sigma(\phi_\alpha(e_{i(1)}), S(b_{(2)}))\sigma(e_{i(3)}, b_{(4)})\sigma(\phi_\alpha(x_{(1)}), b_{(1)})\sigma(x_{(2)}, f_i)a_{(2)}e_{i(2)} \otimes b_{(3)} \otimes a_{(1)} \\
&\quad = \sigma(\phi_\alpha(x_{(2)}), S(b_{(2)}))\sigma(x_{(4)}, b_{(4)})\sigma(\phi_\alpha(x_{(1)}), b_{(1)})a_{(2)}x_{(3)} \otimes b_{(3)} \otimes a_{(1)} \\
&\quad = \sigma(\phi_\alpha(x_{(1)}), b_{(1)}S(b_{(2)}))\sigma(x_{(3)}, b_{(4)})a_{(2)}x_{(2)} \otimes b_{(3)} \otimes a_{(1)} \\
&\quad = \sigma(x_{(2)}, b_{(2)})\; a_{(2)}x_{(1)} \otimes b_{(1)} \otimes a_{(1)}.
\end{aligned}$$

Hence, since the $\sigma(x, \cdot)$ span B^*, Axiom (1.7) is satisfied.

Let us verify Axiom (1.10). Let $\alpha, \beta, \gamma \in \pi$. Since ϕ^* is (σ, ϕ)-compatible, the basis $(\phi_\beta(e_i))_i$ of A and $(\phi^*_\beta(f_i))_i$ of B satisfy $\sigma(\phi_\beta(e_i), \phi^*_\beta(e_j)) = \sigma(e_i, f_j) = \delta_{i,j}$. Therefore we get that:

$$(\varphi_\beta \otimes \varphi_\beta)(R_{\alpha,\gamma}) = \sum_i \phi_\beta(e_i) \otimes 1_B \otimes 1_A \otimes \phi^*_\beta(f_j) = R_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}}.$$

Finally, let us check Axioms (1.8) and (1.9). Let $\alpha, \beta, \gamma \in \pi$. Using (2.15), we have:

$$\begin{aligned}
(\mathrm{id}_{D(A,B;\sigma,\phi_\alpha)} \otimes \Delta_{\beta,\gamma})(R_{\alpha,\beta\gamma}) &= \sum_i e_i \otimes 1_B \otimes 1_A \otimes f_{i(1)} \otimes 1_A \otimes f_{i(2)} \\
&= \sum_{i,j} e_ie_j \otimes 1_B \otimes 1_A \otimes f_j \otimes 1_A \otimes f_i \\
&= (R_{\alpha,\gamma})_{1\beta3} \cdot (R_{\alpha,\beta})_{12\gamma}.
\end{aligned}$$

Likewise, using (2.16) and (1.10), we have:

$$\begin{aligned}(\Delta_{\alpha,\beta} \otimes \mathrm{id}_{D(A,B;\sigma,\phi_\gamma)})(R_{\alpha\beta,\gamma}) &= \sum_i \phi_\beta(e_{i(1)}) \otimes 1_B \otimes e_{i(2)} \otimes 1_B \otimes 1_A \otimes f_i \\ &= \sum_{i,j} \phi_\beta(e_i) \otimes 1_B \otimes e_j \otimes 1_B \otimes 1_A \otimes f_i f_j \\ &= [(\varphi_\beta \otimes \mathrm{id}_{D(A,B;\sigma,\phi_\gamma)})(R_{\beta^{-1}\alpha\beta,\gamma})]_{1\beta 3} \cdot (R_{\beta,\gamma})_{\alpha 23} \\ &= [(\mathrm{id}_{D(A,B;\sigma,\phi_\alpha)} \otimes \varphi_{\beta^{-1}})(R_{\alpha,\beta\gamma\beta^{-1}})]_{1\beta 3} \cdot (R_{\beta,\gamma})_{\alpha 23}.\end{aligned}$$

This completes the proof of the quasitriangularity of $D(A, B; \sigma, \phi)$. □

The next corollary is a direct consequence of Corollary 2.5 and Theorem 2.6.

Corollary 2.8. *Let A be a finite-dimensional Hopf algebra and $\phi : \pi \to \mathrm{Aut}_{\mathrm{Hopf}}(A)$ be an action of π on A by Hopf algebras automorphisms. Recall that the duality bracket $\langle , \rangle_{A\otimes A^*}$ is a non-degenerate Hopf pairing between A and $A^{*\mathrm{cop}}$. Then $D(A, A^{*\mathrm{cop}}; \langle , \rangle_{A\otimes A^*}, \phi)$ is a quasitriangular Hopf π-coalgebra.*

Remark 2.9. The group of Hopf automorphisms of a finite-dimensional semisimple Hopf algebra A over a field of characteristic 0 is finite (see Radford, 1990). To obtain non-trivial examples of (quasitriangular) Hopf π-coalgebras for an infinite group π by using the twisted double method, one has to consider non-semisimple Hopf algebras (at least in characteristic 0).

2.3. The *h*-Adic Case

In this subsection, we develop the h-adic variant of Hopf group-coalgebras. A technical argument for the need of h-adic Hopf group-coalgebras is that they are necessary for a mathematically rigorous treatment of R-matrices for quantized enveloping algebras endowed with a group action.

Recall that if V is a vector space over $\mathbb{C}[[h]]$, the topology on V for which the sets $\{h^n V + v \mid n \in \mathbb{N}\}$ are a neighborhood base of $v \in V$ is called the *h-adic topology*. If V and W are vector spaces over $\mathbb{C}[[h]]$, we shall denote by $V \hat{\otimes} W$ the completion of the tensor product space $V \otimes_{\mathbb{C}[[h]]} W$ in the h-adic topology. Let V be a complex vector space. Then the set $V[[h]]$ of all formal power series $f = \sum_{n=0}^{\infty} v_n h^n$ with coefficients $v_n \in V$ is a vector space over $\mathbb{C}[[h]]$ which is complete in the h-adic topology. Furthermore, $V[[h]] \hat{\otimes} W[[h]] = (V \otimes W)[[h]]$ for any complex vector spaces V and W.

An *h-adic algebra* is a vector space A over $\mathbb{C}[[h]]$, which is complete in the h-adic topology and endowed with a $\mathbb{C}[[h]]$-linear map $m : A \hat{\otimes} A \to A$ and an element $1 \in A$ satisfying $m(\mathrm{id}_A \hat{\otimes} m) = m(m \hat{\otimes} \mathrm{id}_A)$ and $m(a \hat{\otimes} 1) = a = m(1 \hat{\otimes} a)$ for all $a \in A$.

