ON THE CENTER OF FUSION CATEGORIES

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Müger proved in 2003 that the center of a spherical fusion category $\mathscr{C}$ of nonzero dimension over an algebraically closed field is a modular fusion category whose dimension is the square of that of $\mathscr{C}$. We generalize this theorem to a pivotal fusion category $\mathscr{C}$ over an arbitrary commutative ring $\Bbbk$, without any condition on the dimension of the category. (In this generalized setting, modularity is understood as 2-modularity in the sense of Lyubashenko.) Our proof is based on an explicit description of the Hopf algebra structure of the coend of the center of $\mathscr{C}$. Moreover we show that the dimension of $\mathscr{C}$ is invertible in $\Bbbk$ if and only if any object of the center of $\mathscr{C}$ is a retract of a "free" half-braiding. As a consequence, if $\Bbbk$ is a field, then the center of $\mathscr{C}$ is semisimple (as an abelian category) if and only if the dimension of $\mathscr{C}$ is nonzero. If in addition $\Bbbk$ is algebraically closed, then this condition implies that the center is a fusion category, so that we recover Müger's result.

Introduction

Given a monoidal category $\mathscr{C}$, Joyal and Street [1991], Drinfeld (unpublished), and Majid [1991] defined a braided category $\mathscr{Z}(\mathscr{C})$, called the center of $\mathscr{C}$, whose objects are half-braidings of $\mathscr{C}$. Müger [2003] showed that the center $\mathscr{Z}(\mathscr{C})$ of a spherical fusion category $\mathscr{C}$ of nonzero dimension over an algebraically closed field $\Bbbk$ is a modular fusion category, and that the dimension of $\mathscr{Z}(\mathscr{C})$ is the square of that of $\mathscr{C}$. Müger's proof of this remarkable result relies on algebraic constructions due to Ocneanu (such as the "tube" algebra) and involves the construction of a

weak monoidal Morita equivalence between $\mathscr{Z}(\mathscr{C})$ and $\mathscr{C} \otimes \mathscr{C}^{op}$. The modularity of the center is of special interest in three-dimensional quantum topology, since spherical fusion categories and modular categories are respectively the algebraic input for the construction of the Turaev–Viro/Barrett–Westbury invariant and of the Reshetikhin–Turaev invariant. Indeed it has been shown recently in [Turaev and Virelizier 2010] (see also [Balsam 2010]) that, under the hypotheses of Müger's theorem, the Barrett–Westbury generalization of the Turaev–Viro invariant for $\mathscr{C}$ is equal to the Reshetikhin–Turaev invariant for $\mathscr{Z}(\mathscr{C})$.

In this paper, we generalize Müger's theorem to pivotal fusion categories over an arbitrary commutative ring. More precisely, given a pivotal fusion category $\mathscr{C}$ over a commutative ring $\Bbbk$, we prove the following:

(i) the center $\mathscr{Z}(\mathscr{C})$ of $\mathscr{C}$ is always modular (but not necessarily semisimple) and has dimension $\dim(\mathscr{C})^2$;

(ii) the scalar $\dim(\mathscr{C})$ is invertible in $\Bbbk$ if and only if every half braiding is a retract of a so-called *free half braiding*;

(iii) if $\Bbbk$ is a field, then $\mathscr{Z}(\mathscr{C})$ is abelian semisimple if and only if $\dim(\mathscr{C}) \neq 0$;

(iv) if $\Bbbk$ is an algebraically closed field, then $\mathscr{Z}(\mathscr{C})$ is fusion if and only if $\dim(\mathscr{C}) \neq 0$.

Our proof is different from that of Müger. It relies on the principle that if a braided category $\mathscr{B}$ has a coend, then all the relevant information about $\mathscr{B}$ is encoded in its coend, which is a universal Hopf algebra sitting in $\mathscr{B}$ and endowed with a canonical Hopf algebra pairing. For instance, modularity means that the canonical pairing is nondegenerate, and the dimension of $\mathscr{B}$ is that of its coend. In particular we do not need to introduce an auxiliary category.

The center $\mathscr{Z}(\mathscr{C})$ of a pivotal fusion category $\mathscr{C}$ always has a coend. We provide a complete and explicit description of the Hopf algebra structure of this coend, which enables us to exhibit an integral for the coend and an "inverse" to the pairing. Our proofs are based on a "handleslide" property for pivotal fusion categories.

A general description of the coend of the center of a rigid category $\mathscr{C}$, together with its structural morphisms, was given in [Bruguières and Virelizier 2012]. It is an application of the theory of Hopf monads, and in particular, of the notion of double of a Hopf monad, which generalizes the Drinfeld double of a Hopf algebra. It is based on the fact that $\mathscr{Z}(\mathscr{C})$ is the category of modules over a certain quasitriangular Hopf monad Z on $\mathscr{C}$ (generalizing the braided equivalence $\mathscr{Z}(\mathrm{mod}_H) \simeq \mathrm{mod}_{D(H)}$ between the center of the category of modules over a finite-dimensional Hopf algebra H and the category of modules over the Drinfeld double $D(H)$ of H). It turns out that, when $\mathscr{C}$ is a fusion category, we can make this description very explicit and in particular, we can depict the structural morphisms of the coend by means of a graphical formalism for fusion categories.

Part of the results of this paper were announced (without proofs) in [Bruguières and Virelizier 2008], where they were used to define and compute a 3-manifolds invariant of Reshetikhin–Turaev type associated with the center of $\mathscr{C}$, even when the dimension of $\mathscr{C}$ is not invertible.

Organization of the text. In Section 1, we recall definitions, notations and basic results concerning pivotal and fusion categories over a commutative ring. A graphical formalism for representing morphisms in fusion categories is provided. In Section 2, we state the main results of this paper, that is, the description of the coend of the center of a pivotal fusion category and its structural morphisms, the modularity of the center of such a category, its dimension, and a semisimplicity criterion. Section 3 is devoted to coends, Hopf algebras in braided categories, and modular categories. Section 4 contains the proofs of the main results.

1. Pivotal and fusion categories

Monoidal categories are assumed to be strict. This does not lead to any loss of generality, since, in view of Mac Lane's coherence theorem for monoidal categories (see [Mac Lane 1998]), all definitions and statements remain valid for nonstrict monoidal categories after insertion of the suitable canonical isomorphisms.

1A. ***Rigid categories.*** Let $\mathscr{C} = (\mathscr{C}, \otimes, \mathbb{1})$ be a monoidal category. A *left dual* of an object X of $\mathscr{C}$ is an object $^{\vee}X$ of $\mathscr{C}$ together with morphisms $\mathrm{ev}_X : {}^{\vee}X \otimes X \to \mathbb{1}$ and $\mathrm{coev}_X : \mathbb{1} \to X \otimes {}^{\vee}X$ such that

$$(\mathrm{id}_X \otimes \mathrm{ev}_X)(\mathrm{coev}_X \otimes \mathrm{id}_X) = \mathrm{id}_X \quad \text{and} \quad (\mathrm{ev}_X \otimes \mathrm{id}_{^{\vee}X})(\mathrm{id}_{^{\vee}X} \otimes \mathrm{coev}_X) = \mathrm{id}_{^{\vee}X}.$$

Similarly a *right dual* of X is an object $X^{\vee}$ with morphisms $\widetilde{\mathrm{ev}}_X : X \otimes X^{\vee} \to \mathbb{1}$ and $\widetilde{\mathrm{coev}}_X : \mathbb{1} \to X^{\vee} \otimes X$ such that

$$(\widetilde{\mathrm{ev}}_X \otimes \mathrm{id}_X)(\mathrm{id}_X \otimes \widetilde{\mathrm{coev}}_X) = \mathrm{id}_X \quad \text{and} \quad (\mathrm{id}_{X^{\vee}} \otimes \widetilde{\mathrm{ev}}_X)(\widetilde{\mathrm{coev}}_X \otimes \mathrm{id}_{X^{\vee}}) = \mathrm{id}_{X^{\vee}}.$$

The left and right duals of an object, if they exist, are unique up to an isomorphism (preserving the (co)evaluation morphisms).

A monoidal category $\mathscr{C}$ is *rigid* (or *autonomous*) if every object of $\mathscr{C}$ admits a left and a right dual. The choice of left and right duals for each object of a rigid $\mathscr{C}$ defines a left dual functor $^{\vee}? : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}$ and a right dual functor $?^{\vee} : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}$, where $\mathscr{C}^{\mathrm{op}}$ is the opposite category to $\mathscr{C}$ with opposite monoidal structure. The left and right dual functors are strong monoidal. Note that the actual choice of left and right duals is innocuous in the sense that different choices of left (respectively, right) duals define canonically monoidally isomorphic left (respectively, right) dual functors.

There are canonical natural monoidal isomorphisms $^{\vee}(X^{\vee}) \simeq X \simeq (^{\vee}X)^{\vee}$, but in general the left and right dual functors are not monoidally isomorphic.

1B. *Pivotal categories.* A rigid category $\mathscr{C}$ is *pivotal* (or *sovereign*) if it is endowed with a monoidal isomorphism between the left and the right dual functors. We may assume that this isomorphism is the identity without loss of generality. In other words, for each object X of $\mathscr{C}$, we have a *dual object* X^* and four morphisms

$$\mathrm{ev}_X : X^* \otimes X \to \mathbb{1}, \quad \mathrm{coev}_X : \mathbb{1} \to X \otimes X^*,$$

$$\widetilde{\mathrm{ev}}_X : X \otimes X^* \to \mathbb{1}, \quad \widetilde{\mathrm{coev}}_X : \mathbb{1} \to X^* \otimes X,$$

such that $(X^*, \mathrm{ev}_X, \mathrm{coev}_X)$ is a left dual for X, $(X^*, \widetilde{\mathrm{ev}}_X, \widetilde{\mathrm{coev}}_X)$ is a right dual for X, and the induced left and right dual functors coincide as monoidal functors. In particular, the *dual* $f^* : Y^* \to X^*$ of any morphism $f : X \to Y$ in $\mathscr{C}$ is

$$f^* = (\mathrm{ev}_Y \otimes \mathrm{id}_{X^*})(\mathrm{id}_{Y^*} \otimes f \otimes \mathrm{id}_{X^*})(\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X)$$

$$= (\mathrm{id}_{X^*} \otimes \widetilde{\mathrm{ev}}_Y)(\mathrm{id}_{X^*} \otimes f \otimes \mathrm{id}_{Y^*})(\widetilde{\mathrm{coev}}_X \otimes \mathrm{id}_{Y^*}).$$

In what follows, for a pivotal category $\mathscr{C}$, we will suppress the duality constraints $\mathbb{1}^* \cong \mathbb{1}$ and $X^* \otimes Y^* \cong (Y \otimes X)^*$. For example, we will write $(f \otimes g)^* = g^* \otimes f^*$ for morphisms f, g in $\mathscr{C}$.

