Appendix 7

Invariants of 3-dimensional G-manifolds from Hopf coalgebras

by Alexis Virelizier

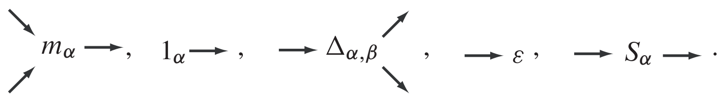
In this appendix we construct invariants of closed oriented 3-dimensional G-manifolds using Hopf G-coalgebras. In contrast to the methods of Chapters VII and VIII, we do not involve representations of the Hopf G-coalgebras. Our invariants generalize the Kuperberg invariant and the Hennings invariant of 3-manifolds corresponding to the case $G = \{1\}$.

Throughout this appendix, G is a group and K is a field.

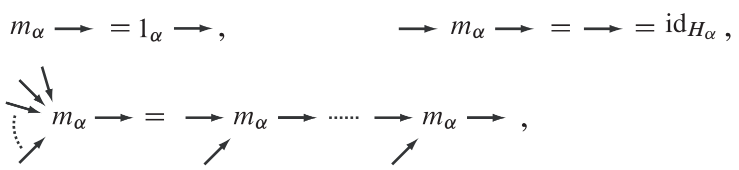
7.1 Kuperberg-type invariants

Kuperberg [Ku] derived an invariant of closed oriented 3-manifolds from any finite-dimensional involutory Hopf algebra. As the main geometric tool, he used Heegaard diagrams of 3-manifolds. Here we generalize Kuperberg's method to construct invariants of closed oriented 3-dimensional G-manifolds from involutory Hopf G-coalgebras.

1.1 Diagrammatic formalism for Hopf G-coalgebras. Let $H = \{H_\alpha\}_{\alpha\in\pi}$ be a Hopf G-coalgebra of finite type. The multiplication $m_\alpha\colon H_\alpha \otimes H_\alpha \to H_\alpha$, the unit element $1_\alpha \in H_\alpha$, the comultiplication $\Delta_{\alpha,\beta}\colon H_{\alpha\beta} \to H_\alpha \otimes H_\beta$, the counit $\varepsilon\colon H_1 \to K$, and the antipode $S_\alpha\colon H_\alpha \to H_{\alpha^{-1}}$ are represented pictorially as follows:



The inputs (incoming arrows) for multiplication are always ordered counterclockwise and the outputs (outgoing arrows) for comultiplication are always ordered clockwise. Furthermore, we adopt the following abbreviations:



and

$$\longrightarrow \Delta_1 = \longrightarrow \varepsilon, \qquad \longrightarrow \Delta_\alpha \longrightarrow = \longrightarrow = \mathrm{id}_{H_\alpha},$$

$$\longrightarrow \Delta_{\alpha_1,\dots,\alpha_n} \;=\; \longrightarrow \Delta_{\alpha_1\dots\alpha_{n-1},\alpha_n} \longrightarrow \cdots\cdots \longrightarrow \Delta_{\alpha_1,\alpha_2} \longrightarrow .$$

The homomorphisms represented by such diagrams may be explicitly computed in terms of the structure constants. In particular, set

$$\Lambda = \Delta_{1,1} \in H_1 \quad \text{and} \quad \lambda_\alpha = m_\alpha \in H_\alpha^*.$$

This means that if $(e_i)_i$ is a basis of H_1 and $C_i^{j,k} \in K$ are the structure constants of $\Delta_{1,1}\colon H_1 \to H_1 \otimes H_1$ defined by $\Delta_{1,1}(e_i) = \sum_{j,k} C_i^{j,k} e_j \otimes e_k$, then $\Lambda = \sum_{i,k} C_i^{i,k} e_k$. Likewise, if $(f_i)_i$ is a basis of H_α and $\mu_{i,j}^k \in K$ are the structure constants of m_α defined by $m_\alpha(f_i \otimes f_j) = \sum_k \mu_{i,j}^k f_k$, then $\lambda_\alpha(f_j) = \sum_k \mu_{k,j}^k$.

