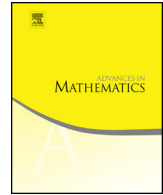




Contents lists available at ScienceDirect

Advances in Mathematics

journal homepage: www.elsevier.com/locate/aim



Monoidal categories graded by crossed modules and 3-dimensional HQFTs



Kürşat Sözer, Alexis Virelizier *

ARTICLE INFO

Article history:

Received 8 August 2022

Received in revised form 26 May 2023

Accepted 28 May 2023

Available online 20 June 2023

Communicated by Ross Street

MSC:

18M05

57K31

57K16

Keywords:

Crossed modules

Fusion categories

State sum

Homotopy quantum field theories

ABSTRACT

Given a crossed module χ , we introduce χ -graded monoidal categories and χ -fusion categories. We use spherical χ -fusion categories to construct (via the state sum method) 3-dimensional Homotopy Quantum Field Theories with target the classifying space $B\chi$ of the crossed module χ (which is a homotopy 2-type).

© 2023 Elsevier Inc. All rights reserved.

1. Introduction

Homotopy Quantum Field Theory (HQFT) is a branch of quantum topology concerned with maps from manifolds to a fixed target space X . The aim is to define and to study homotopy invariants of such maps using methods of quantum topology. The

* Corresponding author.

E-mail address: alexis.virelizier@univ-lille.fr (A. Virelizier).

formal notion of a d -dimensional HQFT with target X was introduced in [16]. Roughly, it is a TQFT for closed oriented $(d - 1)$ -dimensional manifolds and compact oriented d -dimensional cobordisms endowed with maps to X . Such an HQFT yields numerical homotopy invariants of maps from closed oriented d -dimensional manifolds to X . Note that 1-dimensional HQFTs with target X correspond bijectively to finite-dimensional representations of the fundamental group of X or, equivalently, to finite-dimensional flat vector bundles over X . This allows one to view HQFTs as high-dimensional generalizations of flat vector bundles. The case of 2-dimensional HQFTs with target X is quite well understood: they are classified by certain classes of Frobenius algebras which are in particular graded by the fundamental group of X (see [2,16,12,15], and [13] for the extended setting).

The present paper focuses on 3-dimensional HQFTs. The case where the target space X is a point (i.e., a connected homotopy 0-type) corresponds to 3-dimensional TQFTs and has been intensively studied. In particular, there are two fundamental constructions of 3-dimensional TQFTs: the Turaev-Viro TQFT which consists of state sums on triangulations (or more generally on skeletons) of 3-manifolds and uses spherical fusion categories, and the Reshetikhin-Turaev TQFT which is based on a surgery approach and uses modular categories (see for example the monographs [16,20] and the references therein). The case where the target space X is a connected homotopy 1-type, that is, an Eilenberg-MacLane space $K(G, 1)$ where G is a group, has been studied by Turaev and the second author: both the state-sum and the surgery approaches can be extended using spherical and modular fusion G -graded categories (see [17–19]). Here, a G -graded category is a monoidal category whose objects are equipped with a degree in G (which is multiplicative with respect to the monoidal product) and whose morphisms preserve the grading (i.e., there are no nonzero morphisms between objects with distinct degrees). In this paper, we extend the state sum approach to the case where the target space X is a connected homotopy 2-type.

Following MacLane and Whitehead [9], we model pointed connected homotopy 2-types by crossed modules: if X is a homotopy 2-type, then X is homotopy equivalent to the classifying space of a crossed module. Recall that a crossed module is a group homomorphism $\chi: E \rightarrow H$, with H acting on the left on E , such that χ is H -equivariant (where H acts on itself by conjugation) and satisfies the Peiffer identity. Such a crossed module has a classifying space $B\chi$ such that

$$\pi_1(B\chi) = \text{Coker}(\chi), \quad \pi_2(B\chi) = \text{Ker}(\chi), \quad \pi_k(B\chi) = 0 \quad \text{for } k \geq 3.$$

We introduce the notion of a χ -graded monoidal category in which not only objects have a degree but also morphisms. More precisely, each (homogeneous) object of such a category \mathcal{C} is equipped with a degree $|X| \in H$ (which is multiplicative with respect to the monoidal product) and the Hom-sets of \mathcal{C} are E -graded \mathbb{k} -modules:

$$\text{Hom}_{\mathcal{C}}(X, Y) = \bigoplus_{e \in E} \text{Hom}_{\mathcal{C}}^e(X, Y).$$

For homogeneous objects X, Y , we have $\text{Hom}_{\mathcal{C}}^e(X, Y) = 0$ whenever $|Y| \neq \chi(e)|X|$. In particular, there may be nonzero morphisms between homogeneous objects with distinct degrees. The composition of morphisms is multiplicative in degree:

$$(\beta, \alpha) \in \text{Hom}_{\mathcal{C}}^f(Y, Z) \times \text{Hom}_{\mathcal{C}}^e(X, Y) \Rightarrow \beta \circ \alpha \in \text{Hom}_{\mathcal{C}}^{f+e}(X, Z).$$

The monoidal product of morphisms is compatible with the gradings as follows:

$$(\alpha, \beta) \in \text{Hom}_{\mathcal{C}}^e(X, Y) \times \text{Hom}_{\mathcal{C}}^f(Z, T) \Rightarrow \alpha \otimes \beta \in \text{Hom}_{\mathcal{C}}^{e+|X|f}(X \otimes Z, Y \otimes T).$$

For example, the \mathbb{k} -linearization $\mathbb{k}\mathcal{G}_{\chi}$ of the 2-group \mathcal{G}_{χ} associated with χ is a χ -graded monoidal category. Other instances of χ -graded monoidal categories are given by the categories of representations of Hopf χ -(co)algebras (see [14]). Note that if G is a group, then the group homomorphism $1 \rightarrow G$ is a crossed module and $(1 \rightarrow G)$ -graded monoidal categories correspond to G -graded monoidal categories. Any cohomology class in $H^3(B\chi, \mathbb{k}^*)$ can always be represented by a normalized 3-cocycle $\omega: H^3 \times E^3 \rightarrow \mathbb{k}^*$ which allows to twist the composition, the monoidal product, and the associator of any χ -graded monoidal category \mathcal{C} to produce another χ -graded monoidal category \mathcal{C}^{ω} . Also, the push-forward $\phi_*(\mathcal{C})$ of a χ -graded monoidal category \mathcal{C} along a crossed module morphism $\phi: \chi \rightarrow \chi'$ is a χ' -graded monoidal category.

Next, using the state sum approach, we derive a 3-dimensional HQFT $|\cdot|_{\mathcal{C}}$ with target $B\chi$ from any spherical χ -fusion category \mathcal{C} whose neutral component has invertible dimension (see Theorems 7.1 and 9.5). To this end, we represent 3-manifolds by their skeletons and the maps to $B\chi$ by certain H -labels on the faces and E -labels on the edges of the skeletons. In particular, this HQFT induces a scalar invariant $|M, g|_{\mathcal{C}}$ of closed χ -manifolds, that is, of pairs (M, g) where M is closed oriented 3-manifold and g is a homotopy class of maps $M \rightarrow B\chi$. We prove by examples that this invariant is nontrivial. It may even distinguish homotopy classes of phantom maps $M \rightarrow B\chi$ (i.e., of maps inducing trivial homomorphisms on homotopy groups). For the spherical χ -fusion category $\mathbb{k}\mathcal{G}_{\chi}^{\omega}$ obtained by twisting the category $\mathbb{k}\mathcal{G}_{\chi}$ with a 3-cocycle ω for χ , we conjecture that

$$|M, g|_{\mathbb{k}\mathcal{G}_{\chi}^{\omega}} = \langle g^*([\omega]), [M] \rangle,$$

where $g^*([\omega]) \in H^3(M, \mathbb{k}^*)$ is the pullback of the cohomology class $[\omega] \in H^3(B\chi, \mathbb{k}^*)$ induced by the cocycle ω , $[M] \in H_3(M, \mathbb{Z})$ is the fundamental class of M , and the pairing $\langle \cdot, \cdot \rangle: H^3(M, \mathbb{k}^*) \times H_3(M, \mathbb{Z}) \rightarrow \mathbb{k}$ is the Kronecker pairing. Moreover, the push-forward $\phi_*(\mathcal{C})$ of a χ -fusion category \mathcal{C} along a crossed module morphism $\phi: \chi \rightarrow \chi'$ (satisfying some finiteness conditions) is a χ' -fusion category, and we relate the invariant $|M, g'|_{\phi_*(\mathcal{C})}$ of a closed χ' -manifold (M, g') with the invariants $\{|M, g|_{\mathcal{C}}\}_g$ and the fundamental groups $\{\pi_1(\text{TOP}_*(M, B\chi), g)\}_g$, where g runs over homotopy classes of maps $M \rightarrow B\chi$ such that $B\phi \circ g = g'$ (see Theorem 7.2). Here $\text{TOP}_*(M, B\chi)$ is the mapping space of pointed maps and $B\phi: B\chi \rightarrow B\chi'$ is the map induced by ϕ . In particular, if the

morphism ϕ is an equivalence of crossed modules (so that $B\phi$ is a homotopy equivalence), then $|M, g'|_{\phi_*(\mathcal{C})} = |M, (B\phi)^{-1} \circ g'|_{\mathcal{C}}$.

The paper is organized as follows. In Section 2, we review crossed modules and their classifying spaces. Section 3 is dedicated to categories whose Hom-sets are graded by a group. In Section 4, we introduce the notion of a monoidal category graded by a crossed module χ and discuss various classes of such categories including pivotal, spherical, and χ -fusion ones. In Section 5, we first introduce colored χ -cyclic sets and their multiplicity modules, and then we define an isotopy invariant of colored χ -graphs which is used in our state sums (as a replacement of $6j$ -symbols). In Section 6, we discuss skeletons of 3-manifolds and presentations of maps to $B\chi$ by labelings of skeletons. We use these presentations in Section 7 to derive from any spherical χ -fusion category a numerical invariant of closed χ -manifolds. In Section 8, we recall the definition (adapted to our need) of a 3-dimensional HQFT with target $B\chi$. Finally, in Section 9, we extend the state sum invariants of Section 7 to an HQFT with target $B\chi$.

We fix throughout the paper a nonzero commutative ring \mathbb{k} . For topological spaces X and Y , we denote by $[X, Y]$ the set of homotopy classes of maps $X \rightarrow Y$.

2. Crossed modules and their classifying spaces

We recall some standard notions of the theory of crossed modules.

2.1. Crossed modules

A *crossed module* is a group homomorphism $\chi: E \rightarrow H$ together with a left action of H on E (by group automorphisms) denoted

$$(x, e) \in H \times E \mapsto {}^x e \in E$$

such that χ is equivariant with respect to the conjugation action of H on itself:

$$\chi({}^x e) = x\chi(e)x^{-1} \tag{1}$$

and satisfies the Peiffer identity:

$${}^{x(e)}f = efe^{-1} \tag{2}$$

for all $x \in H$ and $e, f \in E$.