By an *h-adic Hopf π-coalgebra*, we shall mean a family $H = \{H_\alpha\}_{\alpha\in\pi}$ of h-adic algebras which is endowed with h-adic algebra homomorphisms $\Delta_{\alpha,\beta} : H_{\alpha\beta} \to H_\alpha \hat{\otimes} H_\beta$ $(\alpha, \beta \in \pi)$ and $\varepsilon : A \to \mathbb{C}[[h]]$ satisfying (1.1) and (1.2), and with $C[[h]]$-linear maps $S_\alpha : H_\alpha \to H_{\alpha^{-1}} (\alpha \in \pi)$ satisfying (1.3). In the previous axioms, one has to replace the algebraic tensor products $\otimes$ by the h-adic completions $\hat{\otimes}$.

The notions of crossed and quasitriangular h-adic Hopf π-coalgebras can be defined similarly as in Sections 1.2 and 1.3.

The definitions of Section 2 and Theorem 2.3 carry over almost verbatim to h-adic Hopf algebras. The only modifications are that $\sigma : A \hat{\otimes} B \to \mathbb{C}[[h]]$ is $\mathbb{C}[[h]]$-linear and that the algebra $D(A, B; \sigma, \phi)$, where ϕ is an h-adic Hopf endomorphism of A, is built over the completion $A \hat{\otimes} B$ of $A \otimes B$ in the h-adic topology. The reasoning of the proof of Theorem 2.6 give the following h-adic version.

Theorem 2.10. *Let $\sigma : A \hat{\otimes} B \to \mathbb{C}[[h]]$ be an h-adic Hopf pairing between two h-adic Hopf algebras A and B, and $\phi : \pi \to \mathrm{Aut}_{\mathrm{Hopf}}(A)$ be an action of π on A by h-adic Hopf automorphisms. Suppose that σ is non-degenerate and that $(e_i)_i$ and $(f_i)_i$ are basis of the vector spaces A and B, respectively, which are dual with respect to the form σ. If $R_{\alpha,\beta} = \sum_i (e_i \otimes 1_B) \otimes (1_A \otimes f_i)$ belongs to the h-adic completion $D(A, B; \sigma, \phi_\alpha) \hat{\otimes} D(A, B; \sigma, \phi_\beta)$, then $R = \{R_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$ is a R-matrix of the crossed h-adic Hopf π-coalgebra $D(A, B; \sigma, \phi) = \{D(A, B; \sigma, \phi_\alpha)\}_{\alpha\in\pi}$.*

3. THE CASE OF ALGEBRAS OF FINITE GROUPS

Let G be a finite group. In this section, we describe Hopf G-coalgebras obtained by the twisted double method from the Hopf algebra $\Bbbk[G]$.

Recall that the Hopf algebra structure of the (finite-dimensional) $\Bbbk$-algebra $\Bbbk[G]$ of G is given by $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$ and $S(g) = g^{-1}$ for all $g \in G$. The dual of $\Bbbk[G]$ is the Hopf algebra $F(G) = \Bbbk^G$ of functions $G \to \Bbbk$. It has a basis $(e_g : G \to \Bbbk)_{g\in G}$ defined by $e_g(h) = \delta_{g,h}$ where $\delta_{g,g} = 1$ and $\delta_{g,h} = 0$ if $g \neq h$. The structure maps of $F(G)$ are given by $e_g e_h = \delta_{g,h} e_g$, $1_{F(G)} = \sum_{g\in G} e_g$, $\Delta(e_g) = \sum_{xy=g} e_x \otimes e_y$, $\varepsilon(e_g) = \delta_{g,1}$, and $S(e_g) = e_{g^{-1}}$ for any $g, h \in G$.

Set $\phi : G \to \mathrm{Aut}_{\mathrm{Hopf}}(\Bbbk[G])$ defined by $\phi_\alpha(h) = \alpha h \alpha^{-1}$. It is a well-defined group homomorphism (since any $\alpha \in G$ is grouplike in $\Bbbk[G]$). By Corollary 2.8, this datum leads to a quasitriangular Hopf G-coalgebra $D(\Bbbk[G], F(G)^{\mathrm{cop}}; \langle, \rangle_{\Bbbk[G]\times F(G)}, \phi)$, which will be denoted by $D_G(G) = \{D_\alpha(G)\}_{\alpha\in G}$.

Let us describe $D_G(G)$ more precisely. Let $\alpha \in G$. Recall that $D_\alpha(G)$ is equal to $\Bbbk[G] \otimes F(G)$ as a $\Bbbk$-space. The unit element and product of $D_\alpha(G)$ are given, for all $g, g', h, h' \in G$, by

$$1_{D_\alpha(G)} = \sum_{g\in G} 1 \otimes e_g \qquad \text{and} \qquad (g \otimes e_h) \cdot (g' \otimes e_{h'}) = \delta_{\alpha g' \alpha^{-1}, h^{-1} g' h'}\, g g' \otimes e_{h'}.$$

The structure maps of $D_G(G)$ are given, for any $\alpha, \beta \in G$ and $g, h \in G$, by

$$\Delta_{\alpha,\beta}(g \otimes e_h) = \sum_{xy=h} \beta g \beta^{-1} \otimes e_y \otimes g \otimes e_x,$$

$$\varepsilon(g \otimes e_h) = \delta_{h,1},$$

$$S_\alpha(g \otimes e_h) = \alpha g^{-1} \alpha^{-1} \otimes e_{\alpha g \alpha^{-1} h^{-1} g^{-1}},$$

$$\varphi_\alpha(g \otimes e_h) = \alpha g \alpha^{-1} \otimes e_{\alpha h \alpha^{-1}}.$$

The crossed Hopf G-coalgebra $D_G(G)$ is quasitriangular and furthermore ribbon with R-matrix and twist given, for any $\alpha, \beta \in G$, by

$$R_{\alpha,\beta} = \sum_{g,h\in G} g \otimes e_h \otimes 1 \otimes e_g \quad \text{and} \quad \theta_\alpha = \sum_{g\in G} \alpha^{-1} g\alpha \otimes e_g.$$

Note that $\theta_\alpha^n = \sum_{g\in G} \alpha^{-n}(g\alpha)^n \otimes e_g$ for any $n \in \mathbb{Z}$.

4. EXAMPLE OF A QUASITRIANGULAR HOPF $\mathrm{GL}_n(\Bbbk)$-COALGEBRA

In this section, $\Bbbk$ is a field whose characteristic is not 2. Fix a positive integer n. We use a (finite dimensional) Hopf algebra whose group of automorphisms is known to be the group $\mathrm{GL}_n(\Bbbk)$ of invertible $n \times n$-matrices with coefficients in $\Bbbk$ (see Radford, 1990) to derive an example of a quasitriangular Hopf $\mathrm{GL}_n(\Bbbk)$-coalgebra.