Assume now that H is involutory as defined in Appendix 6, Section 1.9. By Lemma 4 of [Vir4], $\lambda = (\lambda_\alpha)_{\alpha \in G}$ is a two-sided G-integral for H and Λ is a two-sided integral for H_1 such that

$$\lambda_1(1_1) = \varepsilon(\Lambda) = \lambda_1(\Lambda) = \dim H_1, \quad S_1(\Lambda) = \Lambda, \quad \text{and} \quad \lambda_{\alpha^{-1}} S_\alpha = \lambda_\alpha$$

for all $\alpha \in G$. By Lemma 5 of [Vir4], Λ and λ are symmetric in the following sense: for all $\alpha \in G$ and $x, y \in H_\alpha$,

$$\Delta_{\alpha,\alpha^{-1}}(\Lambda) = \sigma_{H_{\alpha^{-1}},H_\alpha} \Delta_{\alpha^{-1},\alpha}(\Lambda) \quad \text{and} \quad \lambda_\alpha(xy) = \lambda_\alpha(yx).$$

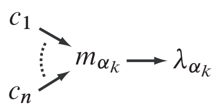
1.2 Construction of the invariant. Let $H = \{H_\alpha\}_{\alpha \in \pi}$ be an involutory Hopf G-coalgebra of finite type such that the characteristic of the ground field K of H does not divide $\dim H_1$. Note that H is then semisimple and cosemisimple (see Appendix 6, Section 1.9).

Let (W, g) be a closed connected oriented 3-dimensional G-manifold. Recall from Section VII.2.1 that W is a closed connected oriented 3-dimensional manifold and g is a free homotopy class of maps from W to $X = K(G, 1)$. We present W by a Heegaard diagram (Σ, u, l), where Σ is an oriented closed surface of genus $g \geq 0$ embedded in W (and cutting W into two genus g handle bodies), $u = \{u_1, \dots, u_g\}$ and $l = \{l_1, \dots, l_g\}$ are two transverse g-tuples of pairwise disjoint circles embedded in Σ such that $\Sigma \setminus \bigcup_k u_k$ and $\Sigma \setminus \bigcup_i l_i$ are connected. We pick $z \in \Sigma \setminus (u \cup l)$ and orient all the circles u_k and l_i in an arbitrary way.

Traveling along each lower circle l_i, we obtain a word $w_i(x_1, \dots, x_g)$ in the alphabet $\{x_1^{\pm 1}, \dots, x_g^{\pm 1}\}$ as follows. Start at any point of l_i not belonging to u and make a round trip along l_i following its orientation. Begin with the empty word and each time l_i intersects some u_k, add on the right of the word the letter x_k if l_i intersects u_k positively[1] and the letter x_k^{-1} otherwise. After a complete turn along l_i, we obtain the word w_i. This word is defined up to conjugation due to the indeterminacy in the choice of the starting point on l_i. Since $\Sigma \setminus u$ is a 2-sphere with $2g$ disks deleted, there exists g loops $\gamma_1, \ldots, \gamma_g$ on (Σ, z) such that each γ_i intersects once positively u_i and does not meet $\bigcup_{j \neq i} u_j$. The homotopy classes $a_i = [\gamma_i] \in \pi_1(W, z)$, $i = 1, \ldots, g$ do not depend on the choice of the loops γ_i. By the Van Kampen theorem,

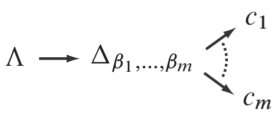
$$\pi_1(W, z) = \langle a_1, \ldots, a_g \mid w_i(a_1, \ldots, a_g) = 1 \text{ for } 1 \leq i \leq g \rangle.$$

For any $1 \leq k \leq g$, we provide the circle u_k with the label $\alpha_k = g_*(a_k) \in G$, where $g_* \colon \pi_1(W, z) \to \pi_1(X, x) = G$ is the homomorphism induced by a map $W \to X$ in the given homotopy class g carrying z to the base point x of $X = K(G, 1)$. To each u_k, we associate the tensor



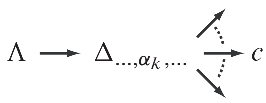
where $c_1, \ldots, c_n$ are the crossings between u_k and the circles l_i which appear in this order when making a round trip along u_k following its orientation. Since this tensor is cyclically symmetric (see Section 1.1), this assignment does not depend on the choice of the starting point on u_k.