These axioms imply that the image $\text{Im}(\chi)$ is normal in H and that the kernel $\text{Ker}(\chi)$ is central in E and is acted on trivially by $\text{Im}(\chi)$. In particular, $\text{Ker}(\chi)$ inherits an action of $H/\text{Im}(\chi) = \text{Coker}(\chi)$.

2.2. Examples

1. Given any normal subgroup E of a group H , the inclusion $E \hookrightarrow H$ is a crossed module with the conjugation action of H on E .
2. For any group E , the homomorphism $E \rightarrow \text{Aut}(E)$ sending any element of E to the corresponding inner automorphism is a crossed module.
3. For any group H and any left H -module E , the trivial map $E \rightarrow H$ is a crossed module.
4. Any group epimorphism $E \rightarrow H$ with kernel contained in the center of E is a crossed module with trivial action of H on E . In particular, if A is an abelian group, then the trivial map $A \rightarrow 1$ is a crossed module.
5. To any map $\omega: S \rightarrow H$, where S is a set and H is a group, one associates the free crossed module $\chi: \text{CM}(\omega) \rightarrow H$ as follows. The group $\text{CM}(\omega)$ is generated by $S \times H$ with the relations

$$(e, x)(f, y) = (f, x\omega(e)x^{-1}y)(e, x)$$

for all $e, f \in S$ and $x, y \in H$. The action of H on $\text{CM}(\omega)$ and the morphism χ are given for all $e \in S$ and $x, y \in H$ by

$${}^x(e, y) = (e, xy) \quad \text{and} \quad \partial(e, x) = x\omega(e)x^{-1}.$$

6. A key geometric example of a crossed module is due to Whitehead. He showed that if (X, A, x) is a pair of pointed topological spaces, then the homotopy boundary map $\partial: \pi_2(X, A, x) \rightarrow \pi_1(A, x)$, together with the standard action of $\pi_1(A, x)$ on $\pi_2(X, A, x)$, is a crossed module.

2.3. Crossed module morphisms

A *morphism* from a crossed module $\chi: E \rightarrow H$ to a crossed module $\chi': E' \rightarrow H'$ is a pair $\phi = (\psi: E \rightarrow E', \varphi: H \rightarrow H')$ of group homomorphisms making the square

$$\begin{array}{ccc} E & \xrightarrow{\chi} & H \\ \psi \downarrow & & \downarrow \varphi \\ E' & \xrightarrow{\chi'} & H' \end{array}$$

commutative and preserving the action in the sense that for all $x \in H$ and $e \in E$,

$$\psi({}^xe) = {}^{\varphi(x)}\psi(e).$$

Crossed modules and crossed module morphisms form a category denoted CRMod . Whitehead's construction (see Example 6 of Section 2.2) extends to a functor

$$\Pi_2: (\text{pair of pointed topological spaces}) \rightarrow \text{CRMod.}$$

2.4. *Classifying spaces of crossed modules*

There is a classifying space functor B (see [3]) which assigns to a crossed module $\chi: E \rightarrow H$ a connected, reduced¹ CW-complex $B\chi$ with the following properties:

- (i) The homotopy groups of the classifying space $B\chi$ are given by

$$\pi_1(B\chi) = \text{Coker}(\chi), \quad \pi_2(B\chi) = \text{Ker}(\chi), \quad \pi_k(B\chi) = 0 \quad \text{for } k \geq 3.$$

- (ii) The classifying space $B\chi$ has a canonical (reduced) subcomplex BH which is a classifying space of the group H , contains all 1-cells of $B\chi$, and satisfies

$$\Pi_2(B\chi, BH) = \chi.$$

In particular, BH is the classifying space $B(1 \rightarrow H)$ of the trivial crossed module $1 \rightarrow H$.

- (iii) Let X be a reduced CW-complex. Let X_1 be the 1-skeleton of X and consider the Whitehead crossed module $\partial: \pi_2(X, X_1) \rightarrow \pi_1(X_1)$. Then there is a map $X \rightarrow B\partial$ inducing isomorphisms on π_1 and π_2 .

It follows that crossed modules model all pointed connected homotopy 2-types (as originally proved by MacLane and Whitehead [9]).

2.5. *2-groups and crossed modules*

Recall that a 2-group is a monoidal small category in which every morphism is invertible and every object has a weak inverse. A strict 2-group is a strict monoidal small category in which every morphism is invertible and every object has a strict inverse.

Strict 2-groups are identified with crossed modules as follows. Any crossed module $\chi: E \rightarrow H$ gives rise to a strict 2-group \mathcal{G}_χ defined as follows. The objects of \mathcal{G}_χ are the elements of H . For any $x, y \in H$,

$$\text{Hom}_{\mathcal{G}_\chi}(x, y) = \{e \in E \mid y = \chi(e)x\}.$$

The composition of morphisms is given by the product of E :

$$\left(y \xrightarrow{f} z\right) \circ \left(x \xrightarrow{e} y\right) = \left(x \xrightarrow{fe} z\right) \quad \text{and} \quad \text{id}_x = 1 \in E.$$

The monoidal product of objects is given by the product of H :

¹ A CW-complex is *reduced* if it has a single 0-cell, serving then as basepoint.

$$x \otimes y = xy \quad \text{and} \quad \mathbb{1} = 1 \in H.$$

The monoidal product of morphisms is induced by the left action of H on E :

$$\left(x \xrightarrow{e} y\right) \otimes \left(z \xrightarrow{f} t\right) = \left(xz \xrightarrow{e^x f} yt\right).$$

Conversely, any strict 2-group \mathcal{G} gives rise to a crossed module $E \rightarrow H$, where H is the set of objects of \mathcal{G} , E is the set of morphisms emanating from the unit object, and the map $E \rightarrow H$ is the target map.

3. Categories with Hom-sets graded by a group

In this section, we study Hom-graded categories which are linear categories whose Hom-sets are graded by a group. Throughout this section, E denotes a group.

3.1. Linear categories

A category \mathcal{C} is \mathbb{k} -linear if for all objects $X, Y \in \mathcal{C}$, the set $\text{Hom}_{\mathcal{C}}(X, Y)$ carries a structure of a left \mathbb{k} -module so that the composition of morphisms is \mathbb{k} -bilinear.

An object X of a \mathbb{k} -linear category \mathcal{C} is *simple* if $\text{End}_{\mathcal{C}}(X)$ is a free \mathbb{k} -module of rank 1 (and so has the basis $\{\text{id}_X\}$). It is clear that an object isomorphic to a simple object is itself simple.

A \mathbb{k} -linear category \mathcal{C} is *semisimple* if

- (a) each object of \mathcal{C} is a finite direct sum of simple objects;
- (b) for any non-isomorphic simple objects i, j of \mathcal{C} , we have $\text{Hom}_{\mathcal{C}}(i, j) = 0$.

Clearly, the Hom spaces in such a \mathcal{C} are free \mathbb{k} -modules of finite rank.

A set I of simple objects of a semisimple \mathbb{k} -linear category \mathcal{C} is *representative* if every simple object of \mathcal{C} is isomorphic to a unique element of I . Each object of \mathcal{C} is then a direct sum of a finite family of elements of I . Also, $\text{Hom}_{\mathcal{C}}(i, j) = 0$ for any distinct elements $i, j \in I$.

A monoidal category is \mathbb{k} -linear if it is \mathbb{k} -linear as a category and the monoidal product of morphisms is \mathbb{k} -bilinear.

3.2. Hom-graded categories

By an *E-Hom-graded category (over \mathbb{k})* we mean a category enriched over the monoidal category of E -graded \mathbb{k} -modules and \mathbb{k} -linear grading-preserving homomorphisms. In other words, an *E-Hom-graded category (over \mathbb{k})* is a \mathbb{k} -linear category \mathcal{C} such that:

(a) The Hom-sets in \mathcal{C} are E -graded \mathbb{k} -modules: for all objects $X, Y \in \mathcal{C}$,

$$\text{Hom}_{\mathcal{C}}(X, Y) = \bigoplus_{e \in E} \text{Hom}_{\mathcal{C}}^e(X, Y).$$

(b) The composition in \mathcal{C} is grading-preserving: for all $X, Y, Z \in \mathcal{C}$ and $e, f \in E$, it sends $\text{Hom}_{\mathcal{C}}^f(Y, Z) \times \text{Hom}_{\mathcal{C}}^e(X, Y)$ into $\text{Hom}_{\mathcal{C}}^{f \circ e}(X, Z)$.

(c) The identities have trivial degree: for all $X \in \mathcal{C}$,

$$\text{id}_X \in \text{End}_{\mathcal{C}}^1(X) = \text{Hom}_{\mathcal{C}}^1(X, X),$$

where 1 denotes the unit element of E .

Note that for any object $X \in \mathcal{C}$, the monoid $\text{End}_{\mathcal{C}}(X) = \text{Hom}_{\mathcal{C}}(X, X)$ is an E -graded \mathbb{k} -algebra.

A morphism $f: X \rightarrow Y$ in an E -Hom-graded category \mathcal{C} is *homogeneous* if

$$f \in \prod_{e \in E} \text{Hom}_{\mathcal{C}}^e(X, Y).$$

It is *homogeneous of degree* $e \in E$ if $f \in \text{Hom}_{\mathcal{C}}^e(X, Y)$. Note that if f is nonzero, then such an $e \in E$ is unique, is called the *degree* of f , and is denoted $e = |f|$. The objects of \mathcal{C} together with the homogenous morphisms of degree 1 form a \mathbb{k} -linear subcategory of \mathcal{C} called the *1-subcategory* of \mathcal{C} and denoted \mathcal{C}^1 .

Given $e \in E$, by an *e-isomorphism* we mean an isomorphism which is homogeneous of degree e . We say that an object X is *e-isomorphic* to an object Y , and we write $X \cong_e Y$, if there is an e -isomorphism $X \rightarrow Y$. It is easy to see that for all $X, Y, Z \in \mathcal{C}$ and $e, f \in E$,

$$X \cong_1 X, \quad X \cong_e Y \implies Y \cong_{e^{-1}} X, \quad X \cong_e Y \text{ and } Y \cong_f Z \implies X \cong_{f \circ e} Z. \tag{3}$$

Also, if $X \cong_e Y$, then for all $d \in E$, there are \mathbb{k} -linear isomorphisms

$$\text{Hom}_{\mathcal{C}}^d(Y, Z) \simeq \text{Hom}_{\mathcal{C}}^{d \circ e}(X, Z) \quad \text{and} \quad \text{Hom}_{\mathcal{C}}^d(Z, X) \simeq \text{Hom}_{\mathcal{C}}^{ed}(Z, Y) \tag{4}$$

which are induced by pre/post-composition with an e -isomorphism $X \rightarrow Y$.