Definition-Proposition 4.1. *For $\alpha = (\alpha_{i,j}) \in GL_n(\Bbbk)$, let $\mathscr{A}_n^\alpha$ be the $\mathbb{C}$-algebra generated g, $x_1, \dots, x_n$, $y_1, \dots, y_n$, subject to the following relations*:

$$g^2 = 1, \quad x_1^2 = \cdots = x_n^2 = 0, \quad gx_i = -x_i g, \quad x_i x_j = -x_j x_i, \tag{4.1}$$

$$y_1^2 = \cdots = y_n^2 = 0, \quad gy_i = -y_i g, \quad y_i y_j = -y_j y_i, \tag{4.2}$$

$$x_i y_j - y_j x_i = (\alpha_{j,i} - \delta_{i,j}) g, \tag{4.3}$$

where $1 \le i, j \le n$. The family $\mathscr{A}_n = \{\mathscr{A}_n^\alpha\}_{\alpha\in\mathrm{GL}_n(\Bbbk)}$ has a structure of a crossed Hopf $GL_n(\Bbbk)$-coalgebra given, for any $\alpha = (\alpha_{i,j}) \in GL_n(\Bbbk)$, $\beta = (\beta_{i,j}) \in GL_n(\Bbbk)$, and $1 \le i \le n$, by:

$$\Delta_{\alpha,\beta}(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S_\alpha(g) = g, \tag{4.4}$$

$$\Delta_{\alpha,\beta}(x_i) = 1 \otimes x_i + \sum_{k=1}^n \beta_{k,i} x_k \otimes g, \quad \varepsilon(x_i) = 0, \quad S_\alpha(x_i) = \sum_{k=1}^n \alpha_{k,i} g x_k, \tag{4.5}$$

$$\Delta_{\alpha,\beta}(y_i) = y_i \otimes 1 + g \otimes y_i, \quad \varepsilon(y_i) = 0, \quad S_\alpha(y_i) = -g y_i, \tag{4.6}$$

$$\varphi_\alpha(g) = g, \quad \varphi_\alpha(x_i) = \sum_{k=1}^n \alpha_{k,i} x_k, \quad \varphi_\alpha(y_i) = \sum_{k=1}^n \tilde{\alpha}_{i,k} y_k, \tag{4.7}$$

where $(\tilde{\alpha}_{i,j}) = \alpha^{-1}$. Moreover $\mathscr{A}_n$ is quasitriangular with R-matrix given, for any $\alpha, \beta \in GL_n(\Bbbk)$, by:

$$R_{\alpha,\beta} = \frac{1}{2} \sum_{S\subseteq[n]} x_S \otimes y_S + x_S \otimes g y_S + g x_S \otimes y_S - g x_S \otimes g y_S.$$

Here $[n] = \{1, \dots, n\}$, $x_\emptyset = 1$, $y_\emptyset = 1$, and, for a nonempty subset S of $[n]$, we let $x_S = x_{i_1} \cdots x_{i_s}$ and $y_S = y_{i_1} \cdots y_{i_s}$ where $i_1 < \cdots < i_s$ are the elements of S.

Remark 4.2. Note that the algebras $\mathscr{A}_n^\alpha$ and $\mathscr{A}_n^\beta$ are in general not isomorphic when $\alpha, \beta \in \mathrm{GL}_n(\Bbbk)$ are such that $\alpha \neq \beta$. For example, we have that $\mathscr{A}_n^\alpha \not\simeq \mathscr{A}_n^1$ for any

$\alpha \in \mathrm{GL}_n(\Bbbk)$ with $\alpha \neq 1$. This can be shown by remarking that:

$$\mathcal{A}_n^\alpha/[\mathcal{A}_n^\alpha, \mathcal{A}_n^\alpha] \not\simeq \mathcal{A}_n^1/[\mathcal{A}_n^1, \mathcal{A}_n^1].$$

Indeed $\mathcal{A}_n^\alpha/[\mathcal{A}_n^\alpha, \mathcal{A}_n^\alpha] = 0$ since $g = \frac{1}{\alpha_{j,i}-\delta_{i,j}}(x_i y_j - y_j x_i) \in [\mathcal{A}_n^\alpha, \mathcal{A}_n^\alpha]$ (for some $1 \leq i, j \leq n$ such that $\alpha_{j,i} \neq \delta_{i,j}$) and so $1 = g^2 \in [\mathcal{A}_n^\alpha, \mathcal{A}_n^\alpha]$. Moreover, in $\mathcal{A}_n^1/[\mathcal{A}_n^1, \mathcal{A}_n^1]$, we have that $x_k = x_k g^2 = 0$ (since $x_k g = g x_k = -x_k g$ and so $x_k g = 0$) and likewise $y_k = 0$. Hence $\mathcal{A}_n^1/[\mathcal{A}_n^1, \mathcal{A}_n^1] = \Bbbk\langle g \mid g^2 = 1\rangle \not\simeq 0$.

Proof. Let A_n be the $\Bbbk$-algebra generated by $g, x_1, \ldots, x_n$, which satisfy the relations (4.1). The algebra A_n is 2^{n+1}-dimensional and is a Hopf algebra with structure maps defined by:

$$\Delta(g) = g \otimes g, \qquad \varepsilon(g) = 1, \qquad S(g) = g,$$
$$\Delta(x_i) = x_i \otimes g + 1 \otimes x_i, \qquad \varepsilon(x_i) = 0, \qquad S(x_i) = g x_i.$$

Radford (1990) showed that the group of Hopf automorphisms of A_n is isomorphic to the group $\mathrm{GL}_n(\Bbbk)$ of invertible $n \times n$-matrices with coefficients in $\Bbbk$. This group automorphism $\phi : \mathrm{GL}_n(\Bbbk) \to \mathrm{Aut}_{\mathrm{Hopf}}(A_n)$ is given by:

$$\phi_\alpha(g) = g \quad \text{ and } \quad \phi_\alpha(x_i) = \sum_{k=1}^{n} \alpha_{k,i} x_k \quad \text{for any } \alpha = (\alpha_{i,j}) \in \mathrm{GL}_n(\Bbbk).$$

