Appendix 6
Algebraic properties of Hopf G-coalgebras
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Let G be a group. The notion of a (ribbon) Hopf G-coalgebra was first introduced by Turaev [Tu4], as the prototype algebraic structure whose category of representations is a (ribbon) G-category (see Section VIII.1). Recall from Chapter VII that ribbon G-categories give rise to invariants of 3-dimensional G-manifolds and to 3-dimensional HQFTs with target $K(G,1)$. Moreover, Hopf G-coalgebras may be used directly (without involving their representations) to construct further topological invariants of 3-dimensional G-manifolds, see Appendix 7.

Here we review the algebraic properties of Hopf G-coalgebras and provide examples. Most of the results are given without proof, see [Vir1]–[Vir4] for details.

In Section 1, we study the algebraic properties of Hopf G-coalgebras, in particular the existence of integrals, the order of the antipode (a generalization of the Radford S^4-formula), and the (co)semisimplicity (a generalization of the Maschke theorem).

In Section 2, we focus on quasitriangular and ribbon Hopf G-coalgebras. In particular we construct G-traces for ribbon Hopf G-coalgebras, which are used to construct invariants of 3-dimensional G-manifolds in Appendix 7.

In Section 3, we give a method for constructing a quasitriangular Hopf G-coalgebra starting from a Hopf algebra endowed with an action of G by Hopf automorphisms. This leads to non-trivial examples of quasitriangular Hopf G-coalgebras for all finite G and for some infinite G such as $\mathrm{GL}_n(K)$. In particular, we define graded quantum groups.

Throughout this appendix, G is a group (with neutral element 1) and K is a field. All algebras are supposed to be over K, associative, and unital. The tensor product $\otimes = \otimes_K$ of K-vector spaces is always taken over K. If U and V are K-vector spaces, then $\sigma_{U,V}: U \otimes V \to V \otimes U$ denotes the flip defined by $\sigma_{U,V}(u \otimes v) = v \otimes u$ for all $u \in U$ and $v \in V$.

6.1 Hopf G-coalgebras

1.1 Hopf G-coalgebras. We recall, for completeness, the definition of a Hopf G-coalgebra from Section VIII.1, but with a minor change: we do not suppose the antipode to be bijective.

A *Hopf G-coalgebra* (over K) is a family $H = \{H_\alpha\}_{\alpha \in G}$ of K-algebras endowed with a family $\Delta = \{\Delta_{\alpha,\beta}: H_{\alpha\beta} \to H_\alpha \otimes H_\beta\}_{\alpha,\beta \in G}$ of algebra homomorphisms (the *comultiplication*), an algebra homomorphism $\varepsilon\colon H_1 \to K$ (the *counit*), and a family $S = \{S_\alpha\colon H_\alpha \to H_{\alpha^{-1}}\}_{\alpha\in G}$ of K-linear maps (the *antipode*) such that, for all $\alpha, \beta, \gamma \in G$,

$$(\Delta_{\alpha,\beta} \otimes \mathrm{id}_{H_\gamma})\Delta_{\alpha\beta,\gamma} = (\mathrm{id}_{H_\alpha} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma},$$

$$(\mathrm{id}_{H_\alpha} \otimes \varepsilon)\Delta_{\alpha,1} = \mathrm{id}_{H_\alpha} = (\varepsilon \otimes \mathrm{id}_{H_\alpha})\Delta_{1,\alpha},$$

$$m_\alpha(S_{\alpha^{-1}} \otimes \mathrm{id}_{H_\alpha})\Delta_{\alpha^{-1},\alpha} = \varepsilon\, 1_\alpha = m_\alpha(\mathrm{id}_{H_\alpha} \otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}},$$

where $m_\alpha\colon H_\alpha \otimes H_\alpha \to H_\alpha$ and $1_\alpha \in H_\alpha$ denote multiplication in H_α and the unit element of H_α.

When $G = 1$, one recovers the usual notion of a Hopf algebra. In particular, H_1 is a Hopf algebra.

Remark that the notion of a Hopf G-coalgebra is not self-dual (the dual notion obtained by reversing the arrows in the definition may be called a Hopf G-algebra).

If $H = \{H_\alpha\}_{\alpha\in G}$ is a Hopf G-coalgebra, then the set $\{\alpha \in G \mid H_\alpha \neq 0\}$ is a subgroup of G. Also, if G' is a subgroup of G, then $H = \{H_\alpha\}_{\alpha\in G'}$ is a Hopf G'-coalgebra.

The antipode S of a Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$ is anti-multiplicative (in the sense that each $S_\alpha\colon H_\alpha \to H_{\alpha^{-1}}$ is an anti-homomorphism of algebras) and anti-comultiplicative in the sense that $\Delta_{\beta^{-1},\alpha^{-1}} S_{\alpha\beta} = \sigma_{H_{\alpha^{-1}},H_{\beta^{-1}}}(S_\alpha \otimes S_\beta)\Delta_{\alpha,\beta}$ for all $\alpha, \beta \in G$ and $\varepsilon S_1 = \varepsilon$; see [Vir2], Lemma 1.1.

A Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$ is said to be of *finite type* if, for all $\alpha \in G$, H_α is finite-dimensional (over K). Note that the direct sum $\bigoplus_{\alpha\in G} H_\alpha$ is finite-dimensional if and only if H is of finite type and $H_\alpha = 0$ for all but a finite number of $\alpha \in G$.

The antipode $S = \{S_\alpha\}_{\alpha\in G}$ of $H = \{H_\alpha\}_{\alpha\in G}$ is said to be *bijective* if each S_α is bijective. Unlike in Section VIII.1, we do not suppose that the antipode of a Hopf G-coalgebra is bijective. As for Hopf algebras, the antipode of a Hopf G-coalgebra H is necessarily bijective if H is of finite type (see Section 1.5) or H is quasitriangular (see Section 2.4).

1.2 The case of finite G. Suppose that G is a finite group. Recall that the Hopf algebra K^G of functions on G has a basis $(e_\alpha\colon G \to K)_{\alpha\in G}$ defined by $e_\alpha(\beta) = \delta_{\alpha,\beta}$ where $\delta_{\alpha,\alpha} = 1$ and $\delta_{\alpha,\beta} = 0$ if $\alpha \neq \beta$. The structure maps of K^G are given by

$$e_\alpha e_\beta = \delta_{\alpha,\beta}\, e_\alpha, \quad 1_{K^G} = \sum_{\alpha\in G} e_\alpha, \quad \Delta(e_\alpha) = \sum_{\beta\gamma=\alpha} e_\beta \otimes e_\gamma, \quad \varepsilon(e_\alpha) = \delta_{\alpha,1},$$

and $S(e_\alpha) = e_{\alpha^{-1}}$. A *central prolongation* of K^G is a Hopf algebra A endowed with a morphism of Hopf algebras $K^G \to A$, called the *central map*, which carries K^G into the center of A.

Since G is finite, any Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$ gives rise to a Hopf algebra $\widetilde{H} = \bigoplus_{\alpha\in G} H_\alpha$ with structure maps given by

$$\widetilde{\Delta}|_{H_\alpha} = \sum_{\beta\gamma=\alpha} \Delta_{\beta,\gamma}, \quad \tilde{\varepsilon}|_{H_\alpha} = \delta_{\alpha,1}\,\varepsilon, \quad \widetilde{m}|_{H_\alpha\otimes H_\beta} = \delta_{\alpha,\beta}\, m_\alpha, \quad \tilde{1} = \sum_{\alpha\in G} 1_\alpha,$$

and $\widetilde{S} = \sum_{\alpha\in G} S_\alpha$. The K-linear map $K^G \to \widetilde{H}$ defined by $e_\alpha \mapsto 1_\alpha$ gives rise to a morphism of Hopf algebras which carries K^G into the center of $\widetilde{H}$. Hence $\widetilde{H}$ is a central prolongation of K^G.