3.3. Example

A \mathbb{k} -module M is *E-graded* if it has a direct sum decomposition

$$M = \bigoplus_{e \in E} M_e$$

by submodules indexed by $e \in E$. We say that such a \mathbb{k} -module M has *finite support* if $M_e = 0$ for all but a finite number of $e \in E$. For example, this is the case if E is finite or if M is free of finite rank. A \mathbb{k} -linear homomorphism $\phi: M \rightarrow N$ between E -graded \mathbb{k} -modules is *homogeneous of degree* $d \in E$ if $\phi(M_e) \subset N_{de}$ for all $e \in E$. Such homomorphisms form a submodule $\text{Hom}_{\mathbb{k}}^d(M, N)$ of the \mathbb{k} -module $\text{Hom}_{\mathbb{k}}(M, N)$ of \mathbb{k} -linear homomorphisms $M \rightarrow N$. For example, the identity $\text{id}_M: M \rightarrow M$ is homogenous of degree 1. Clearly, the composition of homogeneous homomorphisms is multiplicative with respect to the degree. This induces an E -Hom-graded category $\mathcal{M}_{\mathbb{k}}^E$ whose objects are E -graded \mathbb{k} -modules and Hom-sets are

$$\text{Hom}_{\mathcal{M}_{\mathbb{k}}^E}(M, N) = \bigoplus_{d \in E} \text{Hom}_{\mathbb{k}}^d(M, N).$$

The 1-subcategory of $\mathcal{M}_{\mathbb{k}}^E$ is the category of E -graded \mathbb{k} -modules and grading-preserving \mathbb{k} -linear homomorphisms. Note that if the \mathbb{k} -modules M and N have finite support, then $\text{Hom}_{\mathcal{M}_{\mathbb{k}}^E}(M, N) = \text{Hom}_{\mathbb{k}}(M, N)$. Then E -graded \mathbb{k} -modules with finite support and \mathbb{k} -linear homomorphisms form an E -Hom-graded category (as a subcategory of $\mathcal{M}_{\mathbb{k}}^E$). In particular, E -graded free \mathbb{k} -modules of finite rank and \mathbb{k} -linear homomorphisms form an E -Hom-graded category.

3.4. Direct sums in Hom-graded categories

Let \mathcal{C} be an E -Hom-graded category. Given $e \in E$, an object D of \mathcal{C} is an *e -direct sum* of a finite family $(X_{\alpha})_{\alpha \in A}$ of objects of \mathcal{C} if there is a family $(p_{\alpha}, q_{\alpha})_{\alpha \in A}$ of morphisms in \mathcal{C} such that:

- (a) $p_{\alpha}: D \rightarrow X_{\alpha}$ is homogeneous of degree e^{-1} for all $\alpha \in A$;
- (b) $q_{\alpha}: X_{\alpha} \rightarrow D$ is homogeneous of degree e for all $\alpha \in A$;
- (c) $\text{id}_D = \sum_{\alpha \in A} q_{\alpha} p_{\alpha}$;
- (d) $p_{\alpha} q_{\beta} = \delta_{\alpha, \beta} \text{id}_{X_{\alpha}}$ for all $\alpha, \beta \in A$, where $\delta_{\alpha, \beta}$ is the Kronecker symbol defined by $\delta_{\alpha, \beta} = 1$ if $\alpha = \beta$ and $\delta_{\alpha, \beta} = 0$ otherwise.

Such an e -direct sum D , if it exists, is unique up to a 1-isomorphism and is denoted

$$D = \bigoplus_{\alpha \in A}^e X_{\alpha}.$$

It is easy to see that if a finite family $(X_{\alpha})_{\alpha \in A}$ of objects of \mathcal{C} has an e -direct sum and an f -direct sum with $e, f \in E$, then

$$\bigoplus_{\alpha \in A}^e X_{\alpha} \cong_{f e^{-1}} \bigoplus_{\alpha \in A}^f X_{\alpha}.$$

Also, for any finite families $(X_\alpha)_{\alpha \in A}$ and $(Y_\beta)_{\beta \in B}$ of objects of \mathcal{C} and for any $d, e, f \in E$, there are \mathbb{k} -linear isomorphisms

$$\text{Hom}_{\mathcal{C}}^d \left(\bigoplus_{\alpha \in A}^e X_\alpha, \bigoplus_{\beta \in B}^f Y_\beta \right) \simeq \bigoplus_{\substack{\alpha \in A \\ \beta \in B}} \text{Hom}_{\mathcal{C}}^{f^{-1}de} (X_\alpha, Y_\beta). \tag{5}$$

By definition, a direct sum of an empty family of objects of a \mathbb{k} -linear category \mathcal{C} is a *zero object* of \mathcal{C} , that is, an object $\mathbf{0}$ of \mathcal{C} such that $\text{End}_{\mathcal{C}}(\mathbf{0}) = 0$.

An E -Hom-graded category \mathcal{C} is *E -additive* if any finite (possibly empty) family of objects of \mathcal{C} has an e -direct sum in \mathcal{C} for all $e \in E$.

3.5. Semisimple Hom-graded categories

We say that an E -Hom-graded category \mathcal{C} is *E -semisimple* if:

- (a) its 1-subcategory \mathcal{C}^1 is semisimple (see Section 3.1);
- (b) for any $e \in E$, each object of \mathcal{C} is an e -direct sum of a finite family of simple objects of \mathcal{C}^1 .

For $E = 1$, we recover the usual notion of a semisimple category. Note that a simple object of \mathcal{C}^1 is nothing but an object of \mathcal{C} such that $\text{End}_{\mathcal{C}}^1(X)$ is free of rank 1 (with basis id_X). In particular, a simple object of \mathcal{C}^1 may not be simple in \mathcal{C} . Consequently, an E -semisimple E -Hom-graded category is not necessarily semisimple (in the usual sense of Section 3.1).

For example, the category of E -graded free \mathbb{k} -modules of finite rank and all \mathbb{k} -linear homomorphisms (see Section 3.3) is an E -semisimple E -Hom-graded category.

Lemma 3.1. *Let \mathcal{C} be an E -semisimple E -Hom-graded category. Then:*

- (i) *For any simple object i of \mathcal{C}^1 and $e \in E$, there is a simple object j of \mathcal{C}^1 such that $i \cong_e j$.*
- (ii) *For any simple objects i, j of \mathcal{C}^1 and $e \in E$, the \mathbb{k} -module $\text{Hom}_{\mathcal{C}}^e(i, j)$ is free of rank 1 if $i \cong_e j$ and $\text{Hom}_{\mathcal{C}}^e(i, j) = 0$ otherwise.*
- (iii) *Hom-sets in \mathcal{C} are free \mathbb{k} -modules of finite rank.*
- (iv) *For any objects $X, Y \in \mathcal{C}$ and $e, f \in E$, there is a \mathbb{k} -linear isomorphism*

$$\text{Hom}_{\mathcal{C}}^{fe}(X, Y) \simeq \bigoplus_{i \in I} \text{Hom}_{\mathcal{C}}^f(i, Y) \otimes_{\mathbb{k}} \text{Hom}_{\mathcal{C}}^e(X, i),$$

where I is any representative set of simple objects of \mathcal{C}^1 .

Proof. Let us prove Part (i). Pick a decomposition $(p_\alpha: i \rightarrow i_\alpha, q_\alpha: i_\alpha \rightarrow i)_{\alpha \in A}$ of i as an e^{-1} -direct sum of a finite family $(i_\alpha)_{\alpha \in A}$ of simple objects \mathcal{C}^1 . Formula (5) implies that

$$\bigoplus_{\alpha, \beta \in A} \text{Hom}_{\mathcal{C}}^1(i_\alpha, i_\beta) \simeq \text{End}_{\mathcal{C}}^1(i) \quad \text{and so} \quad \bigoplus_{\alpha \in A} \text{End}_{\mathcal{C}}^1(i_\alpha) \subset \text{End}_{\mathcal{C}}^1(i).$$

Since i and each i_α are simple in \mathcal{C}^1 , taking the rank in the latter inclusion gives that the cardinal of A is less than or equal to 1. Since the set A is nonempty (otherwise i would be the zero object which is not simple in \mathcal{C}^1), it is a singleton $A = \{\alpha_0\}$. Set $j = i_{\alpha_0}$. Then $p = p_{\alpha_0}: i \rightarrow j$ is a homogeneous morphism of degree e and $q = q_{\alpha_0}: j \rightarrow i$ is a homogeneous morphism of degree e^{-1} such that $\text{id}_i = qp$ and $pq = \text{id}_j$. Thus p is an e -isomorphism and so $i \cong_e j$.

Let us prove Part (ii). If $i \cong_e j$, then Formula (4) implies that $\text{Hom}_{\mathcal{C}}^e(i, j) \simeq \text{End}_{\mathcal{C}}^1(i)$ is free of rank 1 (since i is simple in \mathcal{C}^1). Assume that $i \not\cong_e j$. By Part (i), there is a simple object k of \mathcal{C}^1 such that $j \cong_{e^{-1}} k$. Then $i \not\cong_1 k$ (indeed, using (3) and since $k \cong_e j$, if $i \cong_1 k$ then $i \cong_e j$). Then the simple objects i, j of \mathcal{C}^1 are not isomorphic in \mathcal{C}^1 and so $\text{Hom}_{\mathcal{C}}^1(i, k) = 0$ (since \mathcal{C}^1 is semisimple). Hence, using (4), we conclude that $\text{Hom}_{\mathcal{C}}^e(i, j) \simeq \text{Hom}_{\mathcal{C}}^1(i, k) = 0$.

Since each object of \mathcal{C} is a 1-direct sum of a finite family of simple objects of \mathcal{C}^1 , Part (iii) follows directly from (5) and Part (ii).