To each circle l_i, we associate the tensor

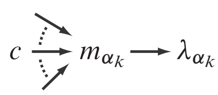


where $c_1, \ldots, c_m$ are the crossings between l_i and the circles u_k which appear in this order when making a round trip along l_i following its orientation; if l_i intersects u_k at c_j, then $\beta_j = \alpha_k \in G$ if the intersection is positive and $\beta_j = \alpha_k^{-1}$ otherwise. Note that $\beta_1 \ldots \beta_m = w_i(\alpha_1, \ldots, \alpha_g) = 1$ and so the tensor associated to l_i is well defined. Since this tensor is cyclically symmetric, this assignment does not depend on the choice of the starting point on l_i.

Let c be a crossing point between some u_k and l_i. If l_i intersects u_k positively at c, then we contract the tensors



and



associated to l_i and u_k as follows:

[1] An oriented curve γ on Σ intersects *positively* another oriented curve ρ on Σ at a point $c \in \Sigma$ if $(d_c\gamma, d_c\rho)$ is a positively-oriented basis for $T_c\Sigma$.

$$\Lambda \longrightarrow \Delta_{\dots,\alpha_k,\dots} \longrightarrow m_{\alpha_k} \longrightarrow \lambda_{\alpha_k}\,.$$

If l_i intersects u_k negatively at c, then we contract the associated tensors

$$\Lambda \longrightarrow \Delta_{\dots,\alpha_k^{-1},\dots} \longrightarrow c \quad \text{and} \quad c \longrightarrow m_{\alpha_k} \longrightarrow \lambda_{\alpha_k}$$

as follows:

$$\Lambda \longrightarrow \Delta_{\dots,\alpha_k^{-1},\dots} \longrightarrow S_{\alpha_k^{-1}} \longrightarrow m_{\alpha_k} \longrightarrow \lambda_{\alpha_k}\,.$$

After having done such contractions at all the crossing points, we obtain an element $Z(\Sigma, u, l)$ of K. Set

$$\mathrm{Ku}_H(W, g) = (\dim H_1)^{-g}\, Z(\Sigma, u, l) \in K.$$

Theorem A ([Vir4], Theorem 9). $\mathrm{Ku}_H(W, g)$ *is a homeomorphism invariant of the G-manifold* (W, g).

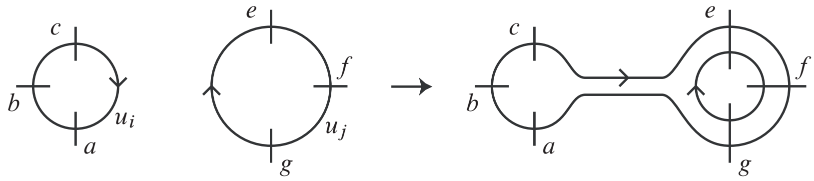
1.3 Examples. 1. If g is the trivial homotopy class of maps $W \to X$ represented by the map $W \to \{x\} \subset X$, then $\mathrm{Ku}_H(W, g)$ is equal to the Kuperberg invariant of W derived from the involutory Hopf algebra H_1. In particular, for $G = \{1\}$, we recover the Kuperberg invariant.