The Hopf algebra $B_n = A_n^{\mathrm{cop}}$ is the $\Bbbk$-algebra generated by the symbols $h, y_1, \ldots, y_n$ which satisfy the relations $h^2 = 1$, $y_i^2 = 0$, $h y_i = -y_i h$, and $y_i y_j = -y_j y_i$. Its Hopf algebra structure is given by:

$$\Delta(h) = h \otimes h, \qquad \varepsilon(h) = 1, \qquad S(h) = h,$$
$$\Delta(y_i) = y_i \otimes 1 + h \otimes y_i, \qquad \varepsilon(y_i) = 0, \qquad S(y_i) = -h y_i.$$

Let us denote the cardinality of a set T by $|T|$. The elements $g^k x_S$ (resp. $h^k y_S$), where $k \in \{0, 1\}$ and $S \subseteq [n]$, form a basis for A_n (resp. B_n). Since Δ is multiplicative, it follows that

$$\Delta(g^k x_S) = \sum_{T \subseteq S} \lambda_{T,S} g^k x_T \otimes g^{k+|T|} x_{S \setminus T} \tag{4.8}$$

and

$$\Delta(h^k y_S) = \sum_{T \subseteq S} \lambda_{T,S} h^{k+|T|} y_{S \setminus T} \otimes h^k y_T, \tag{4.9}$$

where $\lambda_{T,S} = \pm 1$ and $\lambda_{\emptyset,S} = 1 = \lambda_{S,S}$.

By Section 2.1, there exists a (unique) Hopf pairing $\sigma : A_n \times B_n \to \Bbbk$ such that $\sigma(g, h) = -1$, $\sigma(g, y_j) = \sigma(x_i, h) = 0$, and $\sigma(x_i, y_j) = \delta_{i,j}$ for all $1 \leq i, j \leq n$. Using (4.8) and (4.9), one gets (by induction on $|S|$) that

$$\sigma(g^k x_S, h^l y_T) = (-1)^{kl} \delta_{S,T}$$

for any $k, l \in \{0, 1\}$ and $S, T \subseteq [n]$, where $\delta_{S,S} = 1$ and $\delta_{S,T} = 0$ if $S \neq T$. Set $z_0 = (1 + h)/2$ and $z_1 = (1 - h)/2$. The elements $z_k y_S$, where $k \in \{0, 1\}$ and $S \subseteq [n]$, form a basis for B_n such that:

$$\sigma(g^k x_S, z_l y_T) = \delta_{k,l}\delta_{S,T} \tag{4.10}$$

for any $k, l \in \{0, 1\}$ and $S, T \subseteq [n]$. Therefore the pairing σ is non-degenerate. Note that this implies that $A_n^* \cong A_n$ as a Hopf algebra.

By Theorem 2.6, we get a quasitriangular Hopf $\mathrm{GL}_n(\Bbbk)$-coalgebra $D(A_n, B_n; \sigma, \phi)$. For any $\alpha = (\alpha_{i,j}) \in \mathrm{GL}_n(\Bbbk)$, $D(A_n, B_n; \sigma, \phi_\alpha)$ is the algebra generated by g, h, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, subject to the relations $h^2 = 1$, (4.1), (4.2) with g replaced by h, and the following relations:

$$gh = hg, \qquad gy_j = -y_j g, \qquad hx_i = -x_i h, \tag{4.11}$$

$$x_i y_j - y_j x_i = \alpha_{j,i} g - \delta_{i,j} h. \tag{4.12}$$

Indeed $D(A_n, B_n; \sigma, \phi_\alpha)$ is the free algebra generated by the algebras A_n and B_n with cross relation (2.5). Further, it suffices to require the cross relations (2.5) for $(1 \otimes b) \cdot (a \otimes 1)$ with $a = g, x_i$ and $b = h, y_j$. To simplify the notations, we identify of a with $a \otimes 1$ and b with $1 \otimes b$ (recall that these natural maps $A_n \hookrightarrow D(A_n, B_n; \sigma, \phi_\alpha)$ and $B_n \hookrightarrow D(A_n, B_n; \sigma, \phi_\alpha)$ are algebra monomorphisms). For example, let $a = x_i$ and $b = y_j$. Since $\sigma(x_i, 1) = \sigma(g, y_j) = \sigma(x_i, h) = \sigma(1, y_j) = 0$, relation (2.5) gives

$$y_j x_i = \sigma(\phi_\alpha(x_i), y_j h)\sigma(g, 1)g \cdot 1 + \sigma(1, h)\sigma(g, 1)x_i \cdot y_j + \sigma(1, h)\sigma(x_i, y_j)1 \cdot h.$$

Inserting the values $\sigma(g, 1) = \sigma(1, h) = 1$, $\sigma(x_i, y_j) = \delta_{i,j}$, and $\sigma(\phi_\alpha(x_i), y_j h) = -\alpha_{j,i}$, we get (4.12).

From Theorem 2.3, we obtain that the comultiplication $\Delta_{\alpha,\beta}$, the counit ε, the antipode S_α, and the crossing φ_α of $D(A_n, B_n; \sigma, \phi_\alpha)$ are given by

$$\Delta_{\alpha,\beta}(g) = g \otimes g, \qquad \Delta_{\alpha,\beta}(h) = h \otimes h, \tag{4.13}$$

$$\Delta_{\alpha,\beta}(x_i) = 1 \otimes x_i + \sum_{k=1}^{n} \beta_{k,i} x_k \otimes g, \qquad \Delta_{\alpha,\beta}(y_i) = y_i \otimes 1 + h \otimes y_i, \tag{4.14}$$

$$\varepsilon(g) = \varepsilon(h) = 1, \qquad \varepsilon(x_i) = \varepsilon(y_i) = 0, \qquad S_\alpha(g) = g, \tag{4.15}$$

$$S_\alpha(h) = h, \qquad S_\alpha(x_i) = \sum_{k=1}^{n} \alpha_{k,i} g x_k, \qquad S_\alpha(y_i) = -hy_i, \tag{4.16}$$

$$\varphi_\alpha(g) = g, \qquad \varphi_\alpha(h) = h, \qquad \varphi_\alpha(x_i) = \sum_{k=1}^{n} \alpha_{k,i} x_k, \qquad \varphi_\alpha(y_i) = \sum_{k=1}^{n} \tilde{\alpha}_{i,k} y_k, \tag{4.17}$$

where $(\tilde{\alpha}_{i,j}) = \alpha^{-1}$.