The correspondence assigning to every Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$ the central prolongation $K^G \to \widetilde{H}$ is bijective. Given a Hopf algebra $(A, m, 1, \Delta, \varepsilon, S)$ which is a central prolongation of K^G, set $H_\alpha = A1_\alpha$, where $1_\alpha \in A$ is the image of $e_\alpha \in K^G$ under the central map $K^G \to A$. Then the family $\{H_\alpha\}_{\alpha\in G}$ is a Hopf G-coalgebra with structure maps given by

$$m_\alpha = 1_\alpha \cdot m|_{H_\alpha\otimes H_\alpha}, \quad \Delta_{\alpha,\beta} = (1_\alpha \otimes 1_\beta)\cdot \Delta|_{H_{\alpha\beta}}, \quad \varepsilon = \varepsilon|_{H_1}, \quad S_\alpha = 1_{\alpha^{-1}} \cdot S|_{H_\alpha}.$$

1.3 Integrals. Recall that a left (resp. right) integral for a Hopf algebra $(A, \Delta, \varepsilon, S)$ is an element $\Lambda \in A$ such that $x\Lambda = \varepsilon(x)\Lambda$ (resp. $\Lambda x = \varepsilon(x)\Lambda$) for all $x \in A$. A left (resp. right) integral for the dual Hopf algebra A^* is a K-linear form $\lambda \in A^* = \mathrm{Hom}_K(A, K)$ such that $(\mathrm{id}_A \otimes \lambda)\Delta(x) = \lambda(x)1_A$ (resp. $(\lambda \otimes \mathrm{id}_A)\Delta(x) = \lambda(x)1_A$) for all $x \in A$.

A *left* (resp. *right*) *G-integral* for a Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$ is a family of K-linear forms $\lambda = (\lambda_\alpha)_{\alpha\in G} \in \Pi_{\alpha\in G} H_\alpha^*$ such that

$$(\mathrm{id}_{H_\alpha} \otimes \lambda_\beta)\Delta_{\alpha,\beta}(x) = \lambda_{\alpha\beta}(x)1_\alpha \text{ (resp. } (\lambda_\alpha \otimes \mathrm{id}_{H_\beta})\Delta_{\alpha,\beta}(x) = \lambda_{\alpha\beta}(x)1_\beta)$$

for all $\alpha, \beta \in G$ and $x \in H_{\alpha\beta}$. Note that λ_1 is a usual left (resp. right) integral for the Hopf algebra H_1^*.

A G-integral $\lambda = (\lambda_\alpha)_{\alpha\in G}$ is said to be *non-zero* if $\lambda_\beta \neq 0$ for some $\beta \in G$. Given a non-zero G-integral $\lambda = (\lambda_\alpha)_{\alpha\in G}$, we have $\lambda_\alpha \neq 0$ for all $\alpha \in G$ such that $H_\alpha \neq 0$. In particular $\lambda_1 \neq 0$.

It is known that the K-vector space of left (resp. right) integrals for a finite-dimensional Hopf algebra is one-dimensional. This extends to Hopf G-coalgebras as follows.

Theorem A ([Vir2], Theorem 3.6). *Let H be a Hopf G-coalgebra of finite type. Then the vector space of left* (*resp. right*) *G-integrals for H is one-dimensional.*

The proof of this theorem is based on the fact that a Hopf G-comodule has a canonical decomposition generalizing the fundamental decomposition theorem in the theory of Hopf modules.

1.4 Grouplike elements. A family $g = (g_\alpha)_{\alpha\in G} \in \prod_{\alpha\in G} H_\alpha$ such that $\Delta_{\alpha,\beta}(g_{\alpha\beta}) = g_\alpha \otimes g_\beta$ for all $\alpha, \beta \in G$ and $\varepsilon(g_1) = 1_K$ is called a *G-grouplike element* of a Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$. Note that g_1 is then a grouplike element of the Hopf algebra H_1 in the usual sense of the word.

One easily checks that the set $\mathrm{Gr}(H)$ of G-grouplike elements of H is a group with respect to coordinate-wise multiplication in the product monoid $\prod_{\alpha\in G} H_\alpha$. If $g = (g_\alpha)_{\alpha\in G} \in \mathrm{Gr}(H)$, then $g^{-1} = (S_{\alpha^{-1}}(g_{\alpha^{-1}}))_{\alpha\in G}$. The group $\mathrm{Hom}(G, K^*)$ of homomorphisms $G \to K^*$ acts on $\mathrm{Gr}(H)$ by $\phi g = (\phi(\alpha) g_\alpha)_{\alpha\in G}$ for arbitrary $\phi \in \mathrm{Hom}(G, K^*)$ and $g = (g_\alpha)_{\alpha\in G} \in \mathrm{Gr}(H)$.

1.5 The distinguished G-grouplike element. Throughout this subsection, $H = \{H_\alpha\}_{\alpha\in G}$ is a Hopf G-coalgebra of finite type with antipode $S = \{S_\alpha\}_{\alpha\in G}$. Using Theorem A, one verifies that there is a unique G-grouplike element $g = (g_\alpha)_{\alpha\in G}$ of H, called the *distinguished G-grouplike element of H*, such that $(\mathrm{id}_{H_\alpha} \otimes \lambda_\beta)\Delta_{\alpha,\beta} = \lambda_{\alpha\beta}\, g_\alpha$ for any right G-integral $\lambda = (\lambda_\alpha)_{\alpha\in G}$ and all $\alpha, \beta \in G$. Note that g_1 is the distinguished grouplike element of H_1.

Since H_1 is a finite-dimensional Hopf algebra, there exists a unique algebra morphism $\nu\colon H_1 \to K$ such that if Λ is a left integral for H_1, then $\Lambda x = \nu(x)\Lambda$ for all $x \in H_1$. This morphism is a grouplike element of the Hopf algebra H_1^*, called the *distinguished grouplike element of* H_1^*. It is invertible in H_1^* and its inverse ν^{-1} is also an algebra morphism. Moreover, if Λ is a right integral for H_1, then $x\Lambda = \nu^{-1}(x)\Lambda$ for all $x \in H_1$.

For all $\alpha \in G$, we define a left and a right H_1^*-action on H_α by setting, for all $f \in H_1^*$ and $a \in H_\alpha$,

$$f \rightharpoonup a = (\mathrm{id}_{H_\alpha} \otimes f)\Delta_{\alpha,1}(a) \quad \text{and} \quad a \leftharpoonup f = (f \otimes \mathrm{id}_{H_\alpha})\Delta_{1,\alpha}(a).$$

The next assertion generalizes Theorem 3 of [Rad4]. This is a key result in the theory of Hopf G-coalgebras. It is used in particular to prove the existence of traces (see Section 2.8).

Theorem B ([Vir2], Theorem 4.2). *Let $\lambda = (\lambda_\alpha)_{\alpha\in G}$ be a right G-integral for H. Then, for all $\alpha \in G$ and $x, y \in H_\alpha$,*

(a) $\lambda_\alpha(xy) = \lambda_\alpha(S_{\alpha^{-1}} S_\alpha(y \leftharpoonup \nu)\, x)$;
(b) $\lambda_\alpha(xy) = \lambda_\alpha(y\, S_{\alpha^{-1}} S_\alpha(\nu^{-1} \rightharpoonup g_\alpha^{-1} x g_\alpha))$;
(c) $\lambda_{\alpha^{-1}}(S_\alpha(x)) = \lambda_\alpha(g_\alpha x)$.

As a corollary we obtain a generalization of the celebrated Radford S^4-formula to Hopf G-coalgebras:

Corollary C ([Vir2], Lemma 4.6). *Let $H = \{H_\alpha\}_{\alpha\in G}$ be a Hopf G-coalgebra of finite type. Then for all $\alpha \in G$ and $x \in H_\alpha$,*

$$(S_{\alpha^{-1}} S_\alpha)^2(x) = g_\alpha(\nu \rightharpoonup x \leftharpoonup \nu^{-1})g_\alpha^{-1}.$$

This formula implies in particular that the antipode S of H is bijective (i.e., each S_α is bijective).

1.6 The order of the antipode. It is known that the order of the antipode of a finite-dimensional Hopf algebra is finite ([Rad1], Theorem 1) and divides four times the dimension of the algebra ([NZ], Proposition 3.1). We apply this result to study a Hopf G-coalgebra of finite type $H = \{H_\alpha\}_{\alpha\in G}$ with antipode $S = \{S_\alpha\}_{\alpha\in G}$. Let α be an element of G of finite order d. Denote by $\langle\alpha\rangle$ the subgroup of G generated by α. By considering the finite-dimensional Hopf algebra $\bigoplus_{\beta\in\langle\alpha\rangle} H_\beta$ (determined by the Hopf $\langle\alpha\rangle$-coalgebra $\{H_\beta\}_{\beta\in\langle\alpha\rangle}$, see Section 1.2), we obtain that the order of $S_{\alpha^{-1}}S_\alpha \in \mathrm{Aut}_{\mathrm{Alg}}(H_\alpha)$ is finite and divides $2\sum_{\beta\in\langle\alpha\rangle}\dim H_\beta$. From Corollary C, we obtain another upper bound on the order of $S_{\alpha^{-1}}S_\alpha$: if $\alpha\in G$ has a finite order d, then the order of $S_{\alpha^{-1}}S_\alpha$ divides $2d\dim H_1$; see [Vir2], Corollary 4.5.