2. Let G, L be finite groups and $\mathbb{C}^G, \mathbb{C}^L$ be the Hopf algebras of $\mathbb{C}$-valued functions on G and L, respectively. A group homomorphism $\phi\colon L \to G$ induces a Hopf algebra morphism $\mathbb{C}^G \to \mathbb{C}^L$, $f \mapsto f\phi$ whose image is central. By Appendix 6, Section 1.2, this data yields to a Hopf G-coalgebra $H^\phi = \{H^\phi_\alpha\}_{\alpha\in\pi}$, which is involutory and of finite type. Note that $H^\phi_\alpha \cong \mathbb{C}^{\phi^{-1}(\alpha)}$ as an algebra. For every closed connected oriented 3-dimensional G-manifold (W, g),

$$\mathrm{Ku}_{H^\phi}(W, g) = \#\{f : \pi_1(W, z) \to L \mid \phi f = g_*\},$$

where $z \in W$ and $g_*\colon \pi_1(W, z) \to G$ is the homomorphism induced by a map $W \to X$ in the homotopy class of g carrying z to the base point x of $X = K(G, 1)$.

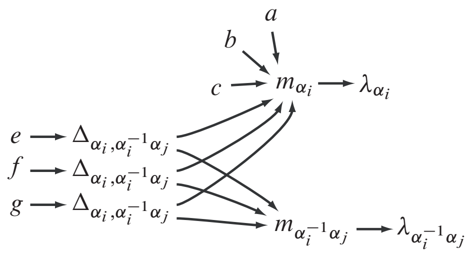
1.4 Proof of Theorem A (sketch). The proof is based on a "G-colored" version of the Reidemeister-Singer theorem which relates any G-colored Heegaard diagrams representing (W, g, z). For example, suppose that a circle u_i (with label α_i) slides across another circle u_j (with label α_j). Assume, as a representative case, that both circles have three crossings with $\bigcup_k l_k$:



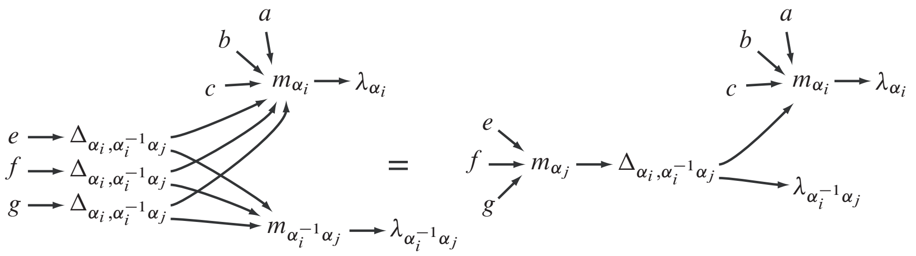
Using the anti-multiplicativity of the antipode (which allows us to consider only the positively-oriented case of the contraction rule), we obtain that the factor

$$\begin{tikzcd}[row sep=small] a \arrow[dr] & & \\ b \arrow[r] & m_{\alpha_i} \arrow[r] & \lambda_{\alpha_i} \\ c \arrow[ur] & & \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] e \arrow[dr] & & \\ f \arrow[r] & m_{\alpha_j} \arrow[r] & \lambda_{\alpha_j} \\ g \arrow[ur] & & \end{tikzcd}$$

of $Z(\Sigma, u, l)$ is replaced under this move by



Since the comultiplication is multiplicative and $\lambda = (\lambda_\alpha)_{\alpha\in G}$ is a left G-integral for H, these two factors are equal:



$$= \begin{tikzcd}[row sep=small] e \arrow[dr] & & \\ f \arrow[r] & m_{\alpha_j} \arrow[r] & \lambda_{\alpha_j} \\ g \arrow[ur] & & \end{tikzcd} \quad \begin{tikzcd}[row sep=small] b \arrow[dr] & a \arrow[d] & \\ c \arrow[r] & m_{\alpha_i} \arrow[r] & \lambda_{\alpha_i} \\ 1_{\alpha_i} \arrow[ur] & & \end{tikzcd} = \begin{tikzcd}[row sep=small] a \arrow[dr] & & \\ b \arrow[r] & m_{\alpha_i} \arrow[r] & \lambda_{\alpha_i} \\ c \arrow[ur] & & \end{tikzcd} \quad \begin{tikzcd}[row sep=small] e \arrow[dr] & & \\ f \arrow[r] & m_{\alpha_j} \arrow[r] & \lambda_{\alpha_j} \\ g \arrow[ur] & & \end{tikzcd}$$

For a detailed proof of Theorem A, we refer to [Vir4] (cf. also the next remark).