1C. *Traces and dimensions.* For an endomorphism f of an object X of a pivotal category $\mathscr{C}$, one defines the *left* and *right traces* $\mathrm{tr}_l(f)$, $\mathrm{tr}_r(f) \in \mathrm{End}_{\mathscr{C}}(\mathbb{1})$ by

$$\mathrm{tr}_l(f) = \mathrm{ev}_X(\mathrm{id}_{X^*} \otimes f)\widetilde{\mathrm{coev}}_X \quad \text{and} \quad \mathrm{tr}_r(f) = \widetilde{\mathrm{ev}}_X(f \otimes \mathrm{id}_{X^*})\mathrm{coev}_X.$$

They satisfy $\mathrm{tr}_l(gh) = \mathrm{tr}_l(hg)$ and $\mathrm{tr}_r(gh) = \mathrm{tr}_r(hg)$ for any morphisms $g : X \to Y$ and $h : Y \to X$ in $\mathscr{C}$. Also we have $\mathrm{tr}_l(f) = \mathrm{tr}_r(f^*) = \mathrm{tr}_l(f^{**})$ for any endomorphism f in $\mathscr{C}$. If

$$(1) \qquad\qquad \alpha \otimes \mathrm{id}_X = \mathrm{id}_X \otimes \alpha \quad \text{for all } \alpha \in \mathrm{End}_{\mathscr{C}}(\mathbb{1}) \text{ and } X \text{ in } \mathscr{C},$$

then tr_l, tr_r are $\otimes$-multiplicative; that is, $\mathrm{tr}_l(f \otimes g) = \mathrm{tr}_l(f)\,\mathrm{tr}_l(g)$ and $\mathrm{tr}_r(f \otimes g) = \mathrm{tr}_r(f)\,\mathrm{tr}_r(g)$ for all endomorphisms f, g in $\mathscr{C}$.

The *left* and the *right dimensions* of an object X of $\mathscr{C}$ are defined by $\dim_l(X) = \mathrm{tr}_l(\mathrm{id}_X)$ and $\dim_r(X) = \mathrm{tr}_r(\mathrm{id}_X)$. Isomorphic objects have the same dimensions, $\dim_l(X) = \dim_r(X^*) = \dim_l(X^{**})$, and $\dim_l(\mathbb{1}) = \dim_r(\mathbb{1}) = \mathrm{id}_{\mathbb{1}}$. If $\mathscr{C}$ satisfies (1), then left and right dimensions are $\otimes$-multiplicative: $\dim_l(X \otimes Y) = \dim_l(X)\dim_l(Y)$ and $\dim_r(X \otimes Y) = \dim_r(X)\dim_r(Y)$ for any X, Y in $\mathscr{C}$.

1D. *Penrose graphical calculus.* We represent morphisms in a category $\mathscr{C}$ by plane diagrams to be read from the bottom to the top. In a pivotal category $\mathscr{C}$, the diagrams are made of oriented arcs colored by objects of $\mathscr{C}$ and of boxes colored by morphisms of $\mathscr{C}$. The arcs connect the boxes and have no mutual intersections or self-intersections. The identity id_X of an object X of $\mathscr{C}$, a morphism $f : X \to Y$,

and the composition of two morphisms $f : X \to Y$ and $g : Y \to Z$ are represented respectively as

$$\mathrm{id}_X = \Big|_X \; , \qquad f = \boxed{f}\; , \qquad gf = \begin{array}{c} \downarrow Z \\ \boxed{g} \\ \downarrow Y. \\ \boxed{f} \\ \uparrow X \end{array}$$

The monoidal product of two morphisms $f : X \to Y$ and $g : U \to V$ is represented by juxtaposition:

$$f \otimes g = \boxed{f}\;\boxed{g}.$$

If an arc colored by X is oriented upwards, then the corresponding object in the source/target of morphisms is X^*. For example, id_{X^*} and a morphism $f : X^* \otimes Y \to U \otimes V^* \otimes W$ may be depicted as

$$\mathrm{id}_{X^*} = \Big|_X = \Big|_{X^*} \quad \text{and} \quad f = \boxed{\quad f \quad}.$$

The duality morphisms are depicted as follows:

$$\mathrm{ev}_X = \bigcap X\,, \quad \mathrm{coev}_X = \bigcup X\,, \quad \widetilde{\mathrm{ev}}_X = \bigcap X\,, \quad \widetilde{\mathrm{coev}}_X = \bigcup X\,.$$

The dual of a morphism $f : X \to Y$ and the traces of a morphism $g : X \to X$ can be depicted as follows:

$$f^* = {}_{Y}\!\left(\boxed{f}\right)^{X} = {}^{X}\!\left(\boxed{f}\right)_{Y} \quad \text{and} \quad \mathrm{tr}_l(g) = {}^{X}\!\left(\boxed{g}\right), \quad \mathrm{tr}_r(g) = \left(\boxed{g}\right)^{X}.$$

In a pivotal category, the morphisms represented by the diagrams are invariant under isotopies of the diagrams in the plane keeping fixed the bottom and top endpoints.

1E. *Spherical categories.* A *spherical category* is a pivotal category whose left and right traces are equal; i.e., $\mathrm{tr}_l(g) = \mathrm{tr}_r(g)$ for every endomorphism g of an object. Then $\mathrm{tr}_l(g)$ and $\mathrm{tr}_r(g)$ are denoted $\mathrm{tr}(g)$ and called the *trace of g*. Similarly, the left and right dimensions of an object X are denoted $\dim(X)$ and called the *dimension of X*.

Note that sphericity can be interpreted in graphical terms: it means that the morphisms represented by closed diagrams are invariant under isotopies of diagrams in the 2-sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$, i.e., are preserved under isotopies pushing arcs of the diagrams across ∞.

1F. *Additive categories.* Let $\Bbbk$ be a commutative ring. A $\Bbbk$-*additive category* is a category where Hom-sets are $\Bbbk$-modules, the composition of morphisms is $\Bbbk$-bilinear, and any finite family of objects has a direct sum. In particular, such a category has a zero object.

An object X of a $\Bbbk$-additive category $\mathscr{C}$ is *scalar* if the map $\Bbbk \to \mathrm{End}_{\mathscr{C}}(X)$, $\alpha \mapsto \alpha \, \mathrm{id}_X$ is bijective.

A $\Bbbk$-*additive monoidal category* is a monoidal category which is $\Bbbk$-additive in such a way that the monoidal product is $\Bbbk$-bilinear. Note that a $\Bbbk$-additive monoidal category whose unit object $\mathbb{1}$ is scalar satisfies (1) and so its traces tr_l, tr_r are $\Bbbk$-linear and $\otimes$-multiplicative.

1G. *Fusion categories.* A *fusion category* over a commutative ring $\Bbbk$ is a $\Bbbk$-additive rigid category $\mathscr{C}$ such that

(a) each object of $\mathscr{C}$ is a finite direct sum of scalar objects;

(b) for any nonisomorphic scalar objects i, j of $\mathscr{C}$, we have $\mathrm{Hom}_{\mathscr{C}}(i, j) = 0$;

(c) the set of isomorphism classes of scalar objects of $\mathscr{C}$ is finite;

(d) the unit object $\mathbb{1}$ is scalar.

Let $\mathscr{C}$ be a fusion category. The Hom spaces in $\mathscr{C}$ are free $\Bbbk$-modules of finite rank. We identify $\mathrm{End}_{\mathscr{C}}(\mathbb{1})$ with $\Bbbk$ via the canonical isomorphism. Given a scalar object i of $\mathscr{C}$, the i-*isotypical component* $X^{(i)}$ of an object X is the largest direct factor of X isomorphic to a direct sum of copies of i. The actual number of copies of i is

$$\nu_i(X) = \mathrm{rank}_{\Bbbk} \, \mathrm{Hom}_{\mathscr{C}}(i, X) = \mathrm{rank}_{\Bbbk} \, \mathrm{Hom}_{\mathscr{C}}(X, i).$$

An i-*decomposition* of X is an explicit direct sum decomposition of $X^{(i)}$ into copies of i, that is, a family $(p_\alpha : X \to i, q_\alpha : i \to X)_{\alpha \in A}$ of pairs of morphisms in $\mathscr{C}$ such that

(a) $p_\alpha \, q_\beta = \delta_{\alpha, \beta} \, \mathrm{id}_i$ for all $\alpha, \beta \in A$,

(b) the set A has $\nu_i(X)$ elements,

where $\delta_{\alpha, \beta}$ is the Kronecker symbol.

A *representative set of scalar objects* of $\mathscr{C}$ is a set I of scalar objects such that $\mathbb{1} \in I$ and every scalar object of $\mathscr{C}$ is isomorphic to exactly one element of I.

Note that if $\Bbbk$ is a field, a fusion category over $\Bbbk$ is abelian and semisimple. Recall that an abelian category is *semisimple* if its objects are direct sums of simple[1] objects.
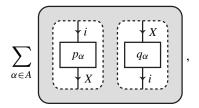
A pivotal fusion category is spherical (see Section 1E) if and only if the left and right dimension of any of its scalar objects coincide.

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[1]An object of an abelian category is *simple* if it is nonzero and has no other subobject than the zero object and itself.
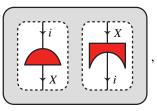
1H. *Graphical calculus in pivotal fusion categories.* Let $\mathscr{C}$ be a pivotal fusion category. Let X be an object of $\mathscr{C}$ and i be a scalar object of $\mathscr{C}$. Then the tensor

$$\sum_{\alpha \in A} p_\alpha \otimes_\Bbbk q_\alpha \in \operatorname{Hom}_{\mathscr{C}}(X, i) \otimes_\Bbbk \operatorname{Hom}_{\mathscr{C}}(i, X),$$
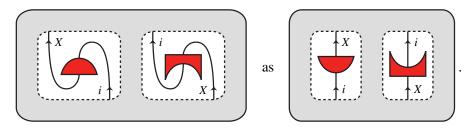
where $(p_\alpha, q_\alpha)_{\alpha \in A}$ is an i-decomposition of X, does not depend on the choice of the i-decomposition $(p_\alpha, q_\alpha)_{\alpha \in A}$ of X. Consequently, a sum of the type

where $(p_\alpha, q_\alpha)_{\alpha \in A}$ is an i-decomposition of an object X and the gray area does not involve α, represents a morphism in $\mathscr{C}$ which is independent of the choice of the i-decomposition. We depict it as

(2)

where the two curvilinear boxes should be shaded with the same color. If several such pairs of boxes appear in a picture, they must have different colors. We will also depict

 as .