Let us prove Part (iv). Pick a decomposition $(p_\alpha: X \rightarrow i_\alpha, q_\alpha: i_\alpha \rightarrow X)_{\alpha \in A}$ of X as an e^{-1} -direct sum of a finite family $(i_\alpha)_{\alpha \in A}$ of objects in I . For $i \in I$, consider the subset A_i of A consisting of elements $\alpha \in A$ such that $i_\alpha = i$. Note that A_i is empty for all but a finite number of $i \in I$. It follows from the definition of an e^{-1} -direct sum decomposition (see Section 3.4) that the map

$$\phi \in \text{Hom}_{\mathcal{C}}^{fe}(X, Y) \mapsto \sum_{i \in I} \left(\sum_{\alpha \in A_i} (\phi q_\alpha) \otimes_{\mathbb{k}} p_\alpha \right) \in \bigoplus_{i \in I} \text{Hom}_{\mathcal{C}}^f(i, Y) \otimes_{\mathbb{k}} \text{Hom}_{\mathcal{C}}^e(X, i)$$

is a \mathbb{k} -linear isomorphism with inverse defined on the summands by

$$v \otimes_{\mathbb{k}} u \in \text{Hom}_{\mathcal{C}}^f(i, Y) \otimes_{\mathbb{k}} \text{Hom}_{\mathcal{C}}^e(X, i) \mapsto vu \in \text{Hom}_{\mathcal{C}}^{fe}(X, Y). \quad \square$$

Lemma 3.2. *Let \mathcal{C} be an E -Hom-graded category such that its 1-subcategory \mathcal{C}^1 is semisimple. Let I be a representative set of simple objects of \mathcal{C}^1 . Then the following assertions are equivalent:*

- (i) *The category \mathcal{C} is E -semisimple.*
- (ii) *For all $i \in I$ and $e \in E$, there exists a unique $j \in I$ such that $i \cong_e j$.*
- (iii) *There is a left action $E \times I \rightarrow I$, $(e, i) \mapsto e \cdot i$, such that:*

$$i \cong_e j \iff j = e \cdot i \quad \text{for all } i, j \in I \text{ and } e \in E.$$

Proof. Let us prove that (i) implies (ii). Let $i \in I$ and $e \in E$. By Lemma 3.1(i), there is a simple object k of \mathcal{C}^1 such that $i \cong_e k$. Since I is a representative set of simple objects of \mathcal{C}^1 , there is $j \in I$ such that $k \cong_1 j$. Then $i \cong_e j$ by (3). If $j' \in I$ is such that $i \cong_e j'$, then $j \cong_1 j'$ by (3), and so $j = j'$ since I is a representative set of simple objects of \mathcal{C}^1 .

Let us prove that (ii) implies (iii). For any $i \in I$ and $e \in E$, denote by $e \cdot i$ the unique element of I verifying $i \cong_e e \cdot i$. This defines a map $E \times I \rightarrow I$. It follows from (3) and the uniqueness in (ii) that this map is a left action such that $i \cong_e j$ if and only if $j = e \cdot i$.

Let us prove that (iii) implies (i). Let X be an object of \mathcal{C} and $e \in E$. We need to prove that X is an e -direct sum of a finite family of simple objects of \mathcal{C}^1 . Since \mathcal{C}^1 is semisimple, there is a decomposition $(p_\alpha: X \rightarrow j_\alpha, q_\alpha: j_\alpha \rightarrow i)_{\alpha \in A}$ of X as a 1-direct sum of a finite family $(j_\alpha)_{\alpha \in A}$ of simple objects. For any $\alpha \in A$, set $i_\alpha = (e^{-1} \cdot j_\alpha) \in I$. Then $j_\alpha = e \cdot i_\alpha$ and so $i_\alpha \cong_e j_\alpha$. Pick an e -isomorphism $\phi_\alpha: i_\alpha \rightarrow j_\alpha$ and set $P_\alpha = \phi_\alpha^{-1} p_\alpha: X \rightarrow i_\alpha$ and $Q_\alpha = q_\alpha \phi_\alpha: i_\alpha \rightarrow X$. It is easy to verify that $(P_\alpha, Q_\alpha)_{\alpha \in A}$ is a decomposition of X as an e -direct sum of the finite family $(i_\alpha)_{\alpha \in A}$. \square

3.6. Multiplicity numbers and partitions

Let \mathcal{C} be an E -semisimple E -Hom-graded category. Given an object X of \mathcal{C} , a simple object i of \mathcal{C}^1 , and $e \in E$, we denote by $N_X^{i,e} \geq 0$ the number of simple objects 1-isomorphic to i in decomposition of X as an e -direct sum of a finite family of simple objects of \mathcal{C}^1 . By (5) and Lemma 3.1(ii), this number is equal to the ranks of the free \mathbb{k} -modules $\text{Hom}_{\mathcal{C}}^e(i, X)$ and $\text{Hom}_{\mathcal{C}}^{e^{-1}}(X, i)$. We call $N_X^{i,e}$ the *degree e multiplicity index of i in X* . It follows from (5) that if X is a f -direct sum of a finite family $(X_\alpha)_{\alpha \in A}$ of objects of \mathcal{C} with $f \in E$, then

$$N_X^{i,e} = \sum_{\alpha \in A} N_{X_\alpha}^{i,f^{-1}e}.$$

If I is a representative set of simple objects of \mathcal{C}^1 , then for all objects X of \mathcal{C} and $e \in E$, the family $\{N_X^{i,e}\}_{i \in I}$ has only a finite number of nonzero terms, and Lemma 3.1(iv) implies that for all objects X, Y of \mathcal{C} and $e, f \in E$,

$$\text{rank}_{\mathbb{k}}(\text{Hom}_{\mathcal{C}}^{fe}(X, Y)) = \sum_{i \in I} N_Y^{i,f} N_X^{i,e}.$$

Given a simple object i of \mathcal{C}^1 and $e \in E$, an (i, e) -partition of an object X of \mathcal{C} is a family of morphisms $(p_\alpha: X \rightarrow i, q_\alpha: i \rightarrow X)_{\alpha \in \Lambda}$ such that $(p_\alpha)_{\alpha \in \Lambda}$ is a basis of $\text{Hom}_{\mathcal{C}}^e(X, i)$, $(q_\alpha)_{\alpha \in \Lambda}$ is a basis of $\text{Hom}_{\mathcal{C}}^e(i, X)$, and $p_\alpha q_\beta = \delta_{\alpha,\beta} \text{id}_i$ for all $\alpha, \beta \in \Lambda$. Note that the cardinality of Λ is equal to the multiplicity index $N_X^{i,e}$. The existence of (i, e) -partitions follows from the fact that each object of \mathcal{C} is an e -direct sum of simple objects of \mathcal{C}^1 .

Given a representative set I of simple objects of \mathcal{C}^1 and $e \in E$, an (I, e) -partition of an object X of \mathcal{C} is a decomposition $(p_\alpha: X \rightarrow i_\alpha, q_\alpha: i_\alpha \rightarrow X)_{\alpha \in A}$ of X as an e -direct sum

of a finite family $(i_\alpha)_{\alpha \in A}$ of objects in I . For such an (I, e) -partition and any $i \in I$, the family $(p_\alpha, q_\alpha)_{\alpha \in A, i_\alpha = i}$ is an (i, e) -partition of X . Conversely, a union of (i, e) -partitions of X over all $i \in I$ is an (I, e) -partition of X .

4. Monoidal categories graded by a crossed module

We introduce categories graded by a crossed module which are linear monoidal categories whose Hom-sets are graded by the source of the crossed module and whose objects are graded by the target of the crossed module in some compatible way (involving the crossed module axioms).

Throughout this section, $\chi: E \rightarrow H$ denotes a crossed module. Recall from Section 2.1 that it comes equipped with a left action $(x, e) \in H \times E \mapsto {}^x e \in E$.

4.1. χ -categories

A χ -graded category (over \mathbb{k}), or shorter a χ -category, is a \mathbb{k} -linear monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1})$ which is E -Hom-graded (see Section 3.2) and is endowed with a subclass \mathcal{C}_{hom} of nonzero objects of \mathcal{C} , called *homogeneous objects*, and with a map $|\cdot|: \mathcal{C}_{\text{hom}} \rightarrow H$, called the *degree map*, such that:

- (a) Each object of \mathcal{C} is a 1-direct sum of a finite family of homogeneous objects.
- (b) For all homogenous objects X, Y and $e \in E$ such that $|Y| \neq \chi(e)|X|$,

$$\text{Hom}_{\mathcal{C}}^e(X, Y) = 0.$$

- (c) The monoidal product $X \otimes Y$ of any two homogeneous objects X, Y is a 1-direct sum of homogeneous objects of degree $|X||Y|$.
- (d) The unit object $\mathbb{1}$ is homogeneous with trivial degree (i.e., $|\mathbb{1}| = 1 \in H$).
- (e) The monoidal product $\alpha \otimes \beta$ of any two nonzero homogeneous morphisms α, β is a homogeneous morphism of degree

$$|\alpha \otimes \beta| = |\alpha|^{|s(\alpha)|}|\beta|,$$

whenever the source $s(\alpha)$ of α is a homogeneous object. In other words, for any objects X, Y, Z, T with X homogeneous and for any morphisms $\alpha \in \text{Hom}_{\mathcal{C}}^e(X, Y)$, $\beta \in \text{Hom}_{\mathcal{C}}^f(Z, T)$ with $e, f \in E$, we have:

$$\alpha \otimes \beta \in \text{Hom}_{\mathcal{C}}^{e|X|f}(X \otimes Z, Y \otimes T).$$

- (f) The associativity constraints $(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$ and the unitality constraints $X \otimes \mathbb{1} \cong X \cong \mathbb{1} \otimes X$ of \mathcal{C} are all homogenous of degree $1 \in E$.

Note that we will suppress in our formulas the associativity and unitality constraints. Indeed, this does not lead to ambiguity because all legitimate ways of inserting these constraints give the same result (by MacLane’s coherence theorem) and do not affect the degree of morphisms (by the above axioms).

Axiom (a) implies that the Hom-sets in \mathcal{C} are fully determined by the Hom-sets between homogeneous objects. Axiom (b) and (5) imply that if $(X_\alpha)_{\alpha \in A}$, $(Y_\beta)_{\beta \in B}$ are finite families of homogeneous objects and $e \in E$ is such that $|Y_\beta| \neq \chi(e)|X_\alpha|$ for all $\alpha \in A$ and $\beta \in B$, then

$$\text{Hom}_{\mathcal{C}}^e \left(\bigoplus_{\alpha \in A} X_\alpha, \bigoplus_{\beta \in B} Y_\beta \right) = 0. \tag{6}$$

Axiom (b) implies that for all homogenous objects X, Y , we have

$$\text{Hom}_{\mathcal{C}}(X, Y) = \bigoplus_{e \in \chi^{-1}(|Y||X|^{-1})} \text{Hom}_{\mathcal{C}}^e(X, Y) \quad \text{and} \quad \text{End}_{\mathcal{C}}(X) = \bigoplus_{e \in \text{Ker}(\chi)} \text{End}_{\mathcal{C}}^e(X).$$

In particular $\text{Hom}_{\mathcal{C}}(X, Y) = 0$ whenever $|Y||X|^{-1} \notin \text{Im}(\chi)$. Also, if $\alpha: X \rightarrow Y$ is a nonzero homogeneous morphism between homogeneous objects, then

$$|Y| = \chi(|\alpha|)|X|, \tag{7}$$

where $|\alpha| \in E$ is the degree of α (see Section 3.2). In particular, 1-isomorphic homogenous objects have the same degree.