1.5 Remark. Since H is of finite type, semisimple, and spherical (with spherical elements $w_\alpha = 1_\alpha \in H_\alpha$), the category $\mathrm{Rep}(H)$ of finite-dimensional representations of H is a finite semisimple spherical G-category. Hence we can consider the state sum invariant $|W,g|_{\mathrm{Rep}(H)}$ introduced in Appendix 2. Adapting the arguments of [BW1], we obtain that

$$\mathrm{Ku}_H(W,g) = |W,g|_{\mathrm{Rep}(H)}$$

for any closed connected oriented 3-dimensional G-manifold (W,g).

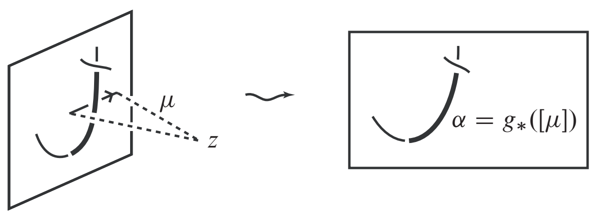
7.2 Hennings–Kauffman–Radford-type invariants

Hennings [He] derived invariants of links and 3-manifolds from right integrals on certain Hopf algebras. His construction was studied and clarified by Kauffman and Radford [KaRa]. In this section, we generalize the Hennings–Kauffman–Radford method to construct an invariant τ_H of 3-dimensional G-manifolds by using a ribbon Hopf G-coalgebra H. When the ribbon G-category Rep(H) of representations of H is modular, we compare τ_H with the Turaev invariant $\tau_{\mathrm{Rep}(H)}$ from Section VII.2.

2.1 The invariant τ_H. Let $H = \{H_\alpha\}_{\alpha\in G}$ be a unimodular ribbon Hopf G-coalgebra of finite type and let $\lambda = (\lambda_\alpha)_{\alpha\in G}$ be a (non-zero) right G-integral for H such that $\lambda_1(\theta_1^{\pm1}) \neq 0$, where $\theta = \{\theta_\alpha\}_{\alpha\in G}$ is the twist of H.

Let (W,g) be a closed connected oriented 3-dimensional G-manifold (see Section VII.2.1). We define $\tau_H(W,g) \in K$ as follows. Present W as the result of surgery on S^3 along a framed link ℓ with $m = \#\ell$ components. Recall that W is obtained by gluing m solid tori to the exterior E_ℓ of $\ell \in S^3$. Take any point $z \in E_\ell \subset W$. Pick in the homotopy class g a map $W \to X$ carrying z to the base point x of X. The restriction of this map to E_ℓ induces a homomorphism $g_*\colon \pi_1(E_\ell, z) \to \pi_1(X,x) = G$. Note that the triple (ℓ, z, g_*) is an unoriented special G-link in the sense of Section VI.5.4.

Regularly project ℓ onto a plane from the base point z, that is, consider a diagram of ℓ such that the base point z corresponds to the eyes of the reader. Without loss of generality, we can assume that the extremal points of the diagram with respect to a chosen height function are isolated. Label the vertical segments of the diagram (delimited by the extremal points of the height function and the under-crossings) by elements of G in the following way: a vertical segment is labeled by $\alpha = g_*([\mu]) \in G$ where μ is a meridional loop of ℓ based at z which encircles the segment once so that its linking number with this segment oriented downwards is +1:



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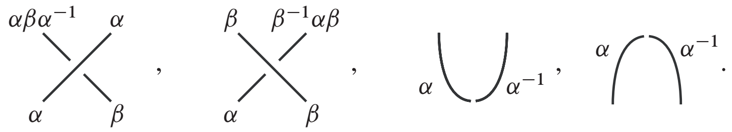
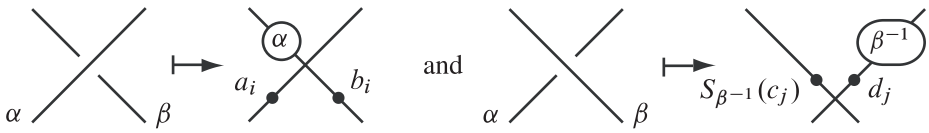
At the crossings and the extremal points the labels are related as follows:



Now, we decorate each crossing of the diagram of ℓ with the R-matrix $R = \{R_{\alpha,\beta}\}_{\alpha,\beta\in G}$ of H and with small disks labeled by elements of G as follows:

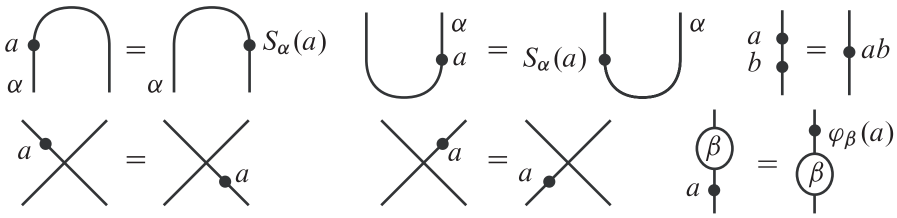


where

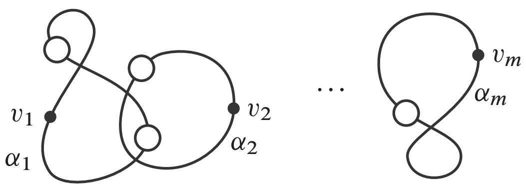
$$R_{\alpha,\beta} = a_i \otimes b_i \quad \text{and} \quad R_{\beta^{-1},\alpha} = c_j \otimes d_j.$$

In this formalism it is understood that there is a summation over all the pairs a_i, b_i and c_j, d_j. The diagram obtained at this step is composed by $m = \#\ell$ transverse closed plane curves (possibly endowed with G-labeled disks), each of them arising from a component of ℓ.

We use the following rules to concentrate the algebraic decoration of each of these plane curves in one point (distinct from the extremal points and the labeled disks):

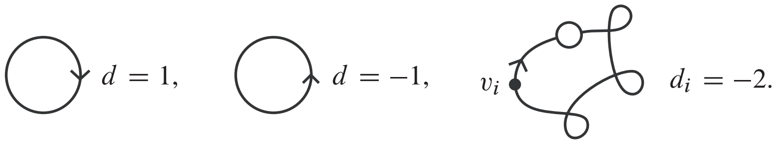


This gives m elements $v_1 \in H_{\alpha_1}, \dots, v_m \in H_{\alpha_m}$:



If there is no algebraic decoration on the i-th curve then, by convention, $v_i = 1_{\alpha_i}$.

For $1 \le i \le m$, let d_i be the Whitney degree of the i-th curve obtained by traversing it upwards from the point where the algebraic decoration has been concentrated. The Whitney degree is the algebraic number of turns of the tangent vector of the curve. For example:



Finally set

$$\tau_H(W, g) = \lambda_1(\theta_1)^{b_-(\ell)-m}\,\lambda_1(\theta_1^{-1})^{-b_-(\ell)}\,\lambda_{\alpha_1}(w_{\alpha_1}^{1+d_1}v_1)\dots\lambda_{\alpha_m}(w_{\alpha_m}^{1+d_m}v_m) \in K,$$

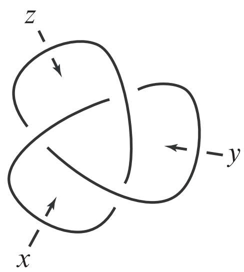
where $w = (w_\alpha)_{\alpha\in G}$ is the spherical G-grouplike element of H (see Appendix 6, Section 2.6) and $b_-(\ell)$ is the number of strictly negative eigenvalues of the linking matrix of ℓ (with the framing numbers on the diagonal).