As usual, the edges labeled with $i = \mathbb{1}$ may be erased and then (2) becomes

.

Note also that tensor products of objects may be depicted as bunches of strands. For example,

where the equality sign means that the pictures represent the same morphism of $\mathscr{C}$.

1I. *Braided and ribbon categories.* A *braiding* in a monoidal category $\mathscr{B}$ is a natural isomorphism $\tau = \{\tau_{X,Y} : X \otimes Y \to Y \otimes X\}_{X,Y \in \mathscr{B}}$ such that

$$\tau_{X,Y \otimes Z} = (\mathrm{id}_Y \otimes \tau_{X,Z})(\tau_{X,Y} \otimes \mathrm{id}_Z) \quad \text{and} \quad \tau_{X \otimes Y, Z} = (\tau_{X,Z} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes \tau_{Y,Z})$$

for all X, Y, Z objects of $\mathscr{C}$. These conditions imply that $\tau_{X,\mathbb{1}} = \tau_{\mathbb{1},X} = \mathrm{id}_X$.

A monoidal category endowed with a braiding is said to be *braided*. The braiding and its inverse are depicted as

$$\tau_{X,Y} = \;\;\raisebox{-0.5em}{\includegraphics{}}\;\; \quad \text{and} \quad \tau_{Y,X}^{-1} = \;\;\raisebox{-0.5em}{\includegraphics{}}\;\;.$$

Note that any braided category satisfies the condition (1) of Section 1C.

For any object X of a braided pivotal category $\mathscr{B}$, the morphism

$$\theta_X = \;\raisebox{-0.5em}{\includegraphics{}}\; = (\mathrm{id}_X \otimes \widetilde{\mathrm{ev}}_X)(\tau_{X,X} \otimes \mathrm{id}_{X^*})(\mathrm{id}_X \otimes \mathrm{coev}_X) : X \to X$$

is called the *twist*. The twist is natural in X and invertible, with inverse

$$\theta_X^{-1} = \;\raisebox{-0.5em}{\includegraphics{}}\; = (\mathrm{ev}_X \otimes \mathrm{id}_X)(\mathrm{id}_{X^*} \otimes \tau_{X,X}^{-1})(\widetilde{\mathrm{coev}}_X \otimes \mathrm{id}_X) : X \to X.$$

It satisfies $\theta_{X \otimes Y} = (\theta_X \otimes \theta_Y)\tau_{Y,X}\tau_{X,Y}$ for all objects X, Y of $\mathscr{B}$ and $\theta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$.

A *ribbon category* is a braided pivotal category $\mathscr{B}$ whose twist θ is self-dual; i.e., $(\theta_X)^* = \theta_{X^*}$ for any object X of $\mathscr{B}$. This is equivalent to the equality

$$\raisebox{-0.5em}{\includegraphics{}} = \raisebox{-0.5em}{\includegraphics{}}.$$

A ribbon category is spherical.

1J. *The center of a monoidal category.* Let $\mathscr{C}$ be a monoidal category. A *half braiding* of $\mathscr{C}$ is a pair (A, σ), where A is an object of $\mathscr{C}$ and

$$\sigma = \{\sigma_X : A \otimes X \to X \otimes A\}_{X \in \mathscr{C}}$$

is a natural isomorphism such that

$$(3) \qquad\qquad \sigma_{X \otimes Y} = (\mathrm{id}_X \otimes \sigma_Y)(\sigma_X \otimes \mathrm{id}_Y)$$

for all X, Y objects of $\mathscr{C}$. This implies that $\sigma_{\mathbb{1}} = \mathrm{id}_A$.

The *center of* $\mathscr{C}$ is the braided category $\mathscr{Z}(\mathscr{C})$ defined as follows. The objects of $\mathscr{Z}(\mathscr{C})$ are half braidings of $\mathscr{C}$. A morphism $(A, \sigma) \to (A', \sigma')$ in $\mathscr{Z}(\mathscr{C})$ is a morphism $f : A \to A'$ in $\mathscr{C}$ such that $(\mathrm{id}_X \otimes f)\sigma_X = \sigma'_X(f \otimes \mathrm{id}_X)$ for any object X of $\mathscr{C}$. The unit object of $\mathscr{Z}(\mathscr{C})$ is $\mathbb{1}_{\mathscr{Z}(\mathscr{C})} = (\mathbb{1}, \{\mathrm{id}_X\}_{X \in \mathscr{C}})$ and the monoidal product is

$$(A, \sigma) \otimes (B, \rho) = \big(A \otimes B, (\sigma \otimes \mathrm{id}_B)(\mathrm{id}_A \otimes \rho)\big).$$

The braiding τ in $\mathscr{Z}(\mathscr{C})$ is defined by

$$\tau_{(A, \sigma), (B, \rho)} = \sigma_B : (A, \sigma) \otimes (B, \rho) \to (B, \rho) \otimes (A, \sigma).$$

There is a *forgetful functor* $\mathscr{U} : \mathscr{Z}(\mathscr{C}) \to \mathscr{C}$ assigning to every half braiding (A, σ) the underlying object A and acting in the obvious way on the morphisms. This is a strict monoidal functor.

If $\mathscr{C}$ satisfies (1), then $\mathrm{End}_{\mathscr{Z}(\mathscr{C})}(\mathbb{1}_{\mathscr{Z}(\mathscr{C})}) = \mathrm{End}_{\mathscr{C}}(\mathbb{1})$.

If $\mathscr{C}$ is rigid, then so is $\mathscr{Z}(\mathscr{C})$. If $\mathscr{C}$ is pivotal, then so is $\mathscr{Z}(\mathscr{C})$ with $(A, \sigma)^* = (A^*, \sigma^{\natural})$, where

$$\sigma_X^{\natural} = \quad \boxed{\sigma_{X^*}} \quad : A^* \otimes X \to X \otimes A^*,$$

and $\mathrm{ev}_{(A,\sigma)} = \mathrm{ev}_A$, $\mathrm{coev}_{(A,\sigma)} = \mathrm{coev}_A$, $\widetilde{\mathrm{ev}}_{(A,\sigma)} = \widetilde{\mathrm{ev}}_A$, $\widetilde{\mathrm{coev}}_{(A,\sigma)} = \widetilde{\mathrm{coev}}_A$. In that case the forgetful functor $\mathscr{U}$ preserves (left and right) traces of morphisms and dimensions of objects.

If $\mathscr{C}$ is a $\Bbbk$-additive monoidal category, then so is $\mathscr{Z}(\mathscr{C})$ and the forgetful functor is $\Bbbk$-linear. If $\mathscr{C}$ is an abelian rigid category, then so is $\mathscr{Z}(\mathscr{C})$, and the forgetful functor is exact.

If $\mathscr{C}$ is a fusion category over the ring $\Bbbk$, then $\mathscr{Z}(\mathscr{C})$ is braided $\Bbbk$-additive rigid category whose monoidal unit is scalar. If in addition $\Bbbk$ is field, then $\mathscr{C}$ is abelian, and so is $\mathscr{Z}(\mathscr{C})$.

2. Main results

In this section, we state our main results concerning the center of a pivotal fusion category. They are proved in Section 4. Let $\mathscr{C}$ be a pivotal fusion category over a commutative ring $\Bbbk$ and I be a representative set of scalar objects of $\mathscr{C}$. Recall from Section 1J that the center $\mathscr{Z}(\mathscr{C})$ of $\mathscr{C}$ is a braided $\Bbbk$-additive pivotal category whose monoidal unit is scalar.

The coend of a rigid braided category is, if it exists, a Hopf algebra in the category which coacts universally on the objects (see Section 3C for details). The center

$\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ has a coend (C, σ), where

$$C = \bigoplus_{i,j \in I} i^* \otimes j^* \otimes i \otimes j$$

and the half braiding $\sigma = \{\sigma_Y\}_{Y \in \mathcal{C}}$ is given by

$$(4) \qquad \sigma_Y = \sum_{i,j,k,\ell,n \in I} \quad \text{} \quad : C \otimes Y \to Y \otimes C.$$

The universal coaction $\delta = \{\delta_{M,\gamma}\}_{(M,\gamma) \in \mathcal{Z}(\mathcal{C})}$ of the coend (C, σ) is

$$(5) \qquad \delta_{(M,\gamma)} = \sum_{i,j \in I} \quad \text{} \quad : (M, \gamma) \to (M, \gamma) \otimes (C, \sigma).$$

The structural morphisms and the canonical pairing of the Hopf algebra (C, σ) can be depicted as follows:

(a) The coproduct $\Delta : C \to C \otimes C$:

$$\Delta = \sum_{i,j,k,\ell,n \in I} \quad \text{}$$

(b) The product $m : C \otimes C \to C$:

$$m = \sum_{i,j,k,\ell,n,a \in I} \quad \text{}$$

(c) The counit $\varepsilon : C \to \mathbb{1}$: $\varepsilon = \sum_{j \in I} \quad \text{}$

(d) The unit $u : \mathbb{1} \to C$: $u = \sum_{i \in I} \quad \text{}$

(e) The antipode $S : C \to C$:

$$S = \sum_{i,j,k,\ell,n \in I}$$

(f) The canonical pairing $\omega : C \otimes C \to \mathbb{1}$:

$$\omega = \sum_{i,j,k,\ell \in I}$$

In the pictures, the dotted lines represent $\mathrm{id}_{\mathbb{1}}$ and serve to indicate which direct factor of C is concerned. Moreover,

(6) $$\Lambda = \sum_{j \in I} \dim_r(j) \quad \bigcup_j \quad : (\mathbb{1}, \mathrm{id}) \to (C, \sigma)$$

is an integral of the Hopf algebra (C, σ), which is invariant under the antipode.

By a modular category we mean a braided pivotal category admitting a coend, and whose canonical pairing is nondegenerate (see Section 3E for details). The dimension of such a category is the dimension of its coend (see Section 3D).