1.7 Semisimplicity. A Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$ is said to be *semisimple* if each algebra H_α is semisimple. For H to be semisimple it is necessary that H_1 be finite-dimensional (since an infinite-dimensional Hopf algebra over a field is not semisimple, see [Sw], Corollary 2.7). When H is of finite type, H is semisimple if and only if H_1 is semisimple, see [Vir2], Lemma 5.1.

1.8 Cosemisimplicity. The notion of a comodule over a coalgebra may be extended to the setting of Hopf G-coalgebras. A *right G-comodule* over a Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$ is a family $M = \{M_\alpha\}_{\alpha\in G}$ of K-vector spaces endowed with a family of K-linear maps

$$\rho = \{\rho_{\alpha,\beta}\colon M_{\alpha\beta} \to M_\alpha \otimes H_\beta\}_{\alpha,\beta\in G}$$

such that

$$(\rho_{\alpha,\beta}\otimes \mathrm{id}_{H_\gamma})\rho_{\alpha\beta,\gamma} = (\mathrm{id}_{M_\alpha}\otimes\Delta_{\beta,\gamma})\rho_{\alpha,\beta\gamma} \quad\text{and}\quad (\mathrm{id}_{M_\alpha}\otimes\varepsilon)\rho_{\alpha,1} = \mathrm{id}_{M_\alpha}$$

for all $\alpha,\beta,\gamma\in G$. A *G-subcomodule* of M is a family $N = \{N_\alpha\}_{\alpha\in G}$, where N_α is a K-subspace of M_α, such that $\rho_{\alpha,\beta}(N_{\alpha\beta}) \subset N_\alpha\otimes H_\beta$ for all $\alpha,\beta\in G$. The sums and direct sums for families of G-subcomodules of a right G-comodule are defined in the obvious way.

A right G-comodule $M = \{M_\alpha\}_{\alpha\in G}$ is said to be *simple* if it is *non-zero* (i.e., $M_\alpha \neq 0$ for some $\alpha\in G$) and if it has no G-subcomodules other than itself and the trivial one $0 = \{0\}_{\alpha\in G}$. A right G-comodule which is a direct sum of a family of simple G-subcomodules is said to be *cosemisimple*. Note that all G-subcomodules and all quotients of a cosemisimple right G-comodule are cosemisimple.

A Hopf G-coalgebra is *cosemisimple* if it is cosemisimple as a right G-comodule over itself (with comultiplication as comodule map). By [Vir2], a Hopf G-coalgebra

$H = \{H_\alpha\}_{\alpha\in G}$ is cosemisimple if and only if every reduced[1] right G-comodule over H is cosemisimple.

We state a Hopf G-coalgebra version of the dual Maschke theorem.

Theorem D ([Vir2], Theorem 5.4). *A Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$ is cosemisimple if and only if there exists a right G-integral $\lambda = (\lambda_\alpha)_{\alpha\in G}$ for H such that $\lambda_\alpha(1_\alpha) = 1_K$ for some $\alpha \in G$ (and then $\lambda_\alpha(1_\alpha) = 1_K$ for all $\alpha \in G$ with $H_\alpha \neq 0$).*

As corollaries, we obtain that a Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$ of finite type is cosemisimple if and only if the Hopf algebra H_1 is cosemisimple, and that the distinguished G-grouplike element of a cosemisimple Hopf G-coalgebra of finite type is trivial.

1.9 Involutory Hopf G-coalgebras. A Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in\pi}$ is *involutory* if its antipode $S = \{S_\alpha\}_{\alpha\in\pi}$ satisfies the identity $S_{\alpha^{-1}} S_\alpha = \mathrm{id}_{H_\alpha}$ for all $\alpha \in \pi$.

Involutory Hopf G-coalgebras of finite type have special properties. For example, each of their G-integrals $\lambda = (\lambda_\alpha)_{\alpha\in G}$ is two sided, S-invariant ($\lambda_{\alpha^{-1}} S_\alpha = \lambda_\alpha$ for all $\alpha \in G$), and symmetric ($\lambda_\alpha(xy) = \lambda_\alpha(yx)$ for all $\alpha \in G$ and $x, y \in H_\alpha$). Also if the ground field K of H is of characteristic 0, then $\dim H_\alpha = \dim H_1$ whenever $H_\alpha \neq 0$.

Finally, if $H = \{H_\alpha\}_{\alpha\in G}$ is an involutory Hopf G-coalgebra of finite type over a field whose characteristic does not divide $\dim H_1$, then H is semisimple and cosemisimple; see [Vir4], Lemma 3.

6.2 Quasitriangular Hopf G-coalgebras

2.1 Crossed Hopf G-coalgebras. A Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$ is *crossed* if it is endowed with a *crossing*, that is, a family of algebra isomorphisms $\varphi = \{\varphi_\beta : H_\alpha \to H_{\beta\alpha\beta^{-1}}\}_{\alpha,\beta\in G}$ such that

$$(\varphi_\beta \otimes \varphi_\beta)\Delta_{\alpha,\gamma} = \Delta_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}}\varphi_\beta, \quad \varepsilon\varphi_\beta = \varepsilon, \quad \text{and} \quad \varphi_{\alpha\beta} = \varphi_\alpha\varphi_\beta$$

for all $\alpha, \beta, \gamma \in G$. One easily verifies that a crossing preserves the antipode, that is, $\varphi_\beta S_\alpha = S_{\beta\alpha\beta^{-1}}\varphi_\beta$ for all $\alpha, \beta \in G$. Therefore this definition of a crossed Hopf G-coalgebra is equivalent to the one in Chapter VIII.

A crossing φ in H yields a group homomorphism $\varphi\colon G \to \mathrm{Aut}_{\mathrm{Hopf}}(H_1)$ and determines thus an action of G on H_1 by Hopf algebra automorphisms. Here for a Hopf algebra A, we denote $\mathrm{Aut}_{\mathrm{Hopf}}(A)$ the group of Hopf automorphisms of A.

If G is an abelian group, then any Hopf G-coalgebra admits a *trivial crossing* $\varphi_\beta = \mathrm{id}$ for all $\beta \in G$.

When G is a finite group, the notion of a crossing can be described in terms of central prolongations of K^G (see Section 1.2): a *crossing* of a central prolongation A

[1]A right G-comodule $M = \{M_\alpha\}_{\alpha\in G}$ over H is *reduced* if $M_\alpha = 0$ whenever $H_\alpha = 0$.

of K^G is a group homomorphism $\varphi\colon G \to \mathrm{Aut}_{\mathrm{Hopf}}(A)$ such that $\varphi_\beta(1_\alpha) = 1_{\beta\alpha\beta^{-1}}$ for all $\alpha, \beta \in G$, where 1_α is the image of $e_\alpha \in K^G$ under the central map $K^G \to A$.

2.2 The distinguished character. Let $H = \{H_\alpha\}_{\alpha\in G}$ be a crossed Hopf G-coalgebra of finite type with crossing φ. Using the uniqueness of G-integrals (see Theorem A), one can show the existence of a unique group homomorphism $\widehat{\varphi}\colon G \to K^*$, called the *distinguished character of* H, such that $\lambda_{\beta\alpha\beta^{-1}}\varphi_\beta = \widehat{\varphi}(\beta)\,\lambda_\alpha$ for any left or right G-integral $\lambda = (\lambda_\alpha)_{\alpha\in G}$ for H and all $\alpha, \beta \in G$.