For any $\alpha \in \mathrm{GL}_n(\Bbbk)$, let I_α be the ideal of $D(A_n, B_n; \sigma, \phi_\alpha)$ generated by $g - h$. Using the above description of the structure maps of $D(A_n, B_n; \sigma, \phi)$, we get that $I = \{I_\alpha\}_{\alpha \in \pi}$ is a crossed Hopf $\mathrm{GL}_n(\Bbbk)$-coideal of $D(A_n, B_n; \sigma, \phi)$. The quotient $D(A_n, B_n; \sigma, \phi)/I = \{D(A_n, B_n; \sigma, \phi_\alpha)/I_\alpha\}_{\alpha \in \mathrm{GL}_n(\Bbbk)}$ is precisely

$\mathscr{A}_n = \{\mathscr{A}_n^\alpha\}_{\alpha \in \mathrm{GL}_n(\Bbbk)}$ and so the latter has a quasitriangular Hopf $\mathrm{GL}_n(\Bbbk)$-coalgebra structure which can be described by replacing h with g in (4.13)–(4.17).

Finally, the R-matrix of $\mathscr{A}_n$ is obtained as the image under the projection maps $D(A_n, B_n; \sigma, \phi_\alpha) \xrightarrow{p_\alpha} D(A_n, B_n; \sigma, \phi_\alpha)/I_\alpha = \mathscr{A}_n^\alpha$ of the R-matrix of $D(A_n, B_n; \sigma, \phi)$, that is, using (4.10),

$$\begin{aligned} R_{\alpha,\beta} &= \sum_{S \subseteq [n]} p_\alpha(x_S) \otimes p_\beta(z_0 y_S) + p_\alpha(g x_S) \otimes p_\beta(z_1 y_S) \\ &= \sum_{S \subseteq [n]} x_S \otimes \left(\frac{1+g}{2}\right) y_S + g x_S \otimes \left(\frac{1-g}{2}\right) y_S \\ &= \frac{1}{2} \sum_{S \subseteq [n]} x_S \otimes y_S + x_S \otimes g y_S + g x_S \otimes y_S - g x_S \otimes g y_S. \end{aligned}$$

This completes the proof of Proposition 4.1. □

5. GRADED QUANTUM GROUPS

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra of rank l with Cartan matrix $(a_{i,j})$. We let d_i be the coprime integers such that the matrix $(d_i a_{i,j})$ is symmetric. Let q be a fixed nonzero complex number and set $q_i = q^{d_i}$. Suppose that $q_i^2 \neq 1$ for $i = 1, 2, \ldots, l$.

Definition-Proposition 5.1. *Set $\pi = (\mathbb{C}^*)^l$. For $\alpha = (\alpha_1, \ldots, \alpha_l) \in \pi$, let $U_q^\alpha(\mathfrak{g})$ be the $\mathbb{C}$-algebra generated by $K_i^{\pm 1}$, E_i, F_i, $1 \leq i \leq l$, subject to the following defining relations*:

$$K_i K_j = K_j K_i, \qquad K_i K_i^{-1} = K_i^{-1} K_i = 1, \tag{5.1}$$

$$K_i E_j = q_i^{a_{i,j}} E_j K_i, \tag{5.2}$$

$$K_i F_j = q_i^{-a_{i,j}} F_j K_i, \tag{5.3}$$

$$E_i F_j - F_j E_i = \delta_{i,j} \frac{\alpha_i K_i - K_i^{-1}}{q_i - q_i^{-1}}, \tag{5.4}$$

$$\sum_{r=0}^{1-a_{i,j}} (-1)^r \begin{bmatrix} 1 - a_{i,j} \\ r \end{bmatrix}_{q_i} E_i^{1-a_{i,j}-r} E_j E_i^r = 0 \quad \text{if } i \neq j, \tag{5.5}$$

$$\sum_{r=0}^{1-a_{i,j}} (-1)^r \begin{bmatrix} 1 - a_{i,j} \\ r \end{bmatrix}_{q_i} F_i^{1-a_{i,j}-r} F_j F_i^r = 0 \quad \text{if } i \neq j. \tag{5.6}$$

The family $U_q^\pi(\mathfrak{g}) = \{U_q^\alpha(\mathfrak{g})\}_{\alpha \in \pi}$ has a structure of a crossed Hopf π-coalgebra given, for $\alpha = (\alpha_1, \ldots, \alpha_l) \in \pi$, $\beta = (\beta_1, \ldots, \beta_l) \in \pi$ and $1 \leq i \leq l$, by:

$$\begin{aligned} \Delta_{\alpha,\beta}(K_i) &= K_i \otimes K_i, \\ \Delta_{\alpha,\beta}(E_i) &= \beta_i E_i \otimes K_i + 1 \otimes E_i, \\ \Delta_{\alpha,\beta}(F_i) &= F_i \otimes 1 + K_i^{-1} \otimes F_i, \end{aligned}$$

$$\begin{aligned}
&\varepsilon(K_i) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0,\\
&S_\alpha(K_i) = K_i^{-1}, \quad S_\alpha(E_i) = -\alpha_i E_i K_i^{-1}, \quad S_\alpha(F_i) = -K_i F_i,\\
&\varphi_\alpha(K_i) = K_i, \quad \varphi_\alpha(E_i) = \alpha_i E_i, \quad \varphi_\alpha(F_i) = \alpha_i^{-1} F_i.
\end{aligned}$$

Remark 5.2. Note that $(U_q^1(\mathfrak{g}), \Delta_{1,1}, \varepsilon, S_1)$ is the usual quantum group $U_q(\mathfrak{g})$.

Proof. Let U_+ be the $\mathbb{C}$-algebra generated by E_i, $K_i^{\pm 1}$, $1 \leq i \leq l$, subject to the relations (5.1), (5.2) and (5.5). Let U_- be the $\mathbb{C}$-algebra generated by F_i, $K_i'^{\pm 1}$, $1 \leq i \leq l$, subject to the relations (5.1), (5.3) and (5.6), where one has to replace K_i with K_i'. The algebras U_+ and U_- have a Hopf algebra structure given by

$$\begin{aligned}
&\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i,\\
&\varepsilon(K_i) = 1, \quad \varepsilon(E_i) = 0, \quad S(K_i) = K_i^{-1}, \quad S(E_i) = -E_i K_i^{-1},\\
&\Delta(K_i') = K_i' \otimes K_i', \quad \Delta(F_i) = F_i \otimes 1 + K_i'^{-1} \otimes F_i,\\
&\varepsilon(K_i') = 1, \quad \varepsilon(F_i) = 0, \quad S(K_i') = K_i'^{-1}, \quad S(F_i) = -K_i' F_i.
\end{aligned}$$

Using the method described in Section 2.1, it can be verified that there exists a (unique) Hopf pairing $\sigma : U_+ \times U_- \to \mathbb{C}$ such that

$$\sigma(E_i, F_j) = \frac{\delta_{i,j}}{q_i - q_i^{-1}}, \quad \sigma(E_i, K_j') = \sigma(K_i, F_j) = 0, \quad \sigma(K_i, K_j') = q_i^{a_{i,j}} = q_j^{a_{j,i}}.$$