Theorem 2.1. *The center $\mathfrak{Z}(\mathscr{C})$ of $\mathscr{C}$ is modular and has dimension $\dim(\mathscr{C})^2$.*

The forgetful functor $\mathfrak{U} : \mathfrak{Z}(\mathscr{C}) \to \mathscr{C}$ has a left adjoint $\mathfrak{F} : \mathscr{C} \to \mathfrak{Z}(\mathscr{C})$. For an object X of $\mathscr{C}$,

$$\mathfrak{F}(X) = \big(Z(X), \varsigma_X = \{\varsigma_{X,Y}\}_{Y \in \mathscr{C}}\big) \quad \text{where} \quad Z(X) = \bigoplus_{i \in I} i^* \otimes X \otimes i \quad \text{and}$$

$$\varsigma_{X,Y} = \sum_{i,j \in I} \qquad : Z(X) \otimes Y \to Y \otimes Z(X).$$

For a morphism f in $\mathscr{C}$,

$$\mathfrak{F}(f) = \sum_{i \in I} \mathrm{id}_{i^*} \otimes f \otimes \mathrm{id}_i.$$

By a *free half braiding* we mean a half braiding of the form $\mathfrak{F}(X)$ for some object X of $\mathscr{C}$.

Theorem 2.2. *The dimension of $\mathscr{C}$ is invertible in $\Bbbk$ if and only if every half braiding is a retract of a free half braiding.*

From Section 1J, if $\Bbbk$ is a field, then $\mathcal{Z}(\mathscr{C})$ is abelian.

Corollary 2.3. *Assume $\Bbbk$ is a field.*

(a) *The center $\mathcal{Z}(\mathscr{C})$ is semisimple (as an abelian category) if and only if $\dim(\mathscr{C}) \neq 0$.*

(b) *Assume $\Bbbk$ is algebraically closed. Then $\mathcal{Z}(\mathscr{C})$ is a fusion category if and only if $\dim(\mathscr{C}) \neq 0$.*

Since the center of a spherical fusion category is ribbon (see, for example, [Turaev and Virelizier 2010, Lemma 10.1]), we recover Müger's theorem:

Corollary 2.4 [Müger 2003, Theorem 1.2]. *If $\mathscr{C}$ is a spherical fusion category over an algebraically closed field and $\dim(\mathscr{C}) \neq 0$, then $\mathcal{Z}(\mathscr{C})$ is a modular ribbon fusion category* (i.e., $\mathcal{Z}(\mathscr{C})$ is modular in the sense of [Turaev 1994]).

Note that by [Etingof et al. 2005], the hypothesis $\dim(\mathscr{C}) \neq 0$ of the previous corollary is automatically fulfilled on a field of characteristic zero.

Example 2.5. Let G be a finite group and $\Bbbk$ be a commutative ring. The category $\mathscr{C}_{G,\Bbbk}$ of G-graded free $\Bbbk$-modules of finite rank is a spherical fusion category. The dimension of $\mathscr{C}_{G,\Bbbk}$ is $\dim(\mathscr{C}_{G,\Bbbk}) = |G|1_{\Bbbk}$, where $|G|$ is the order of G. By Theorem 2.1, the center $\mathcal{Z}(\mathscr{C}_{G,\Bbbk})$ of $\mathscr{C}_{G,\Bbbk}$ is modular of dimension $|G|^2 1_{\Bbbk}$. When $|G|$ is not invertible in $\Bbbk$, by Theorem 2.2, there exist half braidings of $\mathscr{C}_{G,\Bbbk}$ which are not retracts of any free half braiding. If particular, if $\Bbbk$ is a field of characteristic p which divides $|G|$, then $\mathcal{Z}(\mathscr{C}_{G,\Bbbk})$ is not semisimple.

3. Modular categories

In this section, we clarify some notions used in the previous section. More precisely, in Section 3A, we recall the definition of a Hopf algebra in a braided category and provide a criterion for the nondegeneracy of a Hopf algebra pairing. In Section 3B, we recall the definition of a coend. In Section 3C, we describe the Hopf algebra structure of the coend of a braided rigid category. Sections 3D and 3E are devoted to the definition of respectively the dimension and the modularity of a braided category admitting a coend.

3A. *Hopf algebras, pairings, and integrals.* Let $\mathscr{B}$ be a braided category, with braiding τ. Recall that a *bialgebra in $\mathscr{B}$* is an object A of $\mathscr{B}$ endowed with four morphisms $m : A \otimes A \to A$ (the product), $u : \mathbb{1} \to A$ (the unit), $\Delta : A \to A \otimes A$ (the coproduct), and $\varepsilon : A \to \mathbb{1}$ (the counit) such that

$$m(m \otimes \mathrm{id}_A) = m(\mathrm{id}_A \otimes m), \quad m(\mathrm{id}_A \otimes u) = \mathrm{id}_A = m(u \otimes \mathrm{id}_A),$$

$$(\Delta \otimes \mathrm{id}_A)\Delta = (\mathrm{id}_A \otimes \Delta)\Delta, \quad (\mathrm{id}_A \otimes \varepsilon)\Delta = \mathrm{id}_A = (\varepsilon \otimes \mathrm{id}_A)\Delta,$$

$$\Delta m = (m \otimes m)(\mathrm{id}_A \otimes \tau_{A,A} \otimes \mathrm{id}_A)(\Delta \otimes \Delta),$$

$$\Delta u = u \otimes u, \quad \varepsilon m = \varepsilon \otimes \varepsilon, \quad \varepsilon u = \mathrm{id}_{\mathbb{1}}.$$

An *antipode* for a bialgebra A in $\mathscr{B}$ is a morphism $S : A \to A$ in $\mathscr{B}$ such that

$$m(S \otimes \mathrm{id}_A)\Delta = u\varepsilon = m(\mathrm{id}_A \otimes S)\Delta.$$

If it exists, an antipode is unique. A *Hopf algebra in* $\mathscr{B}$ is a bialgebra in $\mathscr{B}$ which admits an invertible antipode.

Let A be a Hopf algebra in $\mathscr{B}$. A *Hopf pairing* for A is a morphism $\omega : A \otimes A \to \mathbb{1}$ such that

$$\omega(m \otimes \mathrm{id}_A) = \omega(\mathrm{id}_A \otimes \omega \otimes \mathrm{id}_A)(\mathrm{id}_{A^{\otimes 2}} \otimes \Delta), \quad \omega(u \otimes \mathrm{id}_A) = \varepsilon,$$
$$\omega(\mathrm{id}_A \otimes m) = \omega(\mathrm{id}_A \otimes \omega \otimes \mathrm{id}_A)(\Delta \otimes \mathrm{id}_{A^{\otimes 2}}), \quad \omega(\mathrm{id}_A \otimes u) = \varepsilon.$$

These axioms imply that $\omega(S \otimes \mathrm{id}_A) = \omega(\mathrm{id}_A \otimes S)$.

A Hopf pairing ω for A is *nondegenerate* if there exists a morphism $\Omega : \mathbb{1} \to A \otimes A$ in $\mathscr{B}$ such that

$$(\omega \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \Omega) = \mathrm{id}_A = (\mathrm{id}_A \otimes \omega)(\Omega \otimes \mathrm{id}_A).$$

If such is the case, the morphism Ω is unique and called the *inverse* of ω.

A *left* (respectively, *right*) *integral* for A is a morphism $\Lambda : \mathbb{1} \to A$ such that

$$m(\mathrm{id}_A \otimes \Lambda) = \Lambda\,\varepsilon \quad \text{(respectively, } m(\Lambda \otimes \mathrm{id}_A) = \Lambda\,\varepsilon\text{)}.$$

A *left* (respectively, *right*) *cointegral* for A is a morphism $\lambda : A \to \mathbb{1}$ such that

$$(\mathrm{id}_A \otimes \lambda)\Delta = u\,\lambda \quad \text{(respectively, } (\lambda \otimes \mathrm{id}_A)\Delta = u\,\lambda\text{)}.$$

A (co)integral is *two-sided* if it is both a left and a right (co)integral.

If Λ is a left (respectively, right) integral for A, then $S\Lambda$ is a right (respectively, left) integral for A. If λ is a left (respectively, right) cointegral for A, then λS is a right (respectively, left) cointegral for A.

Let ω be a Hopf pairing for A and $\Lambda : \mathbb{1} \to A$ be a morphism in $\mathscr{B}$. Assume ω is nondegenerate. Then Λ is a left integral for A if and only if $\lambda = \omega(\mathrm{id}_A \otimes \Lambda)$ is a right cointegral for A, and Λ is a right integral for A if and only if $\lambda = \omega(\Lambda \otimes \mathrm{id}_A)$ is a left cointegral for A.

Lemma 3.1. *Let ω be a Hopf pairing for a Hopf algebra A in a braided category $\mathscr{B}$. Assume there exist morphisms $\Lambda, \Lambda' : \mathbb{1} \to A$ in $\mathscr{B}$ such that*

(a) *$\omega(\Lambda \otimes \mathrm{id}_A)$ and $\omega(\mathrm{id}_A \otimes \Lambda')$ are left cointegrals for A;*

(b) *$\omega(\Lambda \otimes \Lambda')$ is invertible in $\mathrm{End}_{\mathscr{B}}(\mathbb{1})$.*

Then ω is nondegenerate, with inverse

$$\Omega = \omega(\Lambda \otimes \Lambda')^{-1}\,(S \otimes \mathrm{id}_A \otimes \omega)(\mathrm{id}_A \otimes \Delta\Lambda \otimes \mathrm{id}_A)\Delta\Lambda',$$

and Λ and Λ' are right integrals for A.

Proof. Set $e = (S \otimes \mathrm{id}_A \otimes \omega)(\mathrm{id}_A \otimes \Delta\Lambda \otimes \mathrm{id}_A)\Delta\Lambda' : \mathbb{1} \to A \otimes A$. Let us depict the product m, coproduct Δ, antipode S of A, and the morphisms ω, Λ, Λ' as follows:

$$m = \curlywedge, \quad \Delta = \curlyvee, \quad S = \phi, \quad \omega = \frown, \quad \Lambda = \square, \quad \Lambda' = \diamond.$$

Then $(\mathrm{id}_A \otimes \omega)(e \otimes \mathrm{id}_A) = \omega(\Lambda \otimes \Lambda')\,\mathrm{id}_A$ since

We use the product/coproduct axioms of a Hopf pairing in the first and fourth equalities, the unit axiom and the fact that $\omega(\Lambda \otimes \mathrm{id}_A)$ is a left cointegral in the second equality, the compatibility of m and Δ and the axiom of the antipode in the third equality, and finally the fact that $\omega(\mathrm{id}_A \otimes \Lambda')$ is a left cointegral and the unit/counit axiom of a Hopf pairing in the last equality. Similarly one shows that $(\omega \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes e) = \omega(\Lambda \otimes \Lambda')\,\mathrm{id}_A$. Thus $\Omega = \omega(\Lambda \otimes \Lambda')^{-1} e$ is an inverse of ω.