Axiom (e) implies that for any homogenous objects X, Y, Z of \mathcal{C} and $e \in E$,

$$X \cong_e Y \implies X \otimes Z \cong_e Y \otimes Z \quad \text{and} \quad Z \otimes X \cong_{|z|_e} Z \otimes Y.$$

4.2. Relations with group-graded monoidal categories

Recall that a \mathbb{k} -linear monoidal category \mathcal{D} is *graded by the group H* , or shorter *H -graded*, if it decomposes as a direct sum

$$\mathcal{D} = \bigoplus_{h \in H} \mathcal{D}_h,$$

of full subcategories such that $\mathbb{1} \in \mathcal{D}_1$ and $\mathcal{D}_g \otimes \mathcal{D}_h \subset \mathcal{D}_{gh}$ for all $g, h \in H$ (see for example [17] or [5]). The monoidal subcategory \mathcal{D}_1 is called the *neutral component* of \mathcal{D} .

The 1-subcategory \mathcal{C}^1 of a χ -graded category \mathcal{C} is an H -graded \mathbb{k} -linear monoidal category:

$$\mathcal{C}^1 = \bigoplus_{h \in H} \mathcal{C}_h^1$$

where \mathcal{C}_h^1 is the subcategory of \mathcal{C} whose objects are (finite) 1-direct sums of homogeneous objects of \mathcal{C} of degree h and whose morphisms are homogeneous morphisms of degree $1 \in E$.

Also, any χ -graded category \mathcal{C} is $\text{Coker}(\chi)$ -graded:

$$\mathcal{C} = \bigoplus_{p \in \text{Coker}(\chi)} \mathcal{C}_p$$

where \mathcal{C}_p is the full subcategory of \mathcal{C} whose objects are (finite) 1-direct sums of homogeneous objects of \mathcal{C} with degree in the preimage $\pi^{-1}(p) \subset H$ of p under the canonical projection $\pi: H \rightarrow \text{Coker}(\chi) = H/\text{Im}(\chi)$.

4.3. Particular cases

1. Given a group H , the trivial map $1 \rightarrow H$ is a crossed module and the notion of a $(1 \rightarrow H)$ -category agrees with that of an H -graded \mathbb{k} -linear monoidal category (see Section 4.2).

2. Given an abelian group A , the trivial map $A \rightarrow 1$ is a crossed module and the notion of a $(A \rightarrow 1)$ -category corresponds to that of a \mathbb{k} -linear monoidal category whose Hom-sets are A -graded \mathbb{k} -modules so that the identities and monoidal constraints have trivial degree and the composition and monoidal product of morphisms are both multiplicative with respect to the degree: for any $\alpha \in \text{Hom}_{\mathcal{C}}^a(X, Y)$, $\beta \in \text{Hom}_{\mathcal{C}}^b(Y, Z)$, $\gamma \in \text{Hom}_{\mathcal{C}}^c(U, V)$ with $a, b, c \in A$, we have:

$$\beta \circ \alpha \in \text{Hom}_{\mathcal{C}}^{ba}(X, Z) \quad \text{and} \quad \alpha \otimes \gamma \in \text{Hom}_{\mathcal{C}}^{ac}(X \otimes U, Y \otimes V).$$

4.4. Example

Recall from Section 2.5 that the crossed module χ gives rise to a strict 2-group \mathcal{G}_χ . Let us consider the linearization $\mathbb{k}\mathcal{G}_\chi$ of this 2-group. The objects of $\mathbb{k}\mathcal{G}_\chi$ are those of \mathcal{G}_χ , that is, the elements of H . For any $x, y \in H$, the set $\text{Hom}_{\mathbb{k}\mathcal{G}_\chi}(x, y)$ is the free \mathbb{k} -module with basis $\text{Hom}_{\mathcal{G}_\chi}(x, y) = \{e \in E \mid y = \chi(e)x\}$. The composition and monoidal product of $\mathbb{k}\mathcal{G}_\chi$ extend those of \mathcal{G}_χ by bilinearity. Then $\mathbb{k}\mathcal{G}_\chi$ is a \mathbb{k} -linear strict monoidal category. It is χ -graded as follows: each object $x \in H$ is homogeneous with degree $|x| = x$, and for any $x, y \in H$ and $e \in E$,

$$\text{Hom}_{\mathbb{k}\mathcal{G}_\chi}^e(x, y) = \begin{cases} \mathbb{k}e & \text{if } y = \chi(e)x, \\ 0 & \text{otherwise.} \end{cases}$$

4.5. Example

Let E be a normal subgroup of a group H . A \mathbb{k} -linear homomorphism $\alpha: M \rightarrow N$ between H -graded \mathbb{k} -modules is *homogeneous of degree* $e \in E$ if $\alpha(M_h) \subset N_{eh}$ for

all $h \in H$. Such homomorphisms form a submodule $\text{Hom}_{\mathbb{k}}^e(M, N)$ of $\text{Hom}_{\mathbb{k}}(M, N)$. A \mathbb{k} -linear homomorphism $\alpha: M \rightarrow N$ between H -graded \mathbb{k} -modules is E -graded if it is a linear combination of homogeneous \mathbb{k} -linear homomorphisms with degree in E . Such homomorphisms form a submodule $\text{Hom}_{\mathbb{k}}^E(M, N)$ of $\text{Hom}_{\mathbb{k}}(M, N)$. Clearly,

$$\text{Hom}_{\mathbb{k}}^E(M, N) = \bigoplus_{e \in E} \text{Hom}_{\mathbb{k}}^e(M, N).$$

With usual composition and identities, H -graded \mathbb{k} -modules and E -graded \mathbb{k} -linear homomorphisms form an E -Hom-graded category denoted $\mathcal{M}_{\mathbb{k}}^{E,H}$. Note that $\mathcal{M}_{\mathbb{k}}^{E,E}$ is the category $\mathcal{M}_{\mathbb{k}}^E$ of Example 3.3. We endow $\mathcal{M}_{\mathbb{k}}^{E,H}$ with the monoidal product defined on objects by

$$M \otimes N = \bigoplus_{h \in H} (M \otimes N)_h \quad \text{where} \quad (M \otimes N)_h = \bigoplus_{\substack{x,y \in H \\ xy=h}} M_x \otimes_{\mathbb{k}} N_y,$$

and on morphisms by $\alpha \otimes \beta = \alpha \otimes_{\mathbb{k}} \beta$. This gives a \mathbb{k} -linear monoidal category with unit object $\mathbb{1}$ defined by $\mathbb{1}_1 = \mathbb{k}$ and $\mathbb{1}_h = 0$ for all $h \in H \setminus \{1\}$. An H -graded module $M = \bigoplus_{h \in H} M_h$ is *homogeneous* if it is nonzero and there is $h \in H$ such that $M_x = 0$ for all $x \in H \setminus \{h\}$. Such an h is then unique and called the *degree* of M . With these homogenous objects, $\mathcal{M}_{\mathbb{k}}^{E,H}$ becomes a $(E \hookrightarrow H)$ -category, where the inclusion $E \hookrightarrow H$ is a crossed module with the conjugation action of H on E .

4.6. Examples from representations

Recall that the category of representations of a Hopf algebra is monoidal (and closed). Examples of χ -graded categories can be constructed similarly: in [14], we introduce Hopf χ -(co)algebras and prove that their categories of representations are χ -categories.

4.7. Push-forward of χ -categories

Let $\phi = (\psi: E \rightarrow E', \varphi: H \rightarrow H')$ be a morphism from a crossed module $\chi: E \rightarrow H$ to a crossed module $\chi': E' \rightarrow H'$ (see Section 2.3). Every χ -category \mathcal{C} determines a χ' -category $\phi_*(\mathcal{C})$ called the *push-forward* of \mathcal{C} . By definition, $\phi_*(\mathcal{C}) = \mathcal{C}$ as monoidal categories. The E' -Hom-grading of $\phi_*(\mathcal{C})$ is given for any objects X, Y and $e' \in E$ by

$$\text{Hom}_{\phi_*(\mathcal{C})}^{e'}(X, Y) = \bigoplus_{e \in \psi^{-1}(e')} \text{Hom}_{\mathcal{C}}^e(X, Y).$$

Homogenous objects of $\phi_*(\mathcal{C})$ are the same as those of \mathcal{C} . The degree of a homogenous object X of $\phi_*(\mathcal{C})$ is computed from the degree of X in \mathcal{C} by

$$|X|_{\phi_*(\mathcal{C})} = \varphi(|X|_{\mathcal{C}}).$$

Note that 1-subcategory of $\phi_*(\mathcal{C})$ has the same objects as \mathcal{C} and its morphisms are the linear combinations of homogenous morphisms of \mathcal{C} with degree in $\text{Ker}(\psi)$.

4.8. Pivotal χ -categories

Recall that a monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1})$ is pivotal if it is endowed with a pivotal structure, that is, if each object X of \mathcal{C} has a dual object $X^* \in \mathcal{C}$ and four morphisms

$$\begin{aligned} \text{ev}_X : X^* \otimes X &\rightarrow \mathbb{1}, & \text{coev}_X : \mathbb{1} &\rightarrow X \otimes X^*, \\ \widetilde{\text{ev}}_X : X \otimes X^* &\rightarrow \mathbb{1}, & \widetilde{\text{coev}}_X : \mathbb{1} &\rightarrow X^* \otimes X, \end{aligned}$$

called left/right (co)evaluations, satisfying several conditions which say, in summary, that the associated left/right dual functors coincide as monoidal functors (see [20] for details). In particular, each morphism $\alpha : X \rightarrow Y$ in \mathcal{C} has a dual morphism $\alpha^* : Y^* \rightarrow X^*$ computed by

$$\begin{aligned} \alpha^* &= (\text{ev}_Y \otimes \text{id}_{X^*})(\text{id}_{Y^*} \otimes \alpha \otimes \text{id}_{X^*})(\text{id}_{Y^*} \otimes \text{coev}_X) \\ &= (\text{id}_{X^*} \otimes \widetilde{\text{ev}}_Y)(\text{id}_{X^*} \otimes \alpha \otimes \text{id}_{Y^*})(\widetilde{\text{coev}}_X \otimes \text{id}_{Y^*}). \end{aligned}$$

The *left* and *right traces* of an endomorphism α of an object X of a pivotal category \mathcal{C} are defined by

$$\text{tr}_l(\alpha) = \text{ev}_X(\text{id}_{X^*} \otimes \alpha)\widetilde{\text{coev}}_X \quad \text{and} \quad \text{tr}_r(\alpha) = \widetilde{\text{ev}}_X(\alpha \otimes \text{id}_{X^*})\text{coev}_X.$$

Both traces take values in the commutative monoid $\text{End}_{\mathcal{C}}(\mathbb{1})$ and are symmetric: $\text{tr}_l(\beta\gamma) = \text{tr}_l(\gamma\beta)$ for any morphisms $\beta : X \rightarrow Y, \gamma : Y \rightarrow X$ in \mathcal{C} and similarly for tr_r . Also $\text{tr}_l(\alpha^*) = \text{tr}_r(\alpha)$ and $\text{tr}_r(\alpha^*) = \text{tr}_l(\alpha)$ for any endomorphism in \mathcal{C} . The *left* and *right dimensions* of an object X of \mathcal{C} are defined by

$$\dim_l(X) = \text{tr}_l(\text{id}_X) \in \text{End}_{\mathcal{C}}(\mathbb{1}) \quad \text{and} \quad \dim_r(X) = \text{tr}_r(\text{id}_X) \in \text{End}_{\mathcal{C}}(\mathbb{1}).$$

Clearly, $\dim_l(X^*) = \dim_r(X)$ and $\dim_r(X^*) = \dim_l(X)$ for all X .