Let $\phi : \pi \to \mathrm{Aut}_{\mathrm{Hopf}}(U_+)$ and $\psi : \pi \to \mathrm{Aut}_{\mathrm{Hopf}}(U_-)$ be the group homomorphisms defined as follows: for $\beta = (\beta_1, \ldots, \beta_l) \in \pi$ and $1 \leq i \leq l$, set

$$\phi_\beta(K_i) = K_i, \quad \phi_\beta(E_i) = \beta_i E_i, \quad \psi_\beta(K_i') = K_i', \quad \psi_\beta(F_i) = \beta_i^{-1} F_i.$$

It is straightforward to verify that ψ is (σ, ϕ)-compatible. By Lemma 2.4, we can consider the crossed Hopf π-coalgebra $D(U_+, U_-; \sigma, \phi) = \{D(U_+, U_-; \sigma, \phi_\alpha)\}_{\alpha \in \pi}$.

Now, for any $\alpha \in \pi$, $D(U_+, U_-; \sigma, \phi_\alpha)$ is the algebra generated by $K_i^{\pm 1}$, $K_i'^{\pm 1}$, E_i, F_i, where $1 \leq i \leq l$, subject to the relations (5.1), (5.2), (5.5), the relations (5.1), (5.3), (5.6) with K_i replaced by K_i', and the following relations:

$$K_i K_j' = K_j' K_i, \quad K_i F_j = q_i^{-a_{i,j}} F_j K_i, \quad K_i' E_j = q_i^{a_{i,j}} E_j K_i', \tag{5.7}$$

$$E_i F_j - F_j E_i = \delta_{i,j} \frac{\alpha_i K_i - K_i'^{-1}}{q_i - q_i^{-1}}. \tag{5.8}$$

Indeed, $D(U_+, U_-; \sigma, \phi_\alpha)$ is the free algebra generated by the algebras U_+ and U_- with cross relation (2.5). Further, it suffices to require the cross relations (2.5) for $(1 \otimes b) \cdot (a \otimes 1)$ with $a = K_i, E_i$ and $b = K_i', F_i$. To simplify the notations, we identify of a with $a \otimes 1$ and b with $1 \otimes b$ (recall that these natural maps $U_+ \hookrightarrow D(U_+, U_-; \sigma, \phi_\alpha)$ and $U_- \hookrightarrow D(U_+, U_-; \sigma, \phi_\alpha)$ are algebra monomorphisms). For example, let $a = E_i$ and $b = F_j$. Since $\sigma(E_i, 1) = \sigma(K_i, F_j) = \sigma(E_i, K_j'^{-1}) = \sigma(1, F_j) = 0$, relation (2.5) gives

$$F_j E_i = \sigma(\alpha_i E_i, S(F_j))\sigma(K_i, 1)K_i + \sigma(1, K_j')\sigma(K_i, 1)E_i F_j + \sigma(1, K_j')\sigma(E_i, F_j)K_j'^{-1}.$$

Inserting the values $\sigma(K_i, 1) = \sigma(1, K'_j) = 1$, $\sigma(E_i, F_j) = \delta_{i,j}(q_i - q_i^{-1})^{-1}$ and $\sigma(E_i, S(F_j)) = -\delta_{i,j}(q_i - q_i^{-1})^{-1}$, we get (5.8).

From Theorem 2.3, we obtain that the comultiplication $\Delta_{\alpha,\beta}$, the counit ε, the antipode S_α, and the crossing φ_α of $D(U_+, U_-; \sigma, \phi)$ are given, for $1 \leq i \leq l$, by

$$\Delta_{\alpha,\beta}(K_i) = K_i \otimes K_i, \qquad \Delta_{\alpha,\beta}(K'_i) = K'_i \otimes K'_i, \tag{5.9}$$

$$\Delta_{\alpha,\beta}(E_i) = \beta_i E_i \otimes K_i + 1 \otimes E_i, \qquad \Delta_{\alpha,\beta}(F_i) = F_i \otimes 1 + K_i'^{-1} \otimes F_i, \tag{5.10}$$

$$\varepsilon(K_i) = \varepsilon(K'_i) = 1, \qquad \varepsilon(E_i) = \varepsilon(F_i) = 0, \qquad S_\alpha(K_i) = K_i^{-1}, \tag{5.11}$$

$$S_\alpha(K'_i) = K_i'^{-1}, \qquad S_\alpha(E_i) = -\alpha_i E_i K_i^{-1}, \qquad S_\alpha(F_i) = -K'_i F_i, \tag{5.12}$$

$$\varphi_\alpha(K_i) = K_i, \qquad \varphi_\alpha(K'_i) = K'_i, \qquad \varphi_\alpha(E_i) = \alpha_i E_i, \qquad \varphi_\alpha(F_i) = \alpha_i^{-1} F_i. \tag{5.13}$$

Finally, for any $\alpha \in \pi$, let I_α be the ideal of $D(U_+, U_-; \sigma, \phi_\alpha)$ generated by $K_i - K'_i$ and $K_i^{-1} - K_i'^{-1}$, where $1 \leq i \leq l$. Using the above description of the structure maps of $D(U_+, U_-; \sigma, \phi)$, we get that $I = \{I_\alpha\}_{\alpha\in\pi}$ is a crossed Hopf π-coideal of $D(U_+, U_-; \sigma, \phi)$. The quotient $D(U_+, U_-; \sigma, \phi)/I = \{D(U_+, U_-; \sigma, \phi_\alpha)/I_\alpha\}_{\alpha\in\pi}$ is precisely $U_q^\pi(\mathfrak{g}) = \{U_q^\alpha(\mathfrak{g})\}_{\alpha\in\pi}$. Hence the latter has a crossed Hopf π-coalgebra structure given by replacing K'_i with K_i in (5.9)–(5.13). □

Remark 5.3. In the above construction, we use the diagonal Hopf automorphisms of $U_+ = U_q(\mathfrak{b}_+)$. What happens if we use also the Hopf automorphisms coming from diagram automorphisms? Recall that a *diagram automorphism* of $\mathfrak{g}$ is a permutation ω of $\{1, \ldots, l\}$ such that $a_{\omega(i),\omega(j)} = a_{i,j}$ for all $1 \leq i, j \leq l$. Denote by Γ the group of diagram automorphisms of $\mathfrak{g}$. In the following table, we recall the isomorphism class of Γ depending on the type of $\mathfrak{g}$ (see, e.g., Bourbaki, 1981):