Finally, since ω is nondegenerate and $\omega(\Lambda \otimes A)$ and $\omega(A \otimes \Lambda')$ are left cointegrals, we conclude that Λ and Λ' are right integrals. $\square$

3B. *Coends.* Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A *dinatural transformation* from a functor $F : \mathscr{D}^{\mathrm{op}} \times \mathscr{D} \to \mathscr{C}$ to an object A of $\mathscr{C}$ is a family of morphisms in $\mathscr{C}$

$$d = \{d_Y : F(Y, Y) \to A\}_{Y \in \mathscr{D}}$$

such that for every morphism $f : X \to Y$ in $\mathscr{D}$, we have

$$d_X F(f, \mathrm{id}_X) = d_Y F(\mathrm{id}_Y, f) : F(Y, X) \to A.$$

The *composition* of such a d with a morphism $\phi : A \to B$ in $\mathscr{C}$ is the dinatural transformation $\phi \circ d = \{\phi \circ d_X : F(Y, Y) \to B\}_{Y \in \mathscr{D}}$ from F to B. A *coend* of F is a pair (C, ρ) consisting in an object C of $\mathscr{C}$ and a dinatural transformation ρ from F to C satisfying the following universality condition: every dinatural transformation d from F to an object of $\mathscr{C}$ is the composition of ρ with a morphism in $\mathscr{C}$ uniquely determined by d. If F has a coend (C, ρ), then it is unique (up to unique isomorphism). One writes $C = \int^{Y \in \mathscr{D}} F(Y, Y)$. For more on coends, see [Mac Lane 1998].

Remark 3.2. Let $F : \mathscr{D}^{\mathrm{op}} \times \mathscr{D} \to \mathscr{C}$ be a $\Bbbk$-linear functor, where $\mathscr{C}$ is a $\Bbbk$-additive category and $\mathscr{D}$ is a fusion category (over $\Bbbk$). Then F has a coend. More precisely, pick a (finite) representative set I of simple objects of $\mathscr{D}$ and set $C = \bigoplus_{i \in I} F(i, i)$. Let $\rho = \{\rho_Y : F(Y, Y) \to C\}_{Y \in \mathscr{D}}$ be defined by $\rho_Y = \sum_\alpha F(q_Y^\alpha, p_Y^\alpha)$, where $(p_Y^\alpha, q_Y^\alpha)_\alpha$ is any I-partition of Y. Then (C, ρ) is a coend of F and each dinatural transformation d from F to any object A of $\mathscr{C}$ is the composition of ρ with $\bigoplus_{i \in I} d_i : C \to A$.

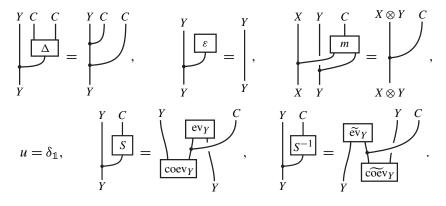**3C.** *The coend of a braided rigid category.* Let $\mathcal{B}$ be braided rigid category. The coend

$$C = \int^{Y \in \mathcal{B}} {}^{\vee}Y \otimes Y,$$

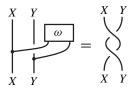if it exists, is called the *coend of $\mathcal{B}$*.

Assume $\mathcal{B}$ has a coend C and denote by $i_Y : {}^{\vee}Y \otimes Y \to C$ the corresponding universal dinatural transformation. The *universal coaction* of C on the objects of $\mathcal{B}$ is the natural transformation δ defined by

(7) $\quad \delta_Y = (\mathrm{id}_Y \otimes i_Y)(\mathrm{coev}_Y \otimes \mathrm{id}_Y) : Y \to Y \otimes C, \quad \text{depicted as } \delta_Y = \vcenter{\hbox{\includegraphics{}}}.$

According to [Majid 1995], C is a Hopf algebra in $\mathcal{B}$. Its coproduct Δ, product m, counit ε, unit u, and antipode S with inverse S^{-1} are characterized by the following equalities, where $X, Y \in \mathcal{B}$:

Furthermore, the morphism $\omega : C \otimes C \to \mathbb{1}$ defined by

is a Hopf pairing for C, called the *canonical pairing*. Moreover this pairing satisfies the following self-duality condition: $\omega \tau_{C,C}(S \otimes S) = \omega$.

3D. *The dimension of a braided pivotal category.* Let $\mathcal{B}$ be a braided pivotal category admitting a coend C.

Lemma 3.3. *The left and right dimensions of C coincide.*

Proof. Let $\upsilon = \{\upsilon_X\}_{X \in \mathcal{B}}$ be the natural transformation defined by

$$\upsilon_X = X \begin{smallmatrix} \circlearrowright \\ \circlearrowleft \end{smallmatrix} : X \to X.$$

Then υ is natural monoidal isomorphism; that is, $\upsilon_{X \otimes Y} = \upsilon_X \otimes \upsilon_Y$ and $\upsilon_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$, which implies that $\upsilon_{X^*} = \upsilon_{X^*}^{-1}$. The full subcategory $\mathcal{B}_0$ of $\mathcal{B}$ made of the objects X of $\mathcal{B}$ satisfying $\tau_X = \mathrm{id}_X$ is a ribbon category. Let us prove that the coend C of $\mathcal{B}$ belongs to $\mathcal{B}_0$. Denote by $i = \{i_X : X^* \otimes X \to C\}_{X \in \mathcal{B}}$ the universal dinatural transformation associated with C. For any object X of $\mathcal{C}$, by naturality and monoidality of υ and dinaturality of i, the following holds:

$$\upsilon_C i_X = i_X \upsilon_{(X^* \otimes X)} = i_X (\upsilon_{X^*} \otimes \upsilon_X) = i_X (\upsilon_X^* \upsilon_{X^*} \otimes \mathrm{id}_X) = i_X.$$

So $\upsilon_C = \mathrm{id}_C$; that is, C belongs to $\mathcal{B}_0$. Hence the left and right dimensions of C coincide, since $\mathcal{B}_0$ is a ribbon category. $\square$

We define the *dimension of $\mathcal{B}$* as $\dim(\mathcal{B}) = \dim_l(C) = \dim_r(C)$.

This definition agrees with the standard definition of the dimension of a pivotal fusion category. Indeed, any pivotal fusion category $\mathcal{C}$ (over the ring $\Bbbk$) admits a coend $C = \bigoplus_{i \in I} i^* \otimes i$, where I is a (finite) representative set of scalar objects of $\mathcal{C}$, and so

$$\dim_l(C) = \dim_r(C) = \sum_{i \in I} \dim_l(i^*) \dim_l(i) = \sum_{i \in I} \dim_r(i) \dim_l(i).$$

3E. *Modular categories.* By a *modular category*, we mean a braided rigid category which admits a coend whose canonical pairing is nondegenerate. Note that when $\mathcal{B}$ is ribbon, this definition coincides with that of a *2-modular category* given in [Lyubashenko 1995].

Remark 3.4. Let $\mathcal{B}$ be a braided pivotal fusion category over $\Bbbk$. Let I be a representative set of the scalar objects of $\mathcal{B}$. Recall that $C = \bigoplus_{i \in I} i^* \otimes i$ is the coend of $\mathcal{B}$. For $i, j \in I$, set

$$S_{i,j} = (\mathrm{ev}_i \otimes \widetilde{\mathrm{ev}}_j)(\mathrm{id}_{i^*} \otimes \tau_{j,i} \tau_{i,j} \otimes \mathrm{id}_{j^*})(\widetilde{\mathrm{coev}}_i \otimes \mathrm{coev}_j) \in \Bbbk.$$

The matrix $S = [S_{i,j}]_{i,j \in I}$, called the *S-matrix* of $\mathcal{B}$, is invertible if and only if the canonical pairing of C is nondegenerate. In particular a modular category in the sense of [Turaev 1994] is a ribbon fusion category which is modular in the above sense.

4. Proofs

The statements of Section 2 derive directly from the theory of Hopf monads, introduced in [Bruguières and Virelizier 2007] and developed in [Bruguières and

Virelizier 2012; Bruguières et al. 2011]. Hopf monads generalize Hopf algebras in the setting of general monoidal categories. In Section 4A, we recall some basic definitions concerning Hopf monads. In Section 4B, we give a Hopf monadic description of the center $\mathcal{Z}(\mathscr{C})$ of a fusion category $\mathscr{C}$, from which is derived the explicit description of the coend of $\mathcal{Z}(\mathscr{C})$. In Section 4C, we prove a "handleslide" property for pivotal fusion categories. In Section 4D, we use the explicit description of the coend of $\mathcal{Z}(\mathscr{C})$ to prove Theorem 2.1 and prove that the morphism Λ of (6) is an integral invariant under the antipode. Sections 4E and 4F are devoted to the proofs of Theorem 2.2 and Corollary 2.3, respectively.

4A. *Hopf monads and their modules.*

Let $\mathscr{C}$ be a category. A *monad* on $\mathscr{C}$ is a monoid in the category of endofunctors of $\mathscr{C}$, that is, a triple (T, μ, η) consisting of a functor $T : \mathscr{C} \to \mathscr{C}$ and two natural transformations

$$\mu = \{\mu_X : T^2(X) \to T(X)\}_{X \in \mathscr{C}} \quad \text{and} \quad \eta = \{\eta_X : X \to T(X)\}_{X \in \mathscr{C}},$$

called the *product* and the *unit* of T, such that, for any object X of $\mathscr{C}$,

$$\mu_X T(\mu_X) = \mu_X \mu_{T(X)} \quad \text{and} \quad \mu_X \eta_{T(X)} = \mathrm{id}_{T(X)} = \mu_X T(\eta_X).$$

Given a monad $T = (T, \mu, \eta)$ on $\mathscr{C}$, a T-*module* in $\mathscr{C}$ is a pair (M, r) where M is an object of $\mathscr{C}$ and $r : T(M) \to M$ is a morphism in $\mathscr{C}$ such that $rT(r) = r\mu_M$ and $r\eta_M = \mathrm{id}_M$. A morphism from a T-module (M, r) to a T-module (N, s) is a morphism $f : M \to N$ in $\mathscr{C}$ such that $fr = sT(f)$. This defines the *category* $\mathscr{C}^T$ *of* T-*modules in* $\mathscr{C}$ with composition induced by that in $\mathscr{C}$. We define a forgetful functor $U_T : \mathscr{C}^T \to \mathscr{C}$ by $U_T(M, r) = M$ and $U_T(f) = f$. The forgetful functor U_T has a left adjoint $F_T : \mathscr{C} \to \mathscr{C}^T$, called the *free module functor*, defined by $F_T(X) = (T(X), \mu_X)$ and $F_T(f) = T(f)$. Note that if $\mathscr{C}$ is $\Bbbk$-additive and T is $\Bbbk$-linear (that is, T induces $\Bbbk$-linear maps on Hom spaces), then the category $\mathscr{C}^T$ is $\Bbbk$-additive and the functors U_T and F_T are $\Bbbk$-linear.