Theorem B ([Vir1], Theorem 4.12). *$\tau_H(W, g)$ is a homeomorphism invariant of the G-manifold (W, g).*

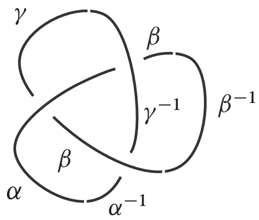
The invariant τ_H is preserved under multiplication of the right G-integral λ by any element of K^*. Since the space of right G-integrals for H is one-dimensional (see Appendix 6, Section 1.1), τ_H does not depend on the choice of λ.

2.2 Examples. 1. If g is the trivial homotopy class of maps $W \to X$ represented by the map $W \to \{x\} \subset X$, then $\tau_H(W, g)$ is the Hennings invariant (in its Kauffman–Radford reformulation) of W derived from the ribbon Hopf algebra H_1. In particular, when $G = \{1\}$, we recover the Hennings invariant.

2. Let P be the closed oriented 3-dimensional manifold obtained by surgery along the trefoil T with framing $+3$:

$T =$ 

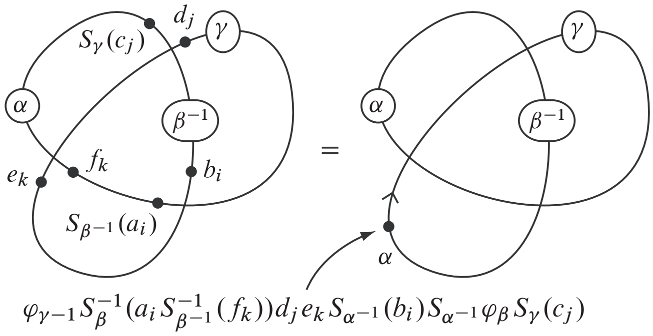
The Wirtinger presentation of the group of T is $\langle x, y, z \mid xy = yz = zx\rangle$ and $\pi_1(P) = \langle x, y, z \mid xy = yz = zx,\ xzy = 1\rangle$. Let g be a free homotopy class of maps $P \to X = K(G, 1)$ inducing a homomorphism $g_*\colon \pi_1(T) \to G$. Set $\alpha = g_*(x)$, $\beta = g_*(y)$, $\gamma = g_*(z)$. The labeling of the vertical segments of the diagram of T is:



Expand

$$R_{\beta^{-1},\alpha^{-1}} = \sum_i a_i \otimes b_i, \quad R_{\gamma,\alpha} = \sum_j c_j \otimes d_j, \quad \text{and} \quad R_{\alpha,\beta} = \sum_k e_k \otimes f_k.$$

The algorithm above gives

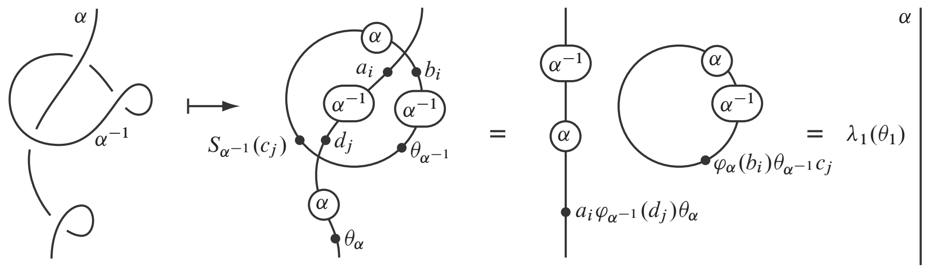


Therefore

$$\tau_H(W,g) = \lambda_1(\theta_1)^{-1} \sum_{i,j,k} \lambda_\alpha\big(w_\alpha^3 \varphi_{\gamma^{-1}} S_\beta^{-1}(a_i S_{\beta^{-1}}^{-1}(f_k)) d_j e_k S_{\alpha^{-1}}(b_i) S_{\alpha^{-1}} \varphi_\beta S_\gamma(c_j)\big).$$

Exercise. Given a finite group G, compute $\tau_{D_G(G)}(W,g)$, where $D_G(G)$ is the ribbon Hopf G-coalgebra defined in Appendix 6, Section 3.3.