Lemma E ([Vir2], Lemma 6.3). *For any $\beta \in G$,*

(a) *If Λ is a left or right integral for H_1, then $\varphi_\beta(\Lambda) = \widehat{\varphi}(\beta)\Lambda$.*
(b) *If ν is the distinguished grouplike element of H_1^*, then $\nu\varphi_\beta = \nu$.*
(c) *If $g = (g_\alpha)_{\alpha\in G}$ is the distinguished G-grouplike element of H, then $\varphi_\beta(g_\alpha) = g_{\beta\alpha\beta^{-1}}$ for all $\alpha \in G$.*

2.3 Quasitriangular Hopf G-coalgebras. Following Chapter VIII, we call a crossed Hopf G-coalgebra $(H = \{H_\alpha\}_{\alpha\in G}, \varphi)$ *quasitriangular* if it is endowed with an *R-matrix*, that is, a family $R = \{R_{\alpha,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in G}$ of invertible elements such that, for all $\alpha, \beta, \gamma \in G$ and $x \in H_{\alpha\beta}$,

$$R_{\alpha,\beta} \cdot \Delta_{\alpha,\beta}(x) = \sigma_{\beta,\alpha}(\varphi_{\alpha^{-1}} \otimes \mathrm{id}_{H_\alpha})\Delta_{\alpha\beta\alpha^{-1},\alpha}(x) \cdot R_{\alpha,\beta},$$
$$(\mathrm{id}_{H_\alpha} \otimes \Delta_{\beta,\gamma})(R_{\alpha,\beta\gamma}) = (R_{\alpha,\gamma})_{1\beta 3} \cdot (R_{\alpha,\beta})_{12\gamma},$$
$$(\Delta_{\alpha,\beta} \otimes \mathrm{id}_{H_\gamma})(R_{\alpha\beta,\gamma}) = [(\mathrm{id}_{H_\alpha} \otimes \varphi_{\beta^{-1}})(R_{\alpha,\beta\gamma\beta^{-1}})]_{1\beta 3} \cdot (R_{\beta,\gamma})_{\alpha 23},$$
$$(\varphi_\beta \otimes \varphi_\beta)(R_{\alpha,\gamma}) = R_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}}.$$

Here $\sigma_{\beta,\alpha}$ denotes the flip $H_\beta \otimes H_\alpha \to H_\alpha \otimes H_\beta$ and, for K-vector spaces P, Q and $r = \sum_j p_j \otimes q_j \in P \otimes Q$, we set

$$r_{12\gamma} = r \otimes 1_\gamma \in P \otimes Q \otimes H_\gamma, \quad r_{\alpha 23} = 1_\alpha \otimes r \in H_\alpha \otimes P \otimes Q,$$

and $r_{1\beta 3} = \sum_j p_j \otimes 1_\beta \otimes q_j \in P \otimes H_\beta \otimes Q$. Note that $R_{1,1}$ is an R-matrix for the Hopf algebra H_1 is the usual sense of the word.

When G is abelian and φ is the trivial crossing, we recover the definition of a quasitriangular G-colored Hopf algebra due to Ohtsuki [Oh1].

An R-matrix for a crossed Hopf G-coalgebra provides a solution of the G-colored Yang–Baxter equation

$$(R_{\beta,\gamma})_{\alpha 23} \cdot (R_{\alpha,\gamma})_{1\beta 3} \cdot (R_{\alpha,\beta})_{12\gamma}$$
$$= (R_{\alpha,\beta})_{12\gamma} \cdot [(\mathrm{id}_{H_\alpha} \otimes \varphi_{\beta^{-1}})(R_{\alpha,\beta\gamma\beta^{-1}})]_{1\beta 3} \cdot (R_{\beta,\gamma})_{\alpha 23}$$

and satisfies the following identities (see [Vir2], Lemma 6.4): for all $\alpha, \beta, \gamma \in G$,

$$(\varepsilon \otimes \mathrm{id}_{H_\alpha})(R_{1,\alpha}) = 1_\alpha = (\mathrm{id}_{H_\alpha} \otimes \varepsilon)(R_{\alpha,1}),$$
$$(S_{\alpha^{-1}}\varphi_\alpha \otimes \mathrm{id}_{H_\beta})(R_{\alpha^{-1},\beta}) = R_{\alpha,\beta}^{-1} \quad \text{and} \quad (\mathrm{id}_{H_\alpha} \otimes S_\beta)(R_{\alpha,\beta}^{-1}) = R_{\alpha,\beta^{-1}},$$
$$(S_\alpha \otimes S_\beta)(R_{\alpha,\beta}) = (\varphi_\alpha \otimes \mathrm{id}_{H_{\beta^{-1}}})(R_{\alpha^{-1},\beta^{-1}}).$$

2.4 The Drinfeld element. The *Drinfeld element* of a quasitriangular Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha \in G}$ is the family $u = (u_\alpha)_{\alpha \in G} \in \Pi_{\alpha \in G} H_\alpha$, where

$$u_\alpha = m_\alpha(S_{\alpha^{-1}}\varphi_\alpha \otimes \mathrm{id}_{H_\alpha})\,\sigma_{\alpha,\alpha^{-1}}(R_{\alpha,\alpha^{-1}}).$$

Observe that u_1 is the Drinfeld element of the quasitriangular Hopf algebra H_1 (see [Drin2]). By [Vir2], Lemma 6.5, each u_α is invertible in H_α and

$$u_\alpha^{-1} = m_\alpha(\mathrm{id}_{H_\alpha} \otimes S_{\alpha^{-1}}S_\alpha)\,\sigma_{\alpha,\alpha}(R_{\alpha,\alpha}).$$

Moreover, for any $\alpha \in G$ and $x \in H$,

$$S_{\alpha^{-1}}S_\alpha(x) = u_\alpha \varphi_{\alpha^{-1}}(x) u_\alpha^{-1},$$

where φ is the crossing in H. This implies that the antipode of H is bijective.

Note also the identities $\varepsilon(u_1) = 1$, $\varphi_\beta(u_\alpha) = u_{\beta\alpha\beta^{-1}}$, and

$$\Delta_{\alpha,\beta}(u_{\alpha\beta}) = [\sigma_{\beta,\alpha}(\mathrm{id}_{H_\beta} \otimes \varphi_\alpha)(R_{\beta,\alpha}) \cdot R_{\alpha,\beta}]^{-1} \cdot (u_\alpha \otimes u_\beta).$$

2.5 Ribbon Hopf G-coalgebras. Following Chapter VIII, we call a quasitriangular Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha \in G}$ *ribbon* if it is endowed with a *twist*, that is, a family of invertible elements $\theta = \{\theta_\alpha \in H_\alpha\}_{\alpha \in G}$ such that for all $\alpha, \beta \in G$ and $x \in H_\alpha$,

$$\varphi_\alpha(x) = \theta_\alpha^{-1} x \theta_\alpha, \quad S_\alpha(\theta_\alpha) = \theta_{\alpha^{-1}}, \quad \varphi_\beta(\theta_\alpha) = \theta_{\beta\alpha\beta^{-1}},$$
$$\Delta_{\alpha,\beta}(\theta_{\alpha\beta}) = (\theta_\alpha \otimes \theta_\beta) \cdot \sigma_{\beta,\alpha}(\mathrm{id}_{H_\beta} \otimes \varphi_\alpha)(R_{\beta,\alpha}) \cdot R_{\alpha,\beta}.$$

Note that θ_1 is a twist of the quasitriangular Hopf algebra H_1, and so $\varepsilon(\theta_1) = 1$. If $\alpha \in G$ has a finite order d, then θ_α^d is a central element of H_α. In particular, θ_1 is central in H_1.

Example. Let G be a group and $c\colon G \times G \to K^*$ be a bicharacter of G, that is, $c(\alpha, \beta\gamma) = c(\alpha, \beta)\, c(\alpha, \gamma)$ and $c(\alpha\beta, \gamma) = c(\alpha, \gamma)\, c(\beta, \gamma)$ for all $\alpha, \beta, \gamma \in G$. Consider the following crossed Hopf algebra K^c: for all $\alpha, \beta \in G$, we have $K_\alpha^c = K$ as an algebra and

$$\Delta_{\alpha,\beta}(1_K) = 1_K \otimes 1_K, \quad \varepsilon(1_K) = 1_K, \quad S_\alpha(1_K) = 1_K, \quad \varphi_\beta(1_K) = 1_K.$$

Then K^c is a ribbon Hopf G-coalgebra of finite type with R-matrix and twist given by $R_{\alpha,\beta} = c(\alpha, \beta)\, 1_K \otimes 1_K$ and $\theta_\alpha = c(\alpha, \alpha)$. The Drinfeld elements of K^c are computed by $u_\alpha = c(\alpha, \alpha)^{-1}$.

2.6 The spherical G-grouplike element. Let $H = \{H_\alpha\}_{\alpha\in G}$ be a ribbon Hopf G-coalgebra with Drinfeld element $u = (u_\alpha)_{\alpha\in G}$. For any $\alpha \in G$, set

$$w_\alpha = \theta_\alpha u_\alpha = u_\alpha \theta_\alpha \in H_\alpha.$$

Then $w = (w_\alpha)_{\alpha\in G}$ is a G-grouplike element, called the *spherical G-grouplike element of H*. It satisfies the identities

$$\varphi_\beta(w_\alpha) = w_{\beta\alpha\beta^{-1}}, \quad S_\alpha(u_\alpha) = w_{\alpha^{-1}}^{-1} u_{\alpha^{-1}} w_{\alpha^{-1}}^{-1}, \quad \text{and} \quad S_{\alpha^{-1}} S_\alpha(x) = w_\alpha x w_\alpha^{-1}$$

for all $\alpha, \beta \in G$ and $x \in H_\alpha$. Conversely, any G-grouplike element $w = (w_\alpha)_{\alpha\in G}$ of a quasitriangular Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$ which satisfies these identities gives rise to a twist $\theta = (\theta_\alpha)_{\alpha\in G}$ in H by $\theta_\alpha = w_\alpha u_\alpha^{-1} = u_\alpha^{-1} w_\alpha$.