Let $\mathscr{C}$ be a monoidal category. A *bimonad* on $\mathscr{C}$ is a monoid in the category of comonoidal endofunctors of $\mathscr{C}$. In other words, a bimonad on $\mathscr{C}$ is a monad (T, μ, η) on $\mathscr{C}$ such that the functor $T : \mathscr{C} \to \mathscr{C}$ and the natural transformations μ and η are comonoidal; that is, T comes equipped with a natural transformation $T_2 = \{T_2(X, Y) : T(X \otimes Y) \to T(X) \otimes T(Y)\}_{X,Y \in \mathscr{C}}$ and a morphism $T_0 : T(\mathbb{1}) \to \mathbb{1}$ such that

$$\big(\mathrm{id}_{T(X)} \otimes T_2(Y, Z)\big) T_2(X, Y \otimes Z) = \big(T_2(X, Y) \otimes \mathrm{id}_{T(Z)}\big) T_2(X \otimes Y, Z);$$

$$(\mathrm{id}_{T(X)} \otimes T_0) T_2(X, \mathbb{1}) = \mathrm{id}_{T(X)} = (T_0 \otimes \mathrm{id}_{T(X)}) T_2(\mathbb{1}, X);$$

$$T_2(X, Y) \mu_{X \otimes Y} = (\mu_X \otimes \mu_Y) T_2(T(X), T(Y)) T(T_2(X, Y));$$

$$T_2(X, Y) \eta_{X \otimes Y} = \eta_X \otimes \eta_Y.$$

For any bimonad T on $\mathscr{C}$, the category of T-modules $\mathscr{C}^T$ has a monoidal structure with unit object $(\mathbb{1}, T_0)$ and with tensor product

$$(M, r) \otimes (N, s) = \big(M \otimes N, (r \otimes s) \, T_2(M, N)\big).$$

Note that the forgetful functor $U_T : \mathscr{C}^T \to \mathscr{C}$ is strict monoidal.

Given a bimonad (T, μ, η) on $\mathscr{C}$ and objects $X, Y \in \mathscr{C}$, one defines the *left fusion operator*

$$H^l_{X,Y} = (T(X) \otimes \mu_Y) T_2(X, T(Y)) : T(X \otimes T(Y)) \to T(X) \otimes T(Y)$$

and the *right fusion operator*

$$H^r_{X,Y} = (\mu_X \otimes T(Y)) T_2(T(X), Y) : T(T(X) \otimes Y) \to T(X) \otimes T(Y).$$

A *Hopf monad* on $\mathscr{C}$ is a bimonad on $\mathscr{C}$ whose left and right fusion operators are isomorphisms for all objects X, Y of $\mathscr{C}$. When $\mathscr{C}$ is a rigid category, a bimonad T on $\mathscr{C}$ is a Hopf monad if and only if the category $\mathscr{C}^T$ is rigid. The structure of a rigid category in $\mathscr{C}^T$ can then be encoded in terms of natural transformations

$$s^l = \{s^l_X : T(^\vee T(X)) \to {}^\vee X\}_{X \in \mathscr{C}} \quad \text{and} \quad s^r = \{s^r_X : T(T(X)^\vee) \to X^\vee\}_{X \in \mathscr{C}},$$

called the *left and right antipodes*. They are computed from the fusion operators:

$$s^l_X = \big(T_0 T(\mathrm{ev}_{T(X)})(H^l_{^\vee T(X),X})^{-1} \otimes {}^\vee \eta_X\big)\big(\mathrm{id}_{T(^\vee T(X))} \otimes \mathrm{coev}_{T(X)}\big);$$
$$s^r_X = \big(\eta^\vee_X \otimes T_0 T(\tilde{\mathrm{e}}\mathrm{v}_{T(X)})(H^r_{X,T(X)^\vee})^{-1}\big)\big(\widetilde{\mathrm{coev}}_{T(X)} \otimes \mathrm{id}_{T(T(X)^\vee)}\big).$$

The left and right duals of any T-module (M, r) are then defined by

$$^\vee(M, r) = \big(^\vee M, s^l_M T(^\vee r)\big) \quad \text{and} \quad (M, r)^\vee = (M^\vee, s^r_M T(r^\vee)).$$

A *quasitriangular Hopf monad* on $\mathscr{C}$ is a Hopf monad T on $\mathscr{C}$ equipped with an R-matrix, that is, a natural transformation

$$R = \{R_{X,Y} : X \otimes Y \to T(Y) \otimes T(X)\}_{X,Y \in \mathscr{C}}$$

satisfying appropriate axioms which ensure that the natural transformation $\tau = \{\tau_{(M,r),(N,s)}\}_{(M,r),(N,s) \in \mathscr{C}^T}$ defined by

$$\tau_{(M,r),(N,s)} = (s \otimes r) R_{M,N} : (M, r) \otimes (N, s) \to (N, s) \otimes (M, r)$$

form a braiding in the category $\mathscr{C}^T$ of T-modules.

4B. *The coend of the center of a fusion category.* Let $\mathscr{C}$ be a pivotal fusion category (over the ring $\Bbbk$), with a representative set of scalar objects I. For each object X of $\mathscr{C}$, by Remark 3.2, the $\Bbbk$-linear functor $\mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$, defined by $(U, V) \mapsto U^* \otimes X \otimes V$, has a coend

$$Z(X) = \bigoplus_{i \in I} i^* \otimes X \otimes i,$$

with dinatural transformation $\rho_X = \{\rho_{X,Y}\}_{Y \in \mathscr{C}}$ given by

$$\rho_{X,Y} = \sum_{i \in I} \quad \vcenter{\hbox{\includegraphics{rho.png}}} \quad : Y^* \otimes X \otimes Y \to Z(X).$$

The correspondence $X \mapsto Z(X)$ extends to a functor $Z : \mathscr{C} \to \mathscr{C}$. By Theorem 6.4 and Section 9.2 of [Bruguières and Virelizier 2012], Z is a quasitriangular Hopf monad on $\mathscr{C}$, with structural morphisms as follows (the dotted lines represent $\mathrm{id}_{\mathbb{1}}$):

$$Z_2(X, Y) = \sum_{i \in I} \quad \vcenter{\hbox{\includegraphics{z2.png}}} \quad : Z(X \otimes Y) \to Z(X) \otimes Z(Y),$$

$$Z_0 = \sum_{i \in I} \quad \vcenter{\hbox{\includegraphics{z0.png}}} \quad i : Z(\mathbb{1}) \to \mathbb{1},$$

$$\mu_X = \sum_{i,j,k \in I} \quad \vcenter{\hbox{\includegraphics{mu.png}}} \quad : Z^2(X) \to Z(X),$$

$$\eta_X = \quad \vcenter{\hbox{\includegraphics{eta.png}}} \quad : X \to X = \mathbb{1}^* \otimes X \otimes \mathbb{1} \hookrightarrow Z(X),$$

$$s_X^l = s_X^r = \sum_{i,j \in I} \quad \vcenter{\hbox{\includegraphics{s.png}}} \quad : Z(Z(X)^*) \to X^*,$$

$$R_{X,Y} = \sum_{i \in I} \quad \vcenter{\hbox{\includegraphics{r.png}}} \quad : X \otimes Y \to Z(Y) \otimes Z(X).$$

In particular, the category $\mathscr{C}^Z$ of Z-modules is a braided pivotal category. By [Bruguières and Virelizier 2012, Theorem 6.5], the functor

$$(8) \qquad \Phi : \begin{cases} \mathscr{C}^Z & \to & \mathscr{Z}(\mathscr{C}) \\ (M, r) & \mapsto & (M, \sigma) \\ f & \mapsto & f \end{cases} \qquad \text{where} \quad \sigma_Y = \sum_{i \in I} \quad \vcenter{\hbox{\includegraphics{sigma.png}}}$$

is an isomorphism of braided pivotal categories. Note that this isomorphism is a "fusion" version of the braided isomorphism $\mathscr{Z}(\mathrm{mod}_H) \simeq \mathrm{mod}_{D(H)}$ between the center of the category of modules over a finite-dimensional Hopf algebra H and the category of modules over the Drinfeld double $D(H)$ of H. Now by [Bruguières and Virelizier 2012, Section 6.3], the coend of $\mathscr{C}^Z$ is (C, α), where

$$C = \bigoplus_{i,j \in I} i^* \otimes j^* \otimes i \otimes j \quad \text{and} \quad \alpha = \sum_{i,j,k,l,n \in I} \quad ,$$

with universal dinatural transformation $\iota = \{\iota_{(M,r)}\}_{(M,r) \in \mathscr{C}^Z}$ given by

$$\iota_{(M,r)} = \sum_{i,j \in I} \qquad : (M,r)^* \otimes (M,r) \to (C, \alpha).$$

Thus $(C, \sigma) = \Phi(C, \alpha)$ is the coend of $\mathscr{Z}(\mathscr{C})$, with universal dinatural transformation $\{\Phi(\iota_{\Phi^{-1}(M,\gamma)})\}_{(M,\gamma) \in \mathscr{Z}(\mathscr{C})}$. Using the description of Φ and the definition of the universal coaction given in (7), we obtain that the half braiding σ is given by (4) and that the universal coaction of (C, σ) is given by (5). Finally, recall from Section 3C that (C, α) is a Hopf algebra in $\mathscr{C}^Z$ endowed with a canonical Hopf algebra pairing. By [Bruguières and Virelizier 2012, Section 9.3], the structural morphisms of (C, α) are those given on pages 10 and 11, items (a)–(f). These structural morphisms are also those of (C, σ), since Φ is the identity on morphisms.