$\mathfrak{g}$	A_1	A_l $(l \geq 2)$	B_l $(l \geq 2)$	C_l $(l \geq 2)$	D_l $(l \geq 3, l \neq 4)$	D_4	E_6	E_7	E_8	F_4	G_2
Γ	1	$\mathbb{Z}_2$	1	1	$\mathbb{Z}_2$	$\mathfrak{S}_3$	$\mathbb{Z}_2$	1	1	1	1

There exists a group morphism $\phi : \Gamma \times (\mathbb{C}^*)^l \to \mathrm{Aut}_{\mathrm{Hopf}}(U_+)$ defined by $\phi_\beta(K_i) = K_{\omega(i)}$ and $\phi_\beta(E_i) = \beta_i E_{\omega(i)}$ for $\beta = (\omega, \beta_1, \ldots, \beta_l) \in \Gamma \times (\mathbb{C}^*)^l$ and $1 \leq i \leq l$. Note that ϕ is in fact a group isomorphism, see Fleury (1997). We can then consider the Hopf $\left(\Gamma \times (\mathbb{C}^*)^l\right)$-coalgebra $D(U_+, U_-; \sigma, \phi)$. Nevertheless, unlike in the proof of Proposition 5.1, there is no natural way to quotient $D(U_+, U_-; \sigma, \phi)$ in order to eliminate the K'_j.

6. *h*-ADIC GRADED QUANTUM GROUPS

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra of rank l with Cartan matrix $(a_{i,j})$. We let d_i be the coprime integers such that the matrix $(d_i a_{i,j})$ is symmetric.

Definition-Proposition 6.1. *Set $\pi = \mathbb{C}[[h]]^l$. For $\alpha = (\alpha_1, \ldots, \alpha_l) \in \pi$, let $U_h^\alpha(\mathfrak{g})$ be the h-adic algebra generated by the elements H_i, E_i, F_i, $1 \leq i \leq l$, subject to the* *following defining relations*:

$$[H_i, H_j] = 0, \tag{6.1}$$

$$[H_i, E_j] = a_{ij}E_j, \tag{6.2}$$

$$[H_i, F_j] = -a_{ij}F_j, \tag{6.3}$$

$$[E_i, F_j] = \delta_{i,j}\frac{e^{d_i h\alpha_i}e^{d_i hH_i} - e^{-d_i hH_i}}{e^{d_i h} - e^{-d_i h}}, \tag{6.4}$$

$$\sum_{r=0}^{1-a_{i,j}} (-1)^r \begin{bmatrix} 1-a_{i,j} \\ r \end{bmatrix}_{e^{d_i h}} E_i^{1-a_{i,j}-r} E_j E_i^r = 0 \quad (i \neq j), \tag{6.5}$$

$$\sum_{r=0}^{1-a_{i,j}} (-1)^r \begin{bmatrix} 1-a_{i,j} \\ r \end{bmatrix}_{e^{d_i h}} F_i^{1-a_{i,j}-r} F_j F_i^r = 0 \quad (i \neq j). \tag{6.6}$$

The family $U_h^\pi(\mathfrak{g}) = \{U_h^\alpha(\mathfrak{g})\}_{\alpha\in\pi}$ has a structure of a crossed h-adic Hopf π-coalgebra given, for $\alpha = (\alpha_1, \ldots, \alpha_l) \in \pi$, $\beta = (\beta_1, \ldots, \beta_l) \in \pi$ and $1 \leq i \leq l$, by:

$$\Delta_{\alpha,\beta}(H_i) = H_i \otimes 1 + 1 \otimes H_i,$$

$$\Delta_{\alpha,\beta}(E_i) = e^{d_i h\beta_i}E_i \otimes e^{d_i hH_i} + 1 \otimes E_i,$$

$$\Delta_{\alpha,\beta}(F_i) = F_i \otimes 1 + e^{-d_i hH_i} \otimes F_i,$$

$$\varepsilon(H_i) = \varepsilon(E_i) = \varepsilon(F_i) = 0,$$

$$S_\alpha(H_i) = -H_i, \quad S_\alpha(E_i) = -e^{d_i h\alpha_i}E_i e^{-d_i hH_i}, \quad S_\alpha(F_i) = -e^{d_i hH_i}F_i,$$

$$\varphi_\alpha(H_i) = H_i, \quad \varphi_\alpha(E_i) = e^{d_i h\alpha_i}E_i, \quad \varphi_\alpha(F_i) = e^{-d_i h\alpha_i}F_i.$$

Remark 6.2. (a) $(U_h^0(\mathfrak{g}), \Delta_{0,0}, \varepsilon, S_0)$ is the usual quantum group $U_h(\mathfrak{g})$.

(b) The element $e^{d_i h} - e^{-d_i h} \in \mathbb{C}[[h]]$ is not invertible in $\mathbb{C}[[h]]$, because the constant term is zero. But the expression of the right hand side of (6.4) is a formal power series $\sum_n p_n(H_i)h^n$ with certain polynomials $p_n(H_i)$, and so it is a well-defined element of the h-adic algebra generated by E_i, F_i, H_i.

Proof. Let U_+ be the h-adic algebra generated by H_i, E_i, $1 \leq i \leq l$, subject to the relations (6.1), (6.2) and (6.5). Let U_- be the h-adic algebra generated by H_i', F_i, $1 \leq i \leq l$, subject to the relations (6.1), (6.3) and (6.6) with H_i replaced by H_i'. The algebras U_+ and U_- have a h-adic Hopf algebra structure given by:

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \Delta(E_i) = E_i \otimes e^{d_i hH_i} + 1 \otimes E_i,$$

$$\varepsilon(H_i) = \varepsilon(E_i) = 0, \quad S(H_i) = -H_i, \quad S(E_i) = -E_i e^{-d_i hH_i},$$

$$\Delta(H_i') = H_i' \otimes 1 + 1 \otimes H_i', \quad \Delta(F_i) = F_i \otimes 1 + e^{-d_i hH_i'} \otimes F_i,$$

$$\varepsilon(H_i') = \varepsilon(F_i) = 0, \quad S(H_i') = -H_i', \quad S(F_i) = -e^{d_i hH_i'}F_i.$$