2.7 Further G-grouplike elements. Let $H = \{H_\alpha\}_{\alpha\in G}$ be a quasitriangular Hopf G-coalgebra of finite type. Set

$$\ell_\alpha = S_{\alpha^{-1}}(u_{\alpha^{-1}})^{-1} u_\alpha = u_\alpha\, S_{\alpha^{-1}}(u_{\alpha^{-1}})^{-1} \in H_\alpha,$$

where $u = (u_\alpha)_{\alpha\in G}$ is the Drinfeld element of H. The properties of u ensure that $\ell = (\ell_\alpha)_{\alpha\in G}$ is a G-grouplike element of H. Let ν be the distinguished grouplike element of H_1^* and $\widehat{\varphi}$ be the distinguished character of H (see Sections 1.5 and 2.2). Denoting $R = \{R_{\alpha,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in G}$ the R-matrix of H, set

$$h_\alpha = (\mathrm{id}_{H_\alpha} \otimes \nu)(R_{\alpha,1}) \in H_\alpha.$$

Theorem F ([Vir2], Theorem 6.9). *The family $h = (h_\alpha)_{\alpha\in G}$ is a G-grouplike element of H. The distinguished G-grouplike element $(g_\alpha)_{\alpha\in G}$ of H is computed by $g_\alpha = \widehat{\varphi}(\alpha)^{-1} \ell_\alpha h_\alpha$ for all $\alpha \in G$.*

For ribbon H, we obtain as a corollary that $g_\alpha = \widehat{\varphi}(\alpha)^{-1} w_\alpha^2 h_\alpha$ for all $\alpha \in G$, where $w = (w_\alpha)_{\alpha\in G}$ is the spherical G-grouplike element of H.

2.8 Traces. Let $H = \{H_\alpha\}_{\alpha\in G}$ be a crossed Hopf G-coalgebra. A *G-trace* for H is a family of K-linear forms $\mathrm{tr} = (\mathrm{tr}_\alpha)_{\alpha\in G} \in \Pi_{\alpha\in G} H_\alpha^*$ such that

$$\mathrm{tr}_\alpha(xy) = \mathrm{tr}_\alpha(yx), \quad \mathrm{tr}_{\alpha^{-1}}(S_\alpha(x)) = \mathrm{tr}_\alpha(x), \quad \text{and} \quad \mathrm{tr}_{\beta\alpha\beta^{-1}}(\varphi_\beta(x)) = \mathrm{tr}_\alpha(x)$$

for all $\alpha, \beta \in G$ and $x, y \in H_\alpha$. Note that tr_1 is a usual trace for the Hopf algebra H_1, which is invariant under the action φ of G.

A Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$ is *unimodular* if the Hopf algebra H_1 is unimodular (that is the spaces of left and right integrals for H_1 coincide). If H_1 is finite-dimensional, then H is unimodular if and only if $\nu = \varepsilon$, where ν is the distinguished grouplike element of H_1^*. For example, any finite type semisimple Hopf G-coalgebra is unimodular.

Consider in more detail a unimodular ribbon Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$ of finite type. Let $\lambda = (\lambda_\alpha)_{\alpha\in G}$ be a non-zero right G-integral for H, $w = (w_\alpha)_{\alpha\in G}$ be the spherical G-grouplike element of H, and $\widehat{\varphi}$ be the distinguished character of H.

Using Theorems B and F, we obtain that the G-traces for H are parameterized by families $z = (z_\alpha)_{\alpha\in G}$ such that $z_\alpha \in H_\alpha$ is central, $S_\alpha(z_\alpha) = \widehat{\varphi}(\alpha)^{-1} z_{\alpha^{-1}}$, and $\varphi_\beta(z_\alpha) = \widehat{\varphi}(\beta) z_{\beta\alpha\beta^{-1}}$ for all $\alpha, \beta \in G$. The G-trace corresponding to such a family z is given by $\mathrm{tr}_\alpha(x) = \lambda_\alpha(w_\alpha z_\alpha x)$. We point out two such families.

Let Λ be a left integral for H_1 such that $\lambda_1(\Lambda) = 1$. Set $z_1 = \Lambda$ and $z_\alpha = 0$ if $\alpha \neq 1$. The resulting family $(z_\alpha)_{\alpha\in G}$ satisfies all the conditions above since H is unimodular (and so Λ is central and $S_1(\Lambda) = \Lambda$) and by Lemma E (a). The corresponding G-trace is given by $\mathrm{tr}_1 = \varepsilon$ and $\mathrm{tr}_\alpha = 0$ for all $\alpha \neq 1$.

If $\widehat{\varphi}(\alpha) = 1$ for all $\alpha \in G$, then another possible choice of a family z is $z_\alpha = 1_\alpha$ for all α. Note that $\widehat{\varphi} = 1$ if H is semisimple or cosemisimple or if $\lambda_1(\theta_1) \neq 0$, where $\theta = \{\theta_\alpha\}_{\alpha\in G}$ is the twist of H. We obtain the following assertion.

Theorem G ([Vir2], Theorem 7.4). *Suppose under the assumptions above that at least one of the following four conditions is satisfied: H is semisimple, or H is cosemisimple, or $\lambda_1(\theta_1) \neq 0$, or $\varphi_\beta|_{H_1} = \mathrm{id}_{H_1}$ for all $\beta \in G$. Then the family of K-linear maps $\mathrm{tr} = (\mathrm{tr}_\alpha)_{\alpha\in G}$, defined by $\mathrm{tr}_\alpha(x) = \lambda_\alpha(w_\alpha x)$ for all $x \in H_\alpha$, is a G-trace for H.*

6.3 The twisted double construction

Starting from a crossed Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$, Zunino [Zu1] constructed a double $Z(H) = \{Z(H)_\alpha\}_{\alpha\in G}$ of H which is a quasitriangular Hopf G-coalgebra containing H as a Hopf G-subcoalgebra. As a vector space, $Z(H)_\alpha = H_\alpha \otimes (\bigoplus_{\beta\in G} H^*_\beta)$. Generally speaking, $Z(H)$ is not of finite type: the components $Z(H)_\alpha$ may be infinite-dimensional.

In this section we provide a method, called the twisted double construction, for deriving quasitriangular Hopf G-coalgebras of finite type from finite-dimensional Hopf algebras endowed with action of G by Hopf automorphisms (cf. Section 2.1). We also give an h-adic version of this construction. This will lead us to non-trivial examples of quasitriangular Hopf G-coalgebras for any finite group G and for infinite groups G such as $\mathrm{GL}_n(K)$. In particular, we define the (h-adic) graded quantum groups.

3.1 Hopf pairings. Recall that a *Hopf pairing* between two Hopf algebras A and B (over K) is a bilinear pairing $\sigma\colon A \times B \to K$ such that, for all $a, a' \in A$ and $b, b' \in B$,

$$\sigma(a, bb') = \sigma(a_{(1)}, b)\,\sigma(a_{(2)}, b'), \quad \sigma(a, 1) = \varepsilon(a),$$
$$\sigma(aa', b) = \sigma(a, b_{(2)})\,\sigma(a', b_{(1)}), \quad \sigma(1, b) = \varepsilon(b).$$

Such a pairing always preserves the antipode: $\sigma(S(a), S(b)) = \sigma(a, b)$ for all $a \in A$ and $b \in B$.

A Hopf pairing $\sigma\colon A \times B \to K$ determines two annihilator ideals $I_A = \{a \in A \mid \sigma(a,b) = 0 \text{ for all } b \in B\}$ and $I_B = \{b \in B \mid \sigma(a,b) = 0 \text{ for all } a \in A\}$. It is easy to check that I_A and I_B are Hopf ideals of A and B, respectively. The pairing σ is *non-degenerate* iff $I_A = I_B = 0$. Any Hopf pairing $\sigma\colon A \times B \to K$ induces a non-degenerate Hopf pairing $\bar\sigma\colon A/I_A \times B/I_B \to K$.