A χ -category is *pivotal* if it is endowed with a pivotal structure such that:

- (a) The dual X^* of any homogenous object X is homogenous of degree

$$|X^*| = |X|^{-1}.$$

- (b) All (co)evaluations morphisms are homogenous of degree $1 \in E$.

The second axiom implies that the 1-subcategory of a pivotal χ -category is itself pivotal. In a pivotal χ -category \mathcal{C} , the dual $\alpha^* : Y^* \rightarrow X^*$ of a nonzero homogenous morphism $\alpha : X \rightarrow Y$ between homogenous objects is homogenous of degree

$$|\alpha^*| = |Y|^{-1}|\alpha| = |X|^{-1}|\alpha|.$$

This follows from the above expressions of the dual morphism. (Note that the latter equality may also be deduced from (7) and the crossed module axioms.) Moreover, for any $e \in E$ and any homogeneous endomorphism $\alpha \in \text{End}_{\mathcal{C}}^e(X)$ of a homogenous object X of \mathcal{C} ,

$$\text{tr}_l(\alpha) \in \text{End}_{\mathcal{C}}^{(|X|^{-1}e)}(\mathbb{1}) \quad \text{and} \quad \text{tr}_r(\alpha) \in \text{End}_{\mathcal{C}}^e(\mathbb{1}).$$

This together with the fact that objects of \mathcal{C} are 1-direct sum of homogeneous objects imply that both traces of homogeneous endomorphisms of degree 1 take values in $\text{End}_{\mathcal{C}}^1(\mathbb{1})$. In particular, both dimensions of objects of \mathcal{C} take values in $\text{End}_{\mathcal{C}}^1(\mathbb{1})$.

4.9. Spherical χ -categories

A spherical χ -category is a pivotal χ -category \mathcal{C} such that its 1-subcategory \mathcal{C}^1 is spherical (as a pivotal category), that is, if the left and right traces of any homogenous endomorphism of degree 1 are equal.

In a spherical χ -category, the *trace* of a homogeneous endomorphism α of degree 1 is defined by

$$\text{tr}(\alpha) = \text{tr}_l(\alpha) = \text{tr}_r(\alpha) \in \text{End}_{\mathcal{C}}^1(\mathbb{1}).$$

Likewise, the *dimension* of an object X is defined by

$$\dim(X) = \text{tr}(\text{id}_X) = \dim_l(X) = \dim_r(X) \in \text{End}_{\mathcal{C}}^1(\mathbb{1}).$$

4.10. χ -fusion categories

A χ -category \mathcal{C} is *pre-fusion* if it is E -semisimple (see Section 3.5) and the unit object $\mathbb{1}$ is simple in the 1-subcategory \mathcal{C}^1 of \mathcal{C} , that is, $\text{End}_{\mathcal{C}}^1(\mathbb{1})$ is free of rank 1. The map $\mathbb{k} \rightarrow \text{End}_{\mathcal{C}}^1(\mathbb{1})$, $k \mapsto k \text{id}_{\mathbb{1}}$ is then a \mathbb{k} -algebra isomorphism which we use to identify $\text{End}_{\mathcal{C}}^1(\mathbb{1}) = \mathbb{k}$. In a pre-fusion χ -category, if an object X is a 1-direct sum of homogeneous objects of degree in $D \subset H$ and i is a homogeneous simple object of \mathcal{C}^1 , then $N_X^{i,e} = 0$ for all $e \in E$ such that $\chi(e)|i| \notin D$. The left and right dimensions of a simple object in the 1-subcategory of a pivotal pre-fusion χ -category are invertible (see for example [20, Lemma 4.2]).

A χ -representative set for a pre-fusion χ -category \mathcal{C} is a set I of simple objects of the 1-subcategory \mathcal{C}^1 such that $\mathbb{1} \in I$, all elements of I are homogeneous, and every simple object of \mathcal{C}^1 is 1-isomorphic to a unique element of I . Note that I is then representative set of simple objects of \mathcal{C}^1 in the sense of Section 3.1. Any such χ -representative set I splits as a disjoint union $I = \coprod_{h \in H} I_h$ where I_h is the set of all elements of I of degree h .

Moreover, any homogeneous object of \mathcal{C} of degree $h \in H$ is a 1-direct sum of a finite family of elements in I_h .

By a χ -fusion category (over \mathbb{k}) we mean a pre-fusion χ -category \mathcal{C} (over \mathbb{k}) such that for any $h \in H$, the set of 1-isomorphism classes of degree h homogeneous simple objects of \mathcal{C}^1 is finite and non-empty. In particular, any χ -representative set $I = \coprod_{h \in H} I_h$ for a χ -fusion category is such that I_h is finite for all $h \in H$.

Let \mathcal{C} be a pivotal χ -fusion category. Recall from Section 4.2 that the 1-subcategory \mathcal{C}^1 of \mathcal{C} decomposes as $\mathcal{C}^1 = \bigoplus_{h \in H} \mathcal{C}_h^1$. Then \mathcal{C}^1 is a pivotal H -fusion category in the sense of [17], and its neutral component \mathcal{C}_1^1 is a pivotal fusion category in the usual sense. The dimension of \mathcal{C}_1^1 (as a pivotal fusion category) is defined by

$$\dim(\mathcal{C}_1^1) = \sum_{i \in J} \dim_l(i) \dim_r(i) \in \text{End}_{\mathcal{C}}^1(\mathbb{1}) = \mathbb{k},$$

where J is any (finite) representative set of simple objects of \mathcal{C}_1^1 . Note that if \mathbb{k} is an algebraically closed field of characteristic zero, then $\dim(\mathcal{C}_1^1) \neq 0$ (see [4]). By [17, Lemma 4.1], if $I = \coprod_{h \in H} I_h$ is a χ -representative set for \mathcal{C} , then for all $h \in H$,

$$\sum_{i \in I_h} \dim_l(i) \dim_r(i) = \dim(\mathcal{C}_h^1). \tag{8}$$

4.11. Remark

If $\chi: E \rightarrow H$ is injective (for example when $E = 1$), then any χ -fusion category \mathcal{C} , endowed with the $\text{Coker}(\chi)$ -grading defined in Section 4.2, is a $\text{Coker}(\chi)$ -fusion category (in the sense of [17]) whose neutral component has dimension $\dim(\mathcal{C}_1^1)$.

4.12. Example

The χ -category $\mathbb{k}\mathcal{G}_\chi$ from Example 4.4 has a canonical pivotal structure: the dual of an object $h \in H$ is $h^* = h^{-1}$ with (co)evaluation morphisms all given by the unit element $1 \in E$. Then $\mathbb{k}\mathcal{G}_\chi$ is a spherical χ -fusion category with χ -representative set $I = H$ and with $\dim((\mathbb{k}\mathcal{G}_\chi)_1^1) = 1_{\mathbb{k}}$.

4.13. Example

Let E be a normal subgroup of a group H . The inclusion $E \hookrightarrow H$ is a crossed module with the conjugation action of H on E . Denote by $(E, H)\text{-vect}_{\mathbb{k}}$ the category of H -graded free \mathbb{k} -modules of finite rank and E -graded \mathbb{k} -linear homomorphisms. This is an $(E \hookrightarrow H)$ -category (as a full subcategory of the category $\mathcal{M}_{\mathbb{k}}^{E, H}$ from Example 4.5). It has a canonical pivotal structure: the dual of an object $M = \bigoplus_{h \in H} M_h$ is

$$M^* = \bigoplus_{h \in H} (M^*)_h \quad \text{where} \quad (M^*)_h = \text{Hom}_{\mathbb{k}}(M_{h^{-1}}, \mathbb{k})$$

with evaluations $ev_M: M^* \otimes M \rightarrow \mathbb{1}$ and $\tilde{ev}_M: M \otimes M^* \rightarrow \mathbb{1}$ given for any $h, k \in H$, $m \in M_h$, and $f \in (M^*)_k$ by

$$ev_M(f \otimes_k m) = \tilde{ev}_M(m \otimes_k f) = \delta_{k, h^{-1}} f(m).$$

Then (E, H) - $\text{vect}_{\mathbb{k}}$ is a spherical $(E \hookrightarrow H)$ -fusion category. For any $h \in H$, let \mathbb{k}_h be the H -graded free \mathbb{k} -module of rank 1 defined by $(\mathbb{k}_h)_h = \mathbb{k}$ and $(\mathbb{k}_h)_x = 0$ for $x \in H \setminus \{h\}$. Note that \mathbb{k}_h is homogeneous of degree h . Then $\{\mathbb{k}_h\}_{h \in H}$ is a $(E \hookrightarrow H)$ -representative set for (E, H) - $\text{vect}_{\mathbb{k}}$. Clearly, $\dim(\mathbb{k}_h) = 1_{\mathbb{k}} \in \text{End}_{\mathcal{C}}^1(\mathbb{1}) = \mathbb{k}$ for all $h \in H$. In particular, $\dim(((E, H)\text{-vect}_{\mathbb{k}})_1^1) = 1_{\mathbb{k}}$. Also, for all $g, h \in H$ and $e \in E$,

$$\mathbb{k}_g \otimes \mathbb{k}_h \cong_1 \mathbb{k}_{gh}, \quad \mathbb{k}_h^* \cong_1 \mathbb{k}_{h^{-1}}, \quad \text{and} \quad \mathbb{k}_g \cong_e \mathbb{k}_h \iff \chi(e) = hg^{-1}.$$

Note that by Remark 4.11, (E, H) - $\text{vect}_{\mathbb{k}}$ is (H/E) -fusion in the sense of [17] with neutral component of dimension $1_{\mathbb{k}}$.