4C. *Slope and handleslide in pivotal fusion categories.* Let $\mathscr{C}$ be a pivotal fusion category. Recall that the left and right dimensions of a scalar object of $\mathscr{C}$ are invertible. The *slope* of a scalar object i is the invertible scalar $\mathrm{sl}(i)$ defined by

$$\mathrm{sl}(i) = \frac{\dim_l(i)}{\dim_r(i)}.$$

The *slope* of an object X of $\mathscr{C}$ is the morphism $\mathrm{SL}_X : X \to X$ defined as

$$\mathrm{SL}_X = \sum_{\alpha \in A} \mathrm{sl}(i_\alpha)\, q_\alpha p_\alpha,$$

where $(p_\alpha : X \to i_\alpha, q_\alpha : i_\alpha \to X)_{\alpha \in A}$ is a decomposition of X as a sum of scalar objects, that is, a family of pairs of morphisms such that i_α is scalar for every $\alpha \in A$, $p_\alpha q_\beta = \delta_{\alpha,\beta} \mathrm{id}_{i_\alpha}$ for all $\alpha, \beta \in A$, and $\mathrm{id}_X = \sum_{\alpha \in A} q_\alpha p_\alpha$. The morphism SL_X does not depend on the choice of the decomposition of X into scalar objects. Note that

SL_X is invertible with inverse

$$SL_X^{-1} = \sum_{\alpha \in A} sl(i_\alpha)^{-1} q_\alpha p_\alpha.$$

The family $SL = \{SL_X : X \to X\}_{X \in \mathscr{C}}$ is a monoidal natural automorphism of the identity functor $1_\mathscr{C}$ of $\mathscr{C}$, called the *slope operator* of $\mathscr{C}$. In particular

$$SL_Y f = f SL_X \quad \text{and} \quad SL_{X \otimes Y} = SL_X \otimes SL_Y$$

for all objects X, Y of $\mathscr{C}$ and all morphism $f : X \to Y$. The slope operator relates the left and right traces: for any endomorphism f of an object of $\mathscr{C}$,

(9) $$\text{tr}_l(f) = \text{tr}_r(f SL_X).$$

Note that $\mathscr{C}$ is spherical if and only its slope operator is the identity.

Lemma 4.1. *Let I be a representative set of scalar objects of $\mathscr{C}$.*

(a) *For any object X of $\mathscr{C}$,*

(b) *For i, $j \in I$ and X, Y objects of $\mathscr{C}$,*

(c) *For $i \in I$ and X, Y objects of $\mathscr{C}$,*

provided there are no j-colored strands in the gray area.

(d) *For all $i, j \in I$,*

Proof. Part (a) follows directly from the definitions. We prove (b). Let $(p_\alpha, q_\alpha)_{\alpha \in A}$ be an i-decomposition of $X^* \otimes j \otimes Y^*$. For $\alpha, \beta \in A$, set

We need to prove that $(P_\alpha, Q_\alpha)_{\alpha \in A}$ is a j-decomposition of $X \otimes i \otimes Y$. Let $\alpha, \beta \in A$. Since $(\mathrm{SL}_X)^* = \mathrm{SL}_{X^*}^{-1}$ and using (9), we obtain

$$P_\alpha Q_\beta = \frac{\mathrm{tr}_r(P_\alpha Q_\beta)}{\dim_r(j)} \, \mathrm{id}_j = \frac{\mathrm{tr}_l(f_{\alpha,\beta} \mathrm{SL}_{X^*}^{-1})}{\dim_r(i)} \, \mathrm{id}_j = \frac{\mathrm{tr}_r(f_{\alpha,\beta})}{\dim_r(i)} \, \mathrm{id}_j$$

$$= \frac{\mathrm{tr}_r(q_\alpha p_\beta)}{\dim_r(i)} \, \mathrm{id}_j = \frac{\mathrm{tr}_r(p_\beta q_\alpha)}{\dim_r(i)} \, \mathrm{id}_j = \frac{\mathrm{tr}_r(\delta_{\alpha,\beta} \, \mathrm{id}_i)}{\dim_r(i)} \, \mathrm{id}_j = \delta_{\alpha,\beta} \, \mathrm{id}_j.$$

We conclude using that $\mathrm{card}(A) = \nu_i(X^* \otimes j \otimes Y^*) = \nu_j(X \otimes i \otimes Y)$.

Part (c) reflects the canonical isomorphisms

$$\mathrm{Hom}_{\mathscr{C}}(X \otimes Y, i) \cong \bigoplus_{j \in I} \mathrm{Hom}_{\mathscr{C}}(X, j) \otimes_{\Bbbk} \mathrm{Hom}_{\mathscr{C}}(j \otimes Y, i)$$

$$\cong \bigoplus_{j \in I} \mathrm{Hom}_{\mathscr{C}}(X \otimes j, i) \otimes_{\Bbbk} \mathrm{Hom}_{\mathscr{C}}(Y, j),$$

and part (d) is a direct consequence of the duality axioms. $\qquad\square$

4D. *Proof of Theorem 2.1 and of the integrality of Λ.* Recall that $\mathscr{Z}(\mathscr{C})$ is a braided pivotal category which has a coend (C, σ) with $C = \bigoplus_{i,j \in I} i^* \otimes j^* \otimes i \otimes j$. Therefore its dimension is well-defined and

$$\dim \mathscr{Z}(\mathscr{C}) = \dim_l(C, \sigma) = \dim_l(C) = \dim_l\left(\sum_{i,j\in I} i^* \otimes j^* \otimes i \otimes j\right)$$

$$= \sum_{i,j\in I} \dim_l(i^*)\dim_l(j^*)\dim_l(i)\dim_l(j)$$

$$= \left(\sum_{i\in I}\dim_r(i)\dim_l(i)\right)\left(\sum_{j\in I}\dim_r(j)\dim_l(j)\right) = \dim(\mathscr{C})^2.$$

Let us prove that the canonical pairing of the coend (C, σ) is nondegenerate. Define the morphism $\lambda : C \to \mathbb{1}$ as follows and recall the definition of the morphism $\Lambda : \mathbb{1} \to C$ of (6):

$$\lambda = \sum_{i\in I}\dim_r(i) \;\; \overset{i}{\frown} \;\; \vdots \quad \text{and} \quad \Lambda = \sum_{j\in I}\dim_r(j) \;\; \vdots\; \underset{j}{\smile}\; \vdots \; .$$

Firstly, Λ is a morphism in $\mathscr{Z}(\mathscr{C})$ from $\mathbb{1}_{\mathscr{Z}(\mathscr{C})} = (\mathbb{1}, \mathrm{id})$ to (C, σ). Indeed, using the description of the half braiding σ given in (4), we obtain that for any object Y of $\mathscr{C}$,

$$\sigma_Y(\Lambda \otimes \mathrm{id}_Y) = \sum_{j,k,\ell,n\in I}\dim_r(j) \quad \text{<diagram>}$$

$$= \sum_{j,k,\ell,n\in I}\frac{\dim_r(\ell)}{\mathrm{sl}(n)} \quad \text{<diagram>} \qquad \text{by Lemma 4.1(b)}$$

$$= \sum_{k,\ell,n\in I}\frac{\dim_r(\ell)}{\mathrm{sl}(n)} \quad \text{<diagram>} \qquad \text{by Lemma 4.1(a)}$$

$$= \sum_{\ell,n\in I}\frac{\dim_r(\ell)}{\mathrm{sl}(n)} \quad \text{<diagram>}$$

$$= \sum_{\ell,n\in I}\dim_r(\ell) \quad \text{<diagram>} \qquad \text{by Lemma 4.1(d)}$$

$$= \mathrm{id}_Y \otimes \Lambda \qquad \text{by Lemma 4.1(a).}$$

Secondly, λ and Λ satisfy $\omega(\mathrm{id}_C \otimes \Lambda) = \lambda = \omega(\Lambda \otimes \mathrm{id}_C)$. Indeed, using the description of the canonical pairing ω given in item (f) on page 11, we obtain

$$\omega(\mathrm{id}_C \otimes \Lambda) = \sum_{i,j,\ell \in I} \dim_r(\ell) \quad$$

$$= \sum_{i,\ell \in I} \dim_r(\ell) \quad$$

$$= \sum_{i,\ell \in I} \frac{\dim_r(\ell)}{\dim_l(i)} \quad \cdots \quad \quad i \qquad \text{by Lemma 4.1(d)}$$

$$= \sum_{i,\ell \in I} \frac{\dim_r(\ell)}{\dim_l(i)} \delta_{\ell,i^*} \quad \cdots \quad \bigcirc i \; = \sum_{i \in I} \dim_r(i) \quad \cdots \; = \lambda,$$

and similarly

$$\omega(\Lambda \otimes \mathrm{id}_C) = \sum_{j,k,\ell \in I} \dim_r(j) \quad$$

$$= \sum_{j,k \in I} \dim_r(j) \, \delta_{j,k^*} \quad$$

$$= \sum_{k \in I} \bigcirc k \quad \cdots \quad \text{by Lemma 4.1(d)}$$

$$= \sum_{k \in I} \dim_r(k) \quad \cdots \; = \lambda.$$

This implies in particular that λ is a morphism in $\mathscr{Z}(\mathscr{C})$ from (C, σ) to $\mathbb{1}_{\mathscr{Z}(\mathscr{C})}$, since ω and Λ are morphisms in $\mathscr{Z}(\mathscr{C})$.

Thirdly, λ is a left cointegral for the Hopf algebra (C, σ) in $\mathscr{Z}(\mathscr{C})$. Indeed, using the description of the coproduct Δ and the unit u — items (a) and (d) on page 10 — we obtain

$$(\mathrm{id}_C \otimes \lambda)\Delta = \sum_{i,k,\ell,n \in I} \dim_r(k) \quad$$

$$= \sum_{i,k,\ell \in I} \dim_r(k) \quad$$

$$= \sum_{i,k,\ell \in I} \dim_r(k) \qquad \text{by Lemma 4.1(d)}$$

$$= \sum_{i,k,\ell \in I} \dim_r(i) \qquad \text{by Lemma 4.1(b)}$$

$$= \sum_{i,\ell \in I} \dim_r(i) \qquad \text{by Lemma 4.1(a)}$$

$$= u\,\lambda.$$

Since $\omega(\Lambda \otimes \Lambda) = \lambda\Lambda = \dim_r(\mathbb{1}) = 1 \in \Bbbk$ is invertible, we conclude by Lemma 3.1 that ω is nondegenerate. Hence $\mathscr{Z}(\mathscr{C})$ is modular.