3.2 The twisted double. Let $\sigma\colon A \times B \to K$ be a Hopf pairing between two Hopf algebras A and B, and let $\phi\colon A \to A$ be a Hopf algebra endomorphism of A. Set

$$D(A, B; \sigma, \phi) = A \otimes B$$

as a K-vector space. We provide $D(A, B; \sigma, \phi)$ with a structure of an algebra with unit $1_A \otimes 1_B$ and multiplication

$$(a \otimes b) \cdot (a' \otimes b') = \sigma(\phi(a'_{(1)}), S(b_{(1)}))\, \sigma(a'_{(3)}, b_{(3)})\; a a'_{(2)} \otimes b_{(2)} b'$$

for any $a, a' \in A$ and $b, b' \in B$.

Note that if ϕ and ϕ' are different Hopf algebra endomorphisms of A, then the algebras $D(A, B; \sigma, \phi)$ and $D(A, B; \sigma, \phi')$ are in general not isomorphic (see Remark in Section 3.4 below).

Theorem H ([Vir3], Theorem 2.6). *Let $\sigma\colon A \times B \to K$ be a Hopf pairing between Hopf algebras A and B, and let ϕ be an action of G on A by Hopf algebra automorphisms, that is, ϕ is a group homomorphism $G \to \mathrm{Aut}_{\mathrm{Hopf}}(A)$. Then the family of algebras*

$$D(A, B; \sigma, \phi) = \{D(A, B; \sigma, \phi_\alpha)\}_{\alpha \in G}$$

has a structure of a Hopf G-coalgebra given by

$$\begin{aligned}
\Delta_{\alpha,\beta}(a \otimes b) &= (\phi_\beta(a_{(1)}) \otimes b_{(1)}) \otimes (a_{(2)} \otimes b_{(2)}),\\
\varepsilon(a \otimes b) &= \varepsilon_A(a)\, \varepsilon_B(b),\\
S_\alpha(a \otimes b) &= \sigma(\phi_\alpha(a_{(1)}), b_{(1)})\, \sigma(a_{(3)}, S(b_{(3)}))\; \phi_\alpha S(a_{(2)}) \otimes S(b_{(2)})
\end{aligned}$$

for all $a \in A$, $b \in B$ and $\alpha, \beta \in G$. Furthermore, if σ is non-degenerate and A, B are finite dimensional, then the Hopf G-coalgebra $D(A, B; \sigma, \phi)$ is quasitriangular with crossing φ and R-matrix $R = \{R_{\alpha,\beta}\}_{\alpha,\beta \in G}$ given by

$$\varphi_\beta(a \otimes b) = \phi_\beta(a) \otimes \phi^*_\beta(b) \quad \text{and} \quad R_{\alpha,\beta} = \sum_i (e_i \otimes 1_B) \otimes (1_A \otimes f_i),$$

where $\phi^*\colon G \to \mathrm{Aut}_{\mathrm{Hopf}}(B)$ is the unique action such that $\sigma(\phi_\beta, \phi^*_\beta) = \sigma$ for all $\beta \in G$, and $(e_i)_i$ and $(f_i)_i$ are dual bases of A and B with respect to σ.

Corollary I. *Let A be a finite-dimensional Hopf algebra and ϕ be an action of G on A by Hopf algebra automorphisms. Then the duality bracket $\langle\, , \rangle_{A \otimes A^*}$ is a non-degenerate Hopf pairing between A and $A^{*\mathrm{cop}}$ and $D(A, A^{*\mathrm{cop}}; \langle\, , \rangle_{A \otimes A^*}, \phi)$ is a quasitriangular Hopf G-coalgebra.*

Note that the group of Hopf automorphisms of a finite-dimensional semisimple Hopf algebra A over a field of characteristic 0 is finite (see [Rad2]). To obtain quasitriangular Hopf G-coalgebras with infinite G using the twisted double method, one has to start from non-semisimple Hopf algebras or from Hopf algebras over fields of non-zero characteristic.

In the next three sections, we use Theorem H to give examples of quasitriangular Hopf G-coalgebras.

3.3 Example: finite G. Let G be a finite group. In this section, we describe the ribbon Hopf G-coalgebras obtained by the twisted double construction from the group algebra $K[G]$. The standard Hopf algebra structure on $K[G]$ is given by $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$, and $S(g) = g^{-1}$ for all $g \in G$. The dual of $K[G]$ is the Hopf algebra $F(G) = K^G$ of functions $G \to K$ with structure maps and basis $(e_g : G \to K)_{g \in G}$ described in Section 2.1. Let $\phi\colon G \to \mathrm{Aut}_{\mathrm{Hopf}}(K[G])$ be the homomorphism defined by $\phi_\alpha(h) = \alpha h \alpha^{-1}$ for $\alpha \in G, h \in K[G]$. Corollary I yields a quasitriangular Hopf G-coalgebra

$$D_G(G) = D(K[G], F(G)^{\mathrm{cop}}; \langle\, , \rangle_{K[G]\times F(G)}, \phi).$$

Let us describe $D_G(G) = \{D_\alpha(G)\}_{\alpha \in G}$ more precisely. For $\alpha \in G$, the algebra $D_\alpha(G)$ is equal to $K[G] \otimes F(G)$ as a K-vector space, has unit $1_{D_\alpha(G)} = \sum_{g \in G} 1 \otimes e_g$ and multiplication

$$(g \otimes e_h) \cdot (g' \otimes e_{h'}) = \delta_{\alpha g' \alpha^{-1}, h^{-1} g' h'}\, gg' \otimes e_{h'}$$

for all $g, g', h, h' \in G$. The structure maps of $D_G(G)$ are

$$\Delta_{\alpha,\beta}(g \otimes e_h) = \sum_{xy=h} \beta g \beta^{-1} \otimes e_y \otimes g \otimes e_x, \quad \varepsilon(g \otimes e_h) = \delta_{h,1},$$

$$S_\alpha(g \otimes e_h) = \alpha g^{-1} \alpha^{-1} \otimes e_{\alpha g \alpha^{-1} h^{-1} g^{-1}}, \quad \varphi_\alpha(g \otimes e_h) = \alpha g \alpha^{-1} \otimes e_{\alpha h \alpha^{-1}}$$

for all $\alpha, \beta, g, h \in G$. The crossed Hopf G-coalgebra $D_G(G)$ is quasitriangular and furthermore ribbon with R-matrix and twist

$$R_{\alpha,\beta} = \sum_{g,h \in G} g \otimes e_h \otimes 1 \otimes e_g \quad \text{and} \quad \theta_\alpha = \sum_{g \in G} \alpha^{-1} g \alpha \otimes e_g$$

for all $\alpha, \beta \in G$. The spherical G-grouplike element of $D_G(G)$ is $w = (1_{D_\alpha(G)})_{\alpha \in G}$. The family $\lambda = (\lambda_\alpha)_{\alpha \in G}$, defined by $\lambda_\alpha(g \otimes e_h) = \delta_{g,1}$, is a two-sided G-integral for $D_G(G)$.

3.4 An example of a quasitriangular Hopf $\mathrm{GL}_n(K)$-coalgebra. In this section, K is a field of characteristic $\neq 2$ and n is a positive integer. Let A be the K-algebra with generators $g, x_1, \ldots, x_n$ subject to the relations

$$g^2 = 1, \quad x_i^2 = 0, \quad g x_i = -x_i g, \quad x_i x_j = -x_j x_i.$$

The algebra A is 2^{n+1}-dimensional and has a Hopf algebra structure given by

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad \Delta(x_i) = x_i \otimes g + 1 \otimes x_i, \quad \varepsilon(x_i) = 0, \quad S(g) = g,$$

and $S(x_i) = gx_i$ for all i. The group of Hopf automorphisms of A is isomorphic to the group $\mathrm{GL}_n(K)$ of invertible $n \times n$-matrices with coefficients in K (see [Rad2]). An explicit isomorphism $\phi\colon \mathrm{GL}_n(K) \to \mathrm{Aut}_{\mathrm{Hopf}}(A)$ carries any $\alpha = (\alpha_{i,j}) \in \mathrm{GL}_n(K)$ to the automorphism ϕ_α of A given by

$$\phi_\alpha(g) = g \quad \text{and} \quad \phi_\alpha(x_i) = \sum_{k=1}^{n} \alpha_{k,i}\, x_k.$$