4.14. *Push forward of fusion χ -categories*

Let $\phi = (\psi: E \rightarrow E', \varphi: H \rightarrow H')$ be a morphism from a crossed module $\chi: E \rightarrow H$ to a crossed module $\chi': E' \rightarrow H'$ (see Section 2.3). Assume that ψ and φ are surjective, $\text{Ker}(\psi) \cap \text{Ker}(\chi) = 1$, and $\text{Ker}(\varphi)$ is finite. These conditions imply that the map χ induces an injective group homomorphism $\text{Ker}(\psi) \hookrightarrow \text{Ker}(\varphi)$, and so $\text{Ker}(\psi)$ is finite and its order divides that of $\text{Ker}(\varphi)$. Then the push-forward $\phi_*(\mathcal{C})$ of any χ -fusion category \mathcal{C} (defined in Section 4.7) is a χ' -fusion category. Also, if \mathcal{C} is pivotal or spherical, then so is $\phi_*(\mathcal{C})$ (with the same pivotal structure) and

$$\dim(\phi_*(\mathcal{C})_1^1) = \frac{\text{card}(\text{Ker}(\varphi))}{\text{card}(\text{Ker}(\psi))} \dim(\mathcal{C}_1^1).$$

Indeed, for any homogeneous object X of \mathcal{C} , we have:

$$\text{End}_{\phi_*(\mathcal{C})}^1(X) = \bigoplus_{e \in \text{Ker}(\psi) \cap \text{Ker}(\chi) = 1} \text{End}_e^{\mathcal{C}}(X) = \text{End}_{\mathcal{C}}^1(X),$$

and so an object is simple in $\phi_*(\mathcal{C})^1$ if and only if it is simple in \mathcal{C}^1 . Then the semisimplicity of $\phi_*(\mathcal{C})^1$ follows from that of \mathcal{C}^1 . Also, the E -semisimplicity of \mathcal{C} together with the surjectivity of ψ imply that $\phi_*(\mathcal{C})$ is E' -semisimple and so a pre-fusion χ' -category. Moreover, the facts that \mathcal{C} is a χ -fusion and φ is surjective with finite kernel imply that $\phi_*(\mathcal{C})$ is a χ' -fusion category. Finally, the formula for the dimension of $\phi_*(\mathcal{C})_1^1$ is proved in Claim 7.5 below.

4.15. *Example*

A crossed module $\chi: E \rightarrow H$ comes with a left action of H on E and so induces the semidirect product group $H \ltimes E = (H \times E, *)$ where

$$(x, e) * (y, f) = (xy, e^x f).$$

Then $E = 1 \times E$ is a normal subgroup of $H \ltimes E$ and so the inclusion $\iota: E \hookrightarrow H \ltimes E$ is a crossed module (with the conjugation action of $H \ltimes E$ on E). Consider spherical ι -fusion category $(E, H \ltimes E)\text{-vect}_{\mathbb{k}}$ from Example 4.13. The pair

$$\phi = (\text{id}_E: E \rightarrow E, \varphi: H \ltimes E \rightarrow H) \quad \text{with} \quad \varphi(h, e) = \chi(e)h$$

is a crossed module morphism from ι to χ . Therefore, by Section 4.7, the push-forward $\phi_*((E, H \ltimes E)\text{-vect}_{\mathbb{k}})$ is a χ -category denoted $\chi\text{-vect}_{\mathbb{k}}$. More explicitly, the objects of $\chi\text{-vect}_{\mathbb{k}}$ are the $(H \ltimes E)$ -graded free \mathbb{k} -modules of finite rank. The degree of a homogeneous object $M = M_{(h,e)}$ is $|M| = \chi(e)h$. The morphisms of $\chi\text{-vect}_{\mathbb{k}}$ are the E -graded \mathbb{k} -linear homomorphisms (see Example 4.5). Note that id_E and φ are surjective, $\text{Ker}(\text{id}_E) \cap \text{Ker}(\chi) = 1$, and there is a group isomorphism

$$\text{Ker}(\varphi) = \{(h, e) \in H \ltimes E \mid \chi(e)h = 1\} \cong E, \quad (h, e) \mapsto e^{-1}.$$

Consequently, by Section 4.14, if E is finite, then $\chi\text{-vect}_{\mathbb{k}}$ is a spherical χ -fusion category and

$$\dim((\chi\text{-vect}_{\mathbb{k}})_1^1) = \text{card}(E)1_{\mathbb{k}}.$$

4.16. *Twisting χ -categories by cocycles*

By a 3-cocycle for the crossed module $\chi: E \rightarrow H$ with values in an abelian group A , we mean a map $\omega: H^3 \times E^3 \rightarrow A$ such that for all $(x, y, z, t) \in H^4$ and $(a, b, c, d, e, f) \in E^6$,

$$\begin{aligned} &\omega(x, y, z, a, b, c) \omega(x, \chi(c)yz, t, ba^x(c^{-1}), f, d) \omega(y, z, t, c, d, e) \\ &= \omega(\chi(a)xy, z, t, b, f, e) \omega(x, y, \chi(e)zt, a, fba^{xy}(e^{-1})a^{-1}, dc^y(e^{-1})). \end{aligned}$$

Note that the map $\tilde{\omega}: H^3 \rightarrow A$, defined by $\tilde{\omega}(x, y, z) = \omega(x, y, z, 1, 1, 1)$, is then a (usual) 3-cocycle for the group H with values in A . A 3-cocycle ω for χ is *normalized* if for all $x, y \in H$ and $e \in E$,

$$\omega(1, x, y, 1, e, 1) = 1_A = \omega(x, 1, y, 1, 1, 1).$$

Normalized 3-cocycles for χ with values in A can be used to describe the third cohomology classes in $H^3(B\chi, A)$, where $B\chi$ is the classifying space of χ (see [6] for the unnormalized case).

Any normalized 3-cocycle ω for χ with values in the group \mathbb{k}^* can be used to twist the composition of morphisms, the monoidal product of morphisms, and the associativity constraints of a χ -category \mathcal{C} (over \mathbb{k}) to obtain a new χ -category \mathcal{C}^ω (over \mathbb{k}). More precisely, the objects, the homogeneous objects, the degree of homogeneous objects, the monoidal product of objects, the unit object $\mathbb{1}$, the Hom-sets (as E -graded \mathbb{k} -modules), and the identity morphisms for \mathcal{C}^ω are the same as those for \mathcal{C} . The composition of morphisms in \mathcal{C}^ω is given, for all $\alpha \in \text{Hom}_{\mathcal{C}}^e(X, Y)$ and $\beta \in \text{Hom}_{\mathcal{C}}^f(Y, Z)$ with X homogeneous, by

$$\beta \circ^\omega \alpha = \omega(|X|, 1, 1, e, f, 1)^{-1} \beta \alpha \in \text{Hom}_{\mathcal{C}}^{fe}(X, Z).$$

The monoidal product of morphisms in \mathcal{C}^ω is given, for all $\alpha \in \text{Hom}_{\mathcal{C}}^e(X, Y)$ and $\beta \in \text{Hom}_{\mathcal{C}}^f(Z, T)$ with X, Y homogeneous, by

$$\alpha \otimes^\omega \beta = \frac{\omega(|X|, |Y|, 1, e^{|X|}f, 1, f)}{\omega(|X|, 1, \chi(f)|Y|, e, 1, 1)} \alpha \otimes \beta \in \text{Hom}_{\mathcal{C}}^{e^{|X|}f}(X \otimes Z, Y \otimes T).$$

For any homogeneous objects X, Y, Z , the monoidal constraint

$$(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$$

of \mathcal{C}^ω is $\tilde{\omega}(|X|, |Y|, |Z|)$ times that of \mathcal{C} , where $\tilde{\omega}: H^3 \rightarrow \mathbb{k}^*$ is the 3-cocycle for H derived from ω as above. Since $\tilde{\omega}$ is normalized (because ω is), the unitality constraints $X \otimes \mathbb{1} \cong X \cong X \otimes \mathbb{1}$ of \mathcal{C}^ω are those of \mathcal{C} .

Note that $(\mathcal{C}^\omega)_1 = \mathcal{C}_1^1$ as \mathbb{k} -linear monoidal categories. Also, recall that the 1-subcategory of a χ -category is H -graded (see Section 4.2). Then $(\mathcal{C}^\omega)^1$ is the twisted H -graded category $(\mathcal{C}^1)^{\tilde{\omega}}$, that is, the H -graded category \mathcal{C}^1 with monoidal constraints twisted by the 3-cocycle $\tilde{\omega}$.

If \mathcal{C} is pivotal, then so is \mathcal{C}^ω with the same dual objects and with the following twisted (co)evaluations: for any homogeneous object X ,

$$\begin{aligned} \text{ev}_X^\omega &= \text{ev}_X, & \text{coev}_X^\omega &= \tilde{\omega}(|X|, |X|^{-1}, |X|)^{-1} \text{coev}_X, \\ \widetilde{\text{coev}}_X^\omega &= \widetilde{\text{coev}}_X, & \tilde{\text{ev}}_X^\omega &= \tilde{\omega}(|X|, |X|^{-1}, |X|) \tilde{\text{ev}}_X. \end{aligned}$$

If \mathcal{C} is spherical, then so is \mathcal{C}^ω . Also, if \mathcal{C} is pre-fusion, then so is \mathcal{C}^ω with the same χ -representative sets. Finally, if \mathcal{C} is χ -fusion, then so is \mathcal{C}^ω .

4.17. Penrose graphical calculus

We will represent morphisms in a category \mathcal{C} by plane diagrams to be read from the bottom to the top. The diagrams are made of oriented arcs colored by objects of \mathcal{C} and of boxes colored by morphisms of \mathcal{C} . The arcs connect the boxes and have no mutual

intersections or self-intersections. The identity id_X of $X \in \mathcal{C}$, a morphism $f: X \rightarrow Y$, and the composition of two morphisms $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are represented as follows:

$$\text{id}_X = \downarrow_X, \quad f = \begin{array}{c} \downarrow_Y \\ \boxed{f} \\ \downarrow_X \end{array}, \quad \text{and} \quad gf = \begin{array}{c} \downarrow_Z \\ \boxed{g} \\ \downarrow_Y \\ \boxed{f} \\ \downarrow_X \end{array}.$$

If \mathcal{C} is monoidal, then the monoidal product of two morphisms $f: X \rightarrow Y$ and $g: Z \rightarrow T$ is represented by juxtaposition:

$$f \otimes g = \begin{array}{cc} \downarrow_Y & \downarrow_T \\ \boxed{f} & \boxed{g} \\ \downarrow_X & \downarrow_Z \end{array}.$$

Suppose that \mathcal{C} is pivotal. By convention, if an arc colored by $X \in \mathcal{C}$ is oriented upwards, then the corresponding object in the source/target of morphisms is X^* . For example, id_{X^*} and a morphism $f: X^* \otimes Y \rightarrow U \otimes V^* \otimes W$ may be depicted as

$$\text{id}_{X^*} = \uparrow_X = \downarrow_{X^*} \quad \text{and} \quad f = \begin{array}{c} \downarrow_U \downarrow_{V^*} \downarrow_W \\ \boxed{f} \\ \uparrow_X \downarrow_Y \end{array}.$$

The duality morphisms are depicted as follows:

$$\text{ev}_X = \frown_X, \quad \text{coev}_X = \smile_X, \quad \widetilde{\text{ev}}_X = \smile_X, \quad \widetilde{\text{coev}}_X = \frown_X.$$

The dual of a morphism $f: X \rightarrow Y$ and the traces of an endomorphism $\alpha: X \rightarrow X$ can be depicted as follows:

$$f^* = \begin{array}{c} \uparrow_Y \\ \boxed{f} \\ \downarrow_X \end{array} = \begin{array}{c} \downarrow_X \\ \boxed{f} \\ \uparrow_Y \end{array} \quad \text{and} \quad \text{tr}_l(\alpha) = \begin{array}{c} \uparrow_X \\ \boxed{\alpha} \\ \downarrow_X \end{array}, \quad \text{tr}_r(\alpha) = \begin{array}{c} \downarrow_X \\ \boxed{\alpha} \\ \uparrow_X \end{array}.$$

The morphisms represented by such diagrams are invariant under isotopies of the diagrams in \mathbb{R}^2 keeping the bottom and top endpoints fixed (see for example [7] or [20, Theorem 2.6]).