2.3 Proof of Theorem B (sketch). The proof uses a "G-colored" version of the Kirby calculus relating G-colored link diagrams representing (W,g). For example, the invariance under the G-colored Fenn–Rourke move with one strand is proven as follows.



where

$$R_{\alpha,\alpha^{-1}} = \sum_i a_i \otimes b_i \quad \text{and} \quad R_{\alpha^{-1},\alpha} = \textstyle\sum_j c_j \otimes d_j .$$

Indeed,

$$\begin{aligned}
&\sum_{i,j} \lambda_{\alpha^{-1}}(\varphi_\alpha(b_i)\theta_{\alpha^{-1}}c_j)\, a_i\varphi_{\alpha^{-1}}(d_j)\theta_\alpha \\
&\qquad = \sum_{i,j} \lambda_{\alpha^{-1}}(\theta_{\alpha^{-1}}b_i c_j)\, \theta_\alpha\varphi_\alpha(a_i)d_j \\
&\qquad = (\lambda_{\alpha^{-1}} \otimes \mathrm{id}_{H_\alpha})((\theta_{\alpha^{-1}} \otimes \theta_\alpha)(\sigma_{\alpha,\alpha^{-1}}(\varphi_\alpha \otimes \mathrm{id}_{H_{\alpha^{-1}}})(R_{\alpha,\alpha^{-1}}))R_{\alpha^{-1},\alpha}) \\
&\qquad = (\lambda_{\alpha^{-1}} \otimes \mathrm{id}_{H_\alpha})\Delta_{\alpha^{-1},\alpha}(\theta_1) = \lambda_1(\theta_1)\, 1_\alpha .
\end{aligned}$$

These equalities follows from the properties of the crossing φ, the twist θ, and the G-integral λ of H (see Appendix 6). We refer to [Vir1] for a detailed proof. A crucial role in the proof is played by the fact that the family of homomorphisms $(H_\alpha \to K,\ x \mapsto \lambda_\alpha(w_\alpha x))_{\alpha\in G}$ is a G-trace for H (Appendix 6, Theorem G).

2.4 Comparison with the Turaev invariant. Let $H = \{H_\alpha\}_{\alpha\in G}$ be a ribbon Hopf G-coalgebra of finite type. Suppose that the ribbon G-category $\mathrm{Rep}(H)$ of representations of H (see Section VIII.1.7) is modular. Then the Turaev invariant $\tau_{\mathrm{Rep}(H)}$ of Section VII.2 is well defined. Moreover, under these assumptions, H is unimodular, since H_1 is then factorizable and so unimodular (see [Sw]). Furthermore, $\lambda_1(\theta_1^{\pm 1}) \neq 0$ for every non-zero right G-integral $\lambda = (\lambda_\alpha)_{\alpha\in G}$ for H (since $\lambda_1(\theta_1^{\pm 1}) = \Delta_\pm^{\mathrm{Rep}(H)}$ up to a non-zero scalar multiple). Hence the invariant of the preceding section τ_H is also defined. The next theorem shows that under these assumptions, the invariants $\tau_{\mathrm{Rep}(H)}$ and τ_H are essentially equivalent.

Theorem C ([Vir1], Theorem 4.18). *Let H be a ribbon Hopf G-coalgebra of finite type such that its ribbon G-category of representations* $\mathrm{Rep}(H)$ *is modular. Then the invariant τ_H is well defined and for every closed connected oriented 3-dimensional G-manifold (W, g),*

$$\tau_{\mathrm{Rep}(H)}(W, g) = D^{-1}\left(\frac{D}{\Delta_-}\right)^{b_1(W)} \tau_H(W, g),$$

where $b_1(W)$ is the first Betti number of W, and D, Δ_- are as in Section VII.1.7.

The proof is based on a description of the Turaev invariant in terms of the coend of the G-category $\mathrm{Rep}(H)$, see [Vir1].

Note that when the category $\mathrm{Rep}(H)$ is not modular (typically, when H is not semisimple, see Appendix 6, Section 1.7) the invariant $\tau_{\mathrm{Rep}(H)}$ is not defined while τ_H may be defined.