We apply Corollary I to these A and ϕ. Observing that $A^* \cong A$ as Hopf algebras, we can quotient the resulting quasitriangular Hopf $\mathrm{GL}_n(K)$-coalgebra to eliminate one copy of the generator g (which appears twice), see [Vir3], Proposition 4.1. This gives a quasitriangular Hopf $\mathrm{GL}_n(K)$-coalgebra $\mathcal{H} = \{\mathcal{H}_\alpha\}_{\alpha \in \mathrm{GL}_n(K)}$. We give here a direct description of $\mathcal{H}$. For $\alpha = (\alpha_{i,j}) \in \mathrm{GL}_n(K)$, let $\mathcal{H}_\alpha$ be the K-algebra generated g, $x_1, \dots x_n, y_1, \dots, y_n$, subject to the relations

$$\begin{gathered} g^2 = 1, \quad x_1^2 = \cdots = x_n^2 = 0, \quad gx_i = -x_i g, \quad x_i x_j = -x_j x_i, \\ y_1^2 = \cdots = y_n^2 = 0, \quad gy_i = -y_i g, \quad y_i y_j = -y_j y_i, \\ x_i y_j - y_j x_i = (\alpha_{j,i} - \delta_{i,j})\, g, \end{gathered}$$

where $1 \le i, j \le n$. The family $\mathcal{H} = \{\mathcal{H}_\alpha\}_{\alpha \in \mathrm{GL}_n(K)}$ has the following structure of a crossed Hopf $\mathrm{GL}_n(K)$-coalgebra:

$$\begin{gathered} \Delta_{\alpha,\beta}(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S_\alpha(g) = g, \\ \Delta_{\alpha,\beta}(x_i) = 1 \otimes x_i + \sum_{k=1}^{n} \beta_{k,i}\, x_k \otimes g, \quad \varepsilon(x_i) = 0, \quad S_\alpha(x_i) = \sum_{k=1}^{n} \alpha_{k,i}\, gx_k, \\ \Delta_{\alpha,\beta}(y_i) = y_i \otimes 1 + g \otimes y_i, \quad \varepsilon(y_i) = 0, \quad S_\alpha(y_i) = -gy_i, \\ \varphi_\alpha(g) = g, \quad \varphi_\alpha(x_i) = \sum_{k=1}^{n} \alpha_{k,i}\, x_k, \quad \varphi_\alpha(y_i) = \sum_{k=1}^{n} \tilde{\alpha}_{i,k}\, y_k, \end{gathered}$$

where $\alpha = (\alpha_{i,j})$, $\beta = (\beta_{i,j})$ run over $\mathrm{GL}_n(K)$, $(\tilde{\alpha}_{i,j}) = \alpha^{-1}$, and $1 \le i \le n$. The crossed Hopf $\mathrm{GL}_n(K)$-coalgebra $\mathcal{H}$ is quasitriangular with R-matrix

$$R_{\alpha,\beta} = \frac{1}{2} \sum_{S \subseteq \{1,\dots n\}} x_S \otimes y_S + x_S \otimes gy_S + gx_S \otimes y_S - gx_S \otimes gy_S$$

for all $\alpha, \beta \in \mathrm{GL}_n(K)$. Here $x_\emptyset = 1$, $y_\emptyset = 1$, and for a nonempty subset S of $\{1, \dots n\}$, we set $x_S = x_{i_1} \dots x_{i_s}$ and $y_S = y_{i_1} \dots y_{i_s}$, where $i_1 < \cdots < i_s$ are the elements of S.

Remark. Generally speaking, for distinct $\alpha, \beta \in \mathrm{GL}_n(K)$, the algebras $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ are not isomorphic. For example, $\mathcal{H}_\alpha \not\simeq \mathcal{H}_1$ for any $\alpha \in \mathrm{GL}_n(K) - \{1\}$. It suffices to prove that

$$\mathcal{H}_\alpha/[\mathcal{H}_\alpha, \mathcal{H}_\alpha] \not\simeq \mathcal{H}_1/[\mathcal{H}_1, \mathcal{H}_1].$$

Indeed, $\mathcal{H}_\alpha/[\mathcal{H}_\alpha, \mathcal{H}_\alpha] = 0$ since $g = \frac{1}{\alpha_{j,i} - \delta_{i,j}}(x_i y_j - y_j x_i) \in [\mathcal{H}_\alpha, \mathcal{H}_\alpha]$ (for some $1 \le i, j \le n$ such that $\alpha_{j,i} \neq \delta_{i,j}$) and so $1 = g^2 \in [\mathcal{H}_\alpha, \mathcal{H}_\alpha]$. In $\mathcal{H}_1/[\mathcal{H}_1, \mathcal{H}_1]$, we have $x_k = x_k g^2 = 0$ (since $x_k g = g x_k = -x_k g$ and so $x_k g = 0$) and likewise $y_k = 0$. Hence $\mathcal{H}_1/[\mathcal{H}_1, \mathcal{H}_1] = K\langle g \mid g^2 = 1\rangle \neq 0$.

3.5 Graded quantum groups. Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra of rank l with Cartan matrix $(a_{i,j})$. Let $\{d_i\}_{i=1}^l$ be coprime integers such that the matrix $(d_i a_{i,j})$ is symmetric. Let q be a fixed non-zero complex number and $q_i = q^{d_i}$ for $i = 1, 2, \ldots, l$. We suppose that $q_i^2 \neq 1$ for all i.

Recall that the (usual) quantum group $U_q(\mathfrak{g})$ can be obtained as a quotient of the quantum double of $U_q(\mathfrak{b}_+)$, where $\mathfrak{b}_+$ is the (positive) Borel subalgebra of $\mathfrak{g}$ (the quotient is needed to eliminate the second copy of the Cartan subalgebra). Applying Theorem H to the Hopf algebra $U_q(\mathfrak{b}_+)$ endowed with an action of $(\mathbb{C}^*)^l$ by Hopf automorphisms, we obtain the "graded quantum group" introduced in [Vir3], Proposition 5.1. It can be directly described as follows.

Set $G = (\mathbb{C}^*)^l$. For $\alpha = (\alpha_1, \ldots, \alpha_l) \in G$, let $U_q^\alpha(\mathfrak{g})$ be the $\mathbb{C}$-algebra generated by $K_i^{\pm 1}$, E_i, F_i, $1 \le i \le l$, subject to the following defining relations:

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1,$$
$$K_i E_j = q_i^{a_{i,j}} E_j K_i,$$
$$K_i F_j = q_i^{-a_{i,j}} F_j K_i,$$
$$E_i F_j - F_j E_i = \delta_{i,j} \frac{\alpha_i K_i - K_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{r=0}^{1-a_{i,j}} (-1)^r \begin{bmatrix} 1-a_{i,j} \\ r \end{bmatrix}_{q_i} E_i^{1-a_{i,j}-r} E_j E_i^r = 0 \quad \text{if } i \neq j,$$
$$\sum_{r=0}^{1-a_{i,j}} (-1)^r \begin{bmatrix} 1-a_{i,j} \\ r \end{bmatrix}_{q_i} F_i^{1-a_{i,j}-r} F_j F_i^r = 0 \quad \text{if } i \neq j.$$

The family $U_q^G(\mathfrak{g}) = \{U_q^\alpha(\mathfrak{g})\}_{\alpha \in G}$ has a structure of a crossed Hopf G-coalgebra given, for $\alpha = (\alpha_1, \ldots, \alpha_l) \in G$, $\beta = (\beta_1, \ldots, \beta_l) \in G$ and $1 \le i \le l$, by:

$$\Delta_{\alpha,\beta}(K_i) = K_i \otimes K_i,$$
$$\Delta_{\alpha,\beta}(E_i) = \beta_i E_i \otimes K_i + 1 \otimes E_i,$$
$$\Delta_{\alpha,\beta}(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$

$$\varepsilon(K_i) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0,$$
$$S_\alpha(K_i) = K_i^{-1}, \quad S_\alpha(E_i) = -\alpha_i E_i K_i^{-1}, \quad S_\alpha(F_i) = -K_i F_i,$$
$$\varphi_\alpha(K_i) = K_i, \quad \varphi_\alpha(E_i) = \alpha_i E_i, \quad \varphi_\alpha(F_i) = \alpha_i^{-1} F_i.$$

Note that $(U_q^1(\mathfrak{g}), \Delta_{1,1}, \varepsilon, S_1)$ is the usual quantum group $U_q(\mathfrak{g})$.

To give a rigorous treatment of R-matrices for the graded quantum groups, we need h-adic versions of Hopf G-coalgebras and of graded quantum groups. This is the content of the next two sections.

3.6 The h-adic case. In this section, we develop an h-adic variant of Hopf G-coalgebras. Roughly speaking, h-adic Hopf G-coalgebras are obtained by taking the ring $\mathbb{C}[[h]]$ of formal power series as the ground ring and requiring that the algebras (resp. the tensor products) are complete (resp. completed) in the h-adic topology.