In order to construct a Hopf pairing adapted to our needs, let us consider the h-adic Hopf algebra $\widetilde{U}_- = \mathbb{C}[[h]]1 + hU_-$. The elements $\widetilde{H}'_i = hH'_i$ and $\widetilde{F}_i = hF_i$ belong to $\widetilde{U}_-$ and satisfy

$$[\widetilde{H}'_i, \widetilde{F}_j] = -ha_{ij}\widetilde{F}_j, \qquad \Delta(\widetilde{H}'_i) = \widetilde{H}'_i \otimes 1 + 1 \otimes \widetilde{H}'_i, \qquad \Delta(\widetilde{F}_i) = \widetilde{F}_i \otimes 1 + e^{-d_i\widetilde{H}'_i} \otimes \widetilde{F}_i.$$

The element $e^{-d_i\widetilde{H}'_i} = 1 + \sum_{k\geq 1} \frac{1}{k!}(-d_ih)^k H_i'^k$ is also in $\widetilde{U}_-$. Note that $e^{-d_i\widetilde{H}'_i}$ is not in the h-adic subalgebra of $\widetilde{U}_-$ generated by $\widetilde{H}'_i$. Using the method described in Section 2.1 (see also Klimyk and Schmudgen, 1997, Proposition 38), it can be verified that there exists a (unique) Hopf pairing $\sigma : U_+ \times \widetilde{U}_- \to \mathbb{C}[[h]]$ such that:

$$\sigma(H_i, \widetilde{H}'_j) = d_i^{-1}a_{j,i}, \qquad \sigma(H_i, \widetilde{F}_j) = \sigma(E_i, \widetilde{H}'_j) = 0, \qquad \sigma(E_i, \widetilde{F}_j) = \frac{\delta_{i,j}\, h}{e^{d_ih} - e^{-d_ih}}.$$

Let $\phi : \pi \to \mathrm{Aut}_{\mathrm{Hopf}}(U_+)$ and $\psi : \pi \to \mathrm{Aut}_{\mathrm{Hopf}}(\widetilde{U}_-)$ defined, for $\alpha = (\alpha_1, \ldots, \alpha_l) \in \pi$ and $1 \leq i \leq l$, by

$$\phi_\alpha(H_i) = H_i, \qquad \phi_\alpha(E_i) = e^{d_ih\alpha_i}E_i, \qquad \psi_\alpha(\widetilde{H}'_i) = \widetilde{H}'_i, \qquad \psi_\alpha(\widetilde{F}_i) = e^{-d_ih\alpha_i}\widetilde{F}_i.$$

It is straightforward to verify that ψ is (σ, ϕ)-compatible. By the h-adic version of Lemma 2.4, we can consider the crossed h-adic Hopf π-coalgebra $D(U_+, \widetilde{U}_-; \sigma, \phi) = \{D(U_+, \widetilde{U}_-; \sigma, \phi_\alpha)\}_{\alpha\in\pi}$ whose structure can be explicitly described as in the proof of Proposition 5.1.

For any $\alpha \in \pi$, let I_α be the h-adic ideal of $D(U_+, \widetilde{U}_-; \sigma, \phi_\alpha)$ generated by $\widetilde{H}'_i - hH_i$ where $1 \leq i \leq l$. Using the description of the structure maps of $D(U_+, \widetilde{U}_-; \sigma, \phi_\alpha)$, we get that $I = \{I_\alpha\}_{\alpha\in\pi}$ is a crossed h-adic Hopf π-coideal of $D(U_+, U_-; \sigma, \phi)$. The quotient $D(U_+, \widetilde{U}_-; \sigma, \phi)/I = \{D(U_+, \widetilde{U}_-; \sigma, \phi_\alpha)/I_\alpha\}_{\alpha\in\pi}$ is precisely $U_h^\pi(\mathfrak{g}) = \{U_h^\alpha(\mathfrak{g})\}_{\alpha\in\pi}$. Hence the latter has a structure of a crossed h-adic Hopf π-coalgebra. □

It is well-know (see, e.g., Klimyk and Schmudgen, 1997) that the Hopf pairing $\sigma : U_+ \times \widetilde{U}_- \to \mathbb{C}[[h]]$ is non-degenerate and that, if $(e_i)_i$ and $(f_i)_i$ are dual basis of the vector spaces U_+ and $\widetilde{U}_-$ with respect to the form σ, then $\sum_i (e_i \otimes 1) \otimes (1 \otimes f_i)$ belongs to the h-adic completion $D(U_+, \widetilde{U}_-; \sigma, \phi_\alpha) \hat{\otimes} D(U_+, \widetilde{U}_-; \sigma, \phi_\beta)$. Therefore, by Theorem 2.10, the crossed h-adic Hopf π-coalgebra $D(U_+, \widetilde{U}_-; \sigma, \phi)$ is quasitriangular. Hence, as a quotient of $D(U_+, \widetilde{U}_-; \sigma, \phi)$, $U_h^\pi(\mathfrak{g})$ is also quasitriangular.

For example, when $\mathfrak{g} = \mathfrak{sl}_2$ and so $\pi = \mathbb{C}[[h]]$, we have that the R-matrix of $U_h^{\mathbb{C}[[h]]}(\mathfrak{sl}_2)$ is given, for any $\alpha, \beta \in \mathbb{C}[[h]]$, by

$$R_{\alpha,\beta} = e^{h(H\otimes H)/2} \sum_{n=0}^{\infty} R_n(h)\, E^n \otimes F^n \in U_h^\alpha(\mathfrak{sl}_2) \hat{\otimes} U_h^\beta(\mathfrak{sl}_2),$$

where $R_n(h) = q^{n(n+1)/2} \frac{(1-q^{-2})^n}{[n]_q!}$ and $q = e^h$.

Let $\alpha \in \mathbb{C}[[h]]$. For any non-negative integer n, consider a $(n+1)$-dimensional $\mathbb{C}$-vector space V_n with basis $\{v_0, \ldots, v_n\}$. The space $V_n^\alpha = V_n[[h]] = V_n \otimes \mathbb{C}[[h]]$ has a structure of a (topological) left $U_h^\alpha(\mathfrak{sl}_2)$-module given, for $0 \leq i \leq n$, as follows:

$$H \cdot v_i = \left(n - 2i - \frac{\alpha}{2}\right) v_i,$$

$$E \cdot v_i = \begin{cases} e^{\frac{h\alpha}{2}}[n-i+1]_q v_{i-1} & \text{if } i > 0, \\ 0 & \text{if } i = 0, \end{cases}$$

$$F \cdot v_i = \begin{cases} [i+1]_q v_{i+1} & \text{if } i < n, \\ 0 & \text{if } i = n. \end{cases}$$

Together with the quasitriangularity of $U_h^{\mathbb{C}[[h]]}(\mathfrak{sl}_2)$, these data lead in particular to a solution of the $\mathbb{C}[[h]]$-colored Yang-Baxter equation.

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