Finally, let us prove that Λ is a two-sided integral of (C, σ) which is invariant under the antipode. The last part of Lemma 3.1 gives that Λ is a right integral of (C, σ). Using the description of the antipode S of (C, σ) in item (e) on page 11, we obtain

$$S\Lambda = \sum_{j,k,\ell \in I} \dim_r(j) \quad$$

$$= \sum_{j,k,\ell \in I} \dim_r(j) \qquad \text{by Lemma 4.1(c)}$$

$$= \sum_{\ell \in I} \dim_r(\ell^*) \quad \text{[diagram]}$$

$$= \sum_{\ell \in I} \dim_r(\ell) \quad \text{[diagram]}_j \quad \text{by Lemma 4.1(d)}$$

$$= \Lambda.$$

Hence Λ is S-invariant. This implies in particular that Λ, being a right integral, is also a left integral. Hence Λ is an S-invariant (two-sided) integral.

4E. *Proof of Theorem 2.2.* Consider the Hopf monad Z of Section 4B. Recall from [Bruguières and Virelizier 2007] that the monad Z is said to be *semisimple* if any Z-module is a Z-linear retract of a free Z-module, that is, of $(Z(X), \mu_X)$ for some object X of $\mathscr{C}$. Since the isomorphism $\Phi : \mathscr{C}^Z \to \mathscr{Z}(\mathscr{C})$ defined in (8) sends the free Z-module $(Z(X), \mu_X)$ to the free half braiding $\Phi(Z(X), \mu_X) = \mathscr{F}(X)$, we need to prove that $\dim(\mathscr{C})$ is invertible if and only if Z is semisimple. Now Theorem 6.5 of [Bruguières and Virelizier 2007] provides an analogue of Maschke's semisimplicity criterion for Hopf monads: the Hopf monad Z is semisimple if and only if there exists a morphism $\alpha : \mathbb{1} \to Z(\mathbb{1})$ in $\mathscr{C}$ such that

$$(10) \qquad\qquad \mu_{\mathbb{1}} \alpha = \alpha Z_0 \quad \text{and} \quad Z_0 \alpha = 1.$$

Let $\alpha : \mathbb{1} \to Z(\mathbb{1}) = \bigoplus_{i \in I} i^* \otimes i$ be a morphism in $\mathscr{C}$. Since $\mathscr{C}$ is a fusion category, α decomposes uniquely as $\alpha = \sum_{i \in I} \alpha_i \, \widetilde{\mathrm{coev}}_i$ where $\alpha_i \in \Bbbk$. From the structural morphisms of the Hopf monad Z (page 19), we obtain

$$\alpha Z_0 = \sum_{j,k \in I} \alpha_k \quad \text{[diagram]}$$

and

$$\mu_{\mathbb{1}} Z(\alpha) = \sum_{i,j,k \in I} \alpha_i \, \frac{\dim_r(k)}{\dim_r(i)} \quad \text{[diagram]}.$$

Thus, by duality, $\alpha Z_0 = \mu_{\mathbb{1}} Z(\alpha)$ if and only if

$$\sum_{j,k \in I} \alpha_k \quad \text{[diagram]} = \sum_{i,j,k \in I} \alpha_i \quad \text{[diagram]} \quad \text{in } \mathrm{End}_{\mathscr{C}}\left(\bigoplus_{k,j \in I} k \otimes j^* \right).$$

Now, for $j, k \in I$, by using Lemma 4.1(b),

$$\sum_{i \in I} \alpha_i \left[\text{diagram}\right] = \sum_{i \in I} \alpha_i \frac{\dim_r(k)}{\dim_r(i)} \left[\text{diagram}\right].$$

Therefore $\alpha Z_0 = \mu_{\mathbb{1}} Z(\alpha)$ if and only if

$$(11) \qquad \alpha_k \operatorname{id}_{k \otimes j^*} = \sum_{i \in I} \alpha_i \left[\text{diagram}\right] \in \operatorname{End}_{\mathscr{C}}(k \otimes j^*) \quad \text{for all } k, j \in I.$$

In particular, if $\alpha Z_0 = \mu_{\mathbb{1}} Z(\alpha)$, then for any $i \in I$, setting $k = \mathbb{1}$ and $j = i^*$ we obtain $\alpha_i = \alpha_{\mathbb{1}} \dim_r(i)$. Conversely, if $\alpha_i = \alpha_{\mathbb{1}} \dim_r(i)$ for all $i \in I$, then (11) holds by Lemma 4.1(a), and so $\alpha Z_0 = \mu_{\mathbb{1}} Z(\alpha)$. In conclusion, $\alpha Z_0 = \mu_{\mathbb{1}} Z(\alpha)$ if and only if $\alpha = \alpha_{\mathbb{1}} \kappa$, where

$$\kappa = \sum_{i \in I} \dim_r(i) \, \widetilde{\operatorname{coev}}_i : \mathbb{1} \to Z(\mathbb{1}).$$

In that case,

$$Z_0 \alpha = \alpha_{\mathbb{1}} Z_0 \kappa = \sum_{i \in I} \dim_r(i) Z_0 \widetilde{\operatorname{coev}}_i = \alpha_{\mathbb{1}} \sum_{i \in I} \dim_r(i) \dim_l(i) = \alpha_{\mathbb{1}} \dim(\mathscr{C}).$$

Hence there exists α satisfying (10) if and only if $\dim(\mathscr{C})$ is invertible in $\Bbbk$. This concludes the proof of Theorem 2.2. $\qquad \square$

4F. *Proof of Corollary 2.3.* Let $\mathscr{A}$ be an abelian category. If $\mathscr{A}$ is semisimple (see Section 2), then every object of $\mathscr{A}$ is projective[2]. The converse is true if in addition we assume that all objects of $\mathscr{A}$ have finite length[3].

Assume $\Bbbk$ is a field and let $\mathscr{C}$ be a pivotal fusion category over $\Bbbk$. Then $\mathscr{C}$ is abelian semisimple and its objects have finite length. The center $\mathscr{Z}(\mathscr{C})$ of $\mathscr{C}$ is then an abelian category and the forgetful functor $\mathscr{U} : \mathscr{Z}(\mathscr{C}) \to \mathscr{C}$ is $\Bbbk$-linear, faithful, and exact. This implies that all objects of $\mathscr{Z}(\mathscr{C})$ have finite length and the Hom spaces in $\mathscr{Z}(\mathscr{C})$ are finite-dimensional. As a result, $\mathscr{Z}(\mathscr{C})$ is semisimple if and only if all of its objects are projective.

---

[2]An object P of $\mathscr{A}$ is *projective* if the functor $\operatorname{Hom}_{\mathscr{A}}(P, -) : \mathscr{A} \to \mathrm{Ab}$ is exact, where Ab is the category of abelian groups.

[3]An object A of $\mathscr{A}$ has *finite length* if there exists a finite sequence of subobjects $A = X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_n = \mathbf{0}$ such that each quotient X_i / X_{i+1} is simple.

We identify $\mathcal{Z}(\mathscr{C})$ with the category $\mathscr{C}^Z$ of Z-modules via the isomorphism (8). Recall from the proof of Theorem 2.2 (see the beginning of Section 4E) that the monad Z is semisimple if and only if $\dim(\mathscr{C})$ is invertible in $\Bbbk$. The following lemma relates the notions of semisimplicity for monads and for categories.

Lemma 4.2. *Let $\mathscr{C}$ be an abelian category and T be a right exact monad on $\mathscr{C}$, so that $\mathscr{C}^T$ is abelian and the forgetful functor $U_T : \mathscr{C}^T \to \mathscr{C}$ is exact.*

(a) *If all the T-modules are projective, then T is semisimple.*

(b) *If T is semisimple and all the objects of $\mathscr{C}$ are projective, then all the T-modules are projective.*

(c) *If the objects of $\mathscr{C}$ have finite length, then the same holds in $\mathscr{C}^T$. If in addition $\mathscr{C}$ has finitely many isomorphy classes of simple objects, then so does $\mathscr{C}^T$.*

Proof. Let us prove assertion (a). Denote by $F_T : \mathscr{C} \to \mathscr{C}^T$ the free module functor (see Section 4A). Let (M, r) be a T-module. The action r defines an epimorphism $F_T(M) \to (M, r)$ in $\mathscr{C}^T$. In particular, if (M, r) is projective, it is a retract of $F_T(M)$. Therefore if all the T-modules are projective, the monad T is semisimple.

Let us prove assertion (b). Note that if X is a projective object of $\mathscr{C}$, then $F_T(X)$ is a projective T-module. Indeed,

$$\mathrm{Hom}_{\mathscr{C}^T}(F_T(X), ?) \simeq \mathrm{Hom}_{\mathscr{C}}(X, U_T)$$

by adjunction, and $\mathrm{Hom}_{\mathscr{C}}(X, U_T)$ is an exact functor when X is projective. In particular, if all objects are projective in $\mathscr{C}$ then all free T-modules are projective. If in addition T is semisimple, then any T-module, being a retract of a free T-module, is projective.

Finally, let us prove assertion (c). The first part results from the fact that U_T is faithful exact. Now if S is a simple object of $\mathscr{C}^T$ and Σ is a simple subobject of $U_T(S)$, then by adjunction the inclusion $\Sigma \subset U_T(S)$ defines a nonzero morphism $F_T(\Sigma) \to S$, which is an epimorphism because S is simple. This proves the second part of assertion (c), because under the assumptions made there are finitely many possibilities for Σ, and each $F_T(\Sigma)$ has finitely many simple quotients. $\square$

Assertion (a) of Corollary 2.3 results immediately from the first two assertions of Lemma 4.2.

Let us prove assertion (b). A fusion category over a field is semisimple. Now assume $\Bbbk$ is algebraically closed. By assertion (a), we need to show that if $\mathcal{Z}(\mathscr{C})$ is semisimple, then it is a fusion category. Assume $\mathcal{Z}(\mathscr{C})$ is semisimple. Since $\mathscr{C}$ is fusion, by the third assertion of Lemma 4.2, the category $\mathcal{Z}(\mathscr{C})$ has finitely many classes of simple objects and its objects have finite length. So each object of $\mathcal{Z}(\mathscr{C})$ is a finite direct sum of simple objects. Since the unit object of $\mathcal{Z}(\mathscr{C})$ is scalar and any simple object S of $\mathcal{Z}(\mathscr{C})$ is scalar (because $\mathrm{End}(S)$ is a finite extension of $\Bbbk$), we obtain that $\mathcal{Z}(\mathscr{C})$ is a fusion category. This proves Corollary 2.3. $\square$

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