Recall that if V is a (left) module over $\mathbb{C}[[h]]$, then the topology on V for which the sets $\{h^n V + v \mid n \in \mathbb{N}\}$ form a base for neighborhoods of $v \in V$ is called the *h-adic topology*. For $\mathbb{C}[[h]]$-modules V and W, denote by $V \hat{\otimes} W$ the completion of $V \otimes_{\mathbb{C}[[h]]} W$ in the h-adic topology.

If V is a complex vector space, then the set $V[[h]]$ of all formal power series $\sum_{n=0}^{\infty} v_n h^n$ with coefficients $v_n \in V$ is a $\mathbb{C}[[h]]$-module called a *topologically free module*. Topologically free modules are exactly $\mathbb{C}[[h]]$-modules which are complete, separated, and torsion-free. Furthermore, $V[[h]] \hat{\otimes} W[[h]] = (V \otimes W)[[h]]$ for any complex vector spaces V and W.

An *h-adic algebra* A is a $\mathbb{C}[[h]]$-module complete in the h-adic topology and endowed with a $\mathbb{C}[[h]]$-linear map $m\colon A \hat{\otimes} A \to A$ and an element $1 \in A$ such that $m(\mathrm{id}_A \hat{\otimes} m) = m(m \hat{\otimes} \mathrm{id}_A)$ and $m(\mathrm{id}_A \hat{\otimes} 1) = \mathrm{id}_A = m(1 \hat{\otimes} \mathrm{id}_A)$.

By an *h-adic Hopf G-coalgebra*, we mean a family $H = \{H_\alpha\}_{\alpha \in G}$ of h-adic algebras endowed with h-adic algebra homomorphisms $\Delta_{\alpha,\beta}\colon H_{\alpha\beta} \to H_\alpha \hat{\otimes} H_\beta$ ($\alpha, \beta \in G$), $\varepsilon\colon A \to \mathbb{C}[[h]]$, and with $C[[h]]$-linear maps $S_\alpha\colon H_\alpha \to H_{\alpha^{-1}}$ ($\alpha \in G$) satisfying formulas of Section 1.1. It is understood that the algebraic tensor product $\otimes$ is replaced everywhere by its h-adic completions $\hat{\otimes}$.

The notions of crossed, quasitriangular, and ribbon h-adic Hopf G-coalgebras can be defined similarly following Sections 2.1 and 2.3.

Theorem H carries over to the h-adic Hopf algebras. The key modifications are that $\sigma\colon A \hat{\otimes} B \to \mathbb{C}[[h]]$ must be $\mathbb{C}[[h]]$-linear and $D(A, B; \sigma, \phi) = A \hat{\otimes} B$.

Theorem J. *Let $\sigma\colon A \hat{\otimes} B \to \mathbb{C}[[h]]$ be an h-adic Hopf pairing between two h-adic Hopf algebras A and B. Let $\phi\colon G \to \mathrm{Aut}_{\mathrm{Hopf}}(A)$ be an action of G on A by h-adic Hopf automorphisms. Then the family $D(A, B; \sigma, \phi) = \{D(A, B; \sigma, \phi_\alpha)\}_{\alpha \in G}$ is an h-adic Hopf G-coalgebra. Assume furthermore that A and B are topologically free, σ is non-degenerate, and $R_{\alpha,\beta} = \sum_i (e_i \otimes 1_B) \otimes (1_A \otimes f_i)$ belongs to the h-adic completion $D(A, B; \sigma, \phi_\alpha) \hat{\otimes} D(A, B; \sigma, \phi_\beta)$, where $(e_i)_i$ and $(f_i)_i$ are bases of A and B dual with respect to σ. Then $D(A, B; \sigma, \phi)$ is quasitriangular with R-matrix $R = \{R_{\alpha,\beta}\}_{\alpha,\beta \in G}$.*

The condition on $R_{\alpha,\beta}$ in the second part of the theorem means the following. Since A and B are topologically free, $A = V[[h]]$ and $B = W[[h]]$ for some complex vector spaces V and W. Then

$$D(A, B; \sigma, \phi_\alpha) \,\hat{\otimes}\, D(A, B; \sigma, \phi_\beta) = (V \otimes W \otimes V \otimes W)[[h]].$$

We require that $R_{\alpha,\beta} = \sum_i (e_i \otimes 1_B) \otimes (1_A \otimes f_i)$ can be expanded as $\sum_{n=0}^{\infty} r_n h^n$ for some $r_n \in V \otimes W \otimes V \otimes W$.

In the next section, we use Theorem J to define h-adic graded quantum groups.

3.7 *h*-adic graded quantum groups. Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra of rank l with Cartan matrix $(a_{i,j})$. Let $\{d_i\}_{i=1}^{l}$ be coprime integers such that the matrix $(d_i a_{i,j})$ is symmetric. Applying Theorem J to the h-adic Hopf algebras $U_h(\mathfrak{b}_+)$ and $\widetilde{U}_h(\mathfrak{b}_-) = \mathbb{C}[[h]]1 + hU_h(\mathfrak{b}_-)$, we obtain (after appropriate quotienting) quasitriangular "h-adic graded quantum groups" (see [Vir3], Proposition 6.1). We give here a direct description of these quantum groups.

Let $G = \mathbb{C}[[h]]^l$ with group operation being addition. For $\alpha = (\alpha_1, \ldots, \alpha_l) \in G$, let $U_h^\alpha(\mathfrak{g})$ be the h-adic algebra generated by the elements H_i, E_i, F_i, $1 \le i \le l$, subject to the following defining relations:

$$\begin{aligned}
&[H_i, H_j] = 0,\\
&[H_i, E_j] = a_{ij} E_j,\\
&[H_i, F_j] = -a_{ij} F_j,\\
&[E_i, F_j] = \delta_{i,j} \frac{e^{d_i h \alpha_i} e^{d_i h H_i} - e^{-d_i h H_i}}{e^{d_i h} - e^{-d_i h}},\\
&\sum_{r=0}^{1-a_{i,j}} (-1)^r \begin{bmatrix} 1-a_{i,j} \\ r \end{bmatrix}_{e^{d_i h}} E_i^{1-a_{i,j}-r} E_j E_i^r = 0 \quad (i \neq j),\\
&\sum_{r=0}^{1-a_{i,j}} (-1)^r \begin{bmatrix} 1-a_{i,j} \\ r \end{bmatrix}_{e^{d_i h}} F_i^{1-a_{i,j}-r} F_j F_i^r = 0 \quad (i \neq j).
\end{aligned}$$

The family $U_h^G(\mathfrak{g}) = \{U_h^\alpha(\mathfrak{g})\}_{\alpha \in G}$ has a structure of a crossed h-adic Hopf G-coalgebra given, for $\alpha = (\alpha_1, \ldots, \alpha_l), \beta = (\beta_1, \ldots, \beta_l) \in G$ and $1 \le i \le l$, by

$$\begin{aligned}
&\Delta_{\alpha,\beta}(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \varepsilon(H_i) = 0,\\
&\Delta_{\alpha,\beta}(E_i) = e^{d_i h \beta_i} E_i \otimes e^{d_i h H_i} + 1 \otimes E_i, \quad \varepsilon(E_i) = 0,\\
&\Delta_{\alpha,\beta}(F_i) = F_i \otimes 1 + e^{-d_i h H_i} \otimes F_i, \quad \varepsilon(F_i) = 0,
\end{aligned}$$

$$S_\alpha(H_i) = -H_i, \quad S_\alpha(E_i) = -e^{d_i h \alpha_i} E_i e^{-d_i h H_i}, \quad S_\alpha(F_i) = -e^{d_i h H_i} F_i,$$

$$\varphi_\alpha(H_i) = H_i, \quad \varphi_\alpha(E_i) = e^{d_i h \alpha_i} E_i, \quad \varphi_\alpha(F_i) = e^{-d_i h \alpha_i} F_i.$$

Furthermore, $U_h^G(\mathfrak{g})$ is quasitriangular by Theorem J (the conditions of this theorem are satisfied by $A = U_h(\mathfrak{b}_+)$ and $B = \widetilde{U}_h(\mathfrak{b}_-)$). For example, for $\mathfrak{g} = sl_2$ and $G = \mathbb{C}[[h]]$, the R-matrix of $U_h^G(sl_2)$ is given by

$$R_{\alpha,\beta} = e^{h(H\otimes H)/2} \sum_{n=0}^{\infty} R_n(h)\, E^n \otimes F^n \in U_h^{\alpha}(sl_2) \mathbin{\widehat{\otimes}} U_h^{\beta}(sl_2)$$

for all $\alpha, \beta \in \mathbb{C}[[h]]$, where $R_n(h) = q^{n(n+1)/2} \frac{(1-q^{-2})^n}{[n]_q!}$ and $q = e^h$.