4.18. Enriched graphical calculus

For a pivotal pre-fusion χ -category \mathcal{C} , we enrich the Penrose graphical calculus of the previous section. Let X be an object of \mathcal{C} , i be a simple object of the 1-subcategory \mathcal{C}^1 of \mathcal{C} , and $e \in E$. Consider a (finite) formal sum of diagrams

$$\sum_{\alpha \in \Lambda} \left(\begin{array}{c} \downarrow i \\ \boxed{p_\alpha} \\ \downarrow X \end{array} \quad \begin{array}{c} \downarrow X \\ \boxed{q_\alpha} \\ \downarrow i \end{array} \right) \tag{9}$$

where $(p_\alpha: X \rightarrow i, q_\alpha: i \rightarrow X)_{\alpha \in \Lambda}$ is an (i, e) -partition of X (see Section 3.6) and the gray area represents a part of these diagrams independent of $\alpha \in \Lambda$ (and, in particular, not involving p_α, q_α). By the Penrose graphical calculus and the \mathbb{k} -linearity of \mathcal{C} , the sum (9) represents a morphism in \mathcal{C} . Since the tensor

$$\sum_{\alpha \in \Lambda} p_\alpha \otimes q_\alpha \in \text{Hom}_{\mathcal{C}}^{e^{-1}}(X, i) \otimes_{\mathbb{k}} \text{Hom}_{\mathcal{C}}^e(i, X) \tag{10}$$

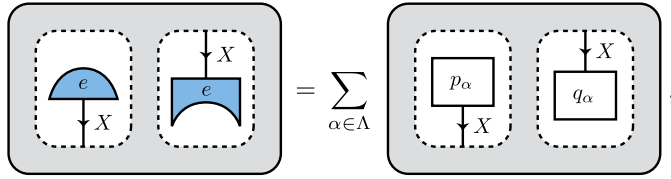
does not depend on the choice of the (i, e) -partition of X , the morphism (9) in \mathcal{C} also does not depend on this choice. We therefore can eliminate the colors p_α, q_α of the two boxes, keeping in mind only the order of the boxes and the fact that they jointly stand for the tensor (10). We will graphically represent this pair of boxes by two curvilinear boxes (a semi-disk and a compressed rectangle) labeled by e and painted with the same color which stand respectively for p_α and q_α where α runs over Λ :

$$\left(\begin{array}{c} \downarrow i \\ \text{semi-disk } e \\ \downarrow X \end{array} \quad \begin{array}{c} \downarrow X \\ \text{compressed rectangle } e \\ \downarrow i \end{array} \right) = \sum_{\alpha \in \Lambda} \left(\begin{array}{c} \downarrow i \\ \boxed{p_\alpha} \\ \downarrow X \end{array} \quad \begin{array}{c} \downarrow X \\ \boxed{q_\alpha} \\ \downarrow i \end{array} \right). \tag{11}$$

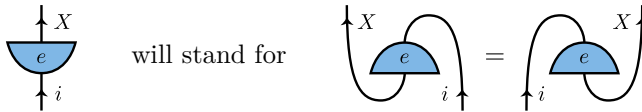
The gray areas in the picture are the same as above. We will also use similar notation obtained from (11) by reorienting the X -colored arcs upward and replacing $(p_\alpha, q_\alpha)_{\alpha \in \Lambda}$ with an (i, e) -partition of X^* , or by reorienting the i -colored arcs upward and replacing $(p_\alpha, q_\alpha)_{\alpha \in \Lambda}$ with an (i^*, e) -partition of X . We will allow several arcs to be attached to the bottom of the semi-disk and to the top of the compressed rectangle in (11). The number of these arcs, their directions (up/down), and their colors should be the same. For example,

$$\left(\begin{array}{c} \uparrow i \\ \text{semi-disk } e \\ \uparrow X \uparrow Y \end{array} \quad \begin{array}{c} \uparrow X \uparrow Y \\ \text{compressed rectangle } e \\ \uparrow i \end{array} \right) = \sum_{\alpha \in \Lambda} \left(\begin{array}{c} \uparrow i \\ \boxed{p_\alpha} \\ \uparrow X \uparrow Y \end{array} \quad \begin{array}{c} \uparrow X \uparrow Y \\ \boxed{q_\alpha} \\ \uparrow i \end{array} \right),$$

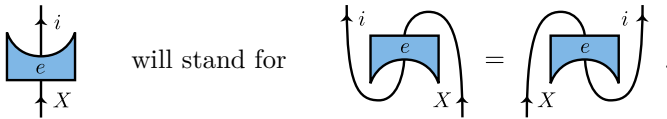
where $(p_\alpha, q_\alpha)_{\alpha \in \Lambda}$ is any (i^*, e) -partition of $X^* \otimes Y$. We will allow to erase i -colored arcs for $i = \mathbb{1}$. In particular,



where $(p_\alpha, q_\alpha)_{\alpha \in \Lambda}$ is any $(\mathbb{1}, e)$ -partition of X . Furthermore, in accordance with the isotopy invariance,



and



Similar notation will be applied when the X -colored or i -colored arcs are oriented downward and when there are several arcs attached to the horizontal sides of the curvilinear boxes.

Every enriched diagram as above represents a morphism in \mathcal{C} which is invariant under ambient isotopies (relative endpoints) of the diagram in the plane.

We now state four properties of the above enriched graphical calculus associated with the pivotal pre-fusion χ -category \mathcal{C} . First, for any object X of \mathcal{C} , any simple object i of \mathcal{C}^1 , and any $e \in E$,

$$\begin{array}{c} \downarrow i \\ \text{blue semi-circle } e \\ \downarrow X \\ \text{blue inverted semi-circle } e \\ \downarrow i \end{array} = N_X^{i,e} \left. \begin{array}{c} | \\ \downarrow i \end{array} \right\} \tag{12}$$

where $N_X^{i,e}$ is the degree e multiplicity index of i in X . This equality follows from the fact that for any (i, e) -partition $(p_\alpha, q_\alpha)_{\alpha \in \Lambda}$ of X , we have $p_\alpha q_\alpha = \text{id}_i$ for all $\alpha \in \Lambda$ and $\text{card}(\Lambda) = N_X^{i,e}$. Second, pick a χ -representative set I for \mathcal{C} . For any object X of \mathcal{C} and $e \in E$,

$$\sum_{i \in I} \begin{array}{c} \downarrow X \\ \text{blue inverted semi-circle } e \\ \downarrow i \\ \text{blue semi-circle } e \\ \downarrow X \end{array} = \left. \begin{array}{c} | \\ \downarrow X \end{array} \right\} \tag{13}$$

This follows from the fact that the union of (i, e) -partitions of X over all $i \in I$ is an (I, e) -partition of X . Third, for any homogenous morphisms $\alpha \in \text{Hom}_{\mathcal{C}}^e(\mathbb{1}, X)$ and $\beta \in \text{Hom}_{\mathcal{C}}^{e^{-1}}(X, \mathbb{1})$ with $e \in E$, we have

$$\begin{array}{c} \downarrow X \\ \text{[e]} \\ \text{[e]} \\ \downarrow X \\ \square \alpha \end{array} = \begin{array}{c} \downarrow X \\ \square \alpha \end{array} \quad \text{and} \quad \begin{array}{c} \square \beta \\ \downarrow X \\ \text{[e]} \\ \text{[e]} \\ \downarrow X \end{array} = \begin{array}{c} \square \beta \\ \downarrow X \end{array} \quad (14)$$

These equalities follow from (13) and the fact $\text{Hom}_{\mathcal{C}}^1(\mathbb{1}, i) = 0 = \text{Hom}_{\mathcal{C}}^1(i, \mathbb{1})$ for all $i \in I \setminus \{\mathbb{1}\}$. Fourth, for any object X of \mathcal{C} and $e \in E$,

$$\dim_l(X) = \sum_{i \in I} \dim_l(i) N_X^{i,e} \quad \text{and} \quad \dim_r(X) = \sum_{i \in I} \dim_r(i) N_X^{i,e}. \quad (15)$$

Here, recall that $N_X^{i,e} = 0$ for all but a finite number of $i \in I$. These equalities follow from Formulas (12) and (13) together with the isotopy invariance of the enriched graphical calculus:

$$\begin{aligned} \dim_l(X) &= \text{circle}_X = \sum_{i \in I} \left(\text{[e]} \downarrow X \text{[e]} \text{---} i \text{---} \right) = \sum_{i \in I} \left(\text{[e]} \downarrow X \text{[e]} \text{---} i \text{---} \right) \\ &= \sum_{i \in I} \left(\text{[e]} \downarrow X \text{[e]} \text{---} i \text{---} \right) = \sum_{i \in I} N_X^{i,e} \text{circle}_i = \sum_{i \in I} \dim_l(i) N_X^{i,e} \end{aligned}$$

and similarly for the right trace. As an application, we prove the following claim.

Lemma 4.1. *Let \mathcal{C} be a pivotal χ -fusion category and let $I = \coprod_{h \in H} I_h$ be a χ -representative set for \mathcal{C} . Then, for any homogeneous objects X, Y of \mathcal{C} , $g, h \in H$, and $e \in E$ such that $\chi(e) = |X|g|Y|h$, we have:*

$$\sum_{\substack{m \in I_g \\ n \in I_h}} \dim_l(m) \dim_l(n) N_{X \otimes m \otimes Y \otimes n}^{1,e} = \dim_r(X) \dim_r(Y) \dim(\mathcal{C}_1^1)$$

and similarly with the subscripts l, r exchanged.

Proof. Observe first that for any object Z of \mathcal{C} ,

$$N_{Z \otimes n}^{1,e} = N_Z^{n*,e}.$$

Indeed the free \mathbb{k} -modules $\text{Hom}_{\mathcal{C}}^e(\mathbb{1}, Z \otimes n)$ and $\text{Hom}_{\mathcal{C}}^e(n^*, Z)$ have the same rank since the map $\alpha \in \text{Hom}_{\mathcal{C}}^e(\mathbb{1}, Z \otimes n) \mapsto (\text{id}_Z \otimes \tilde{v}_n)(\alpha \otimes \text{id}_{n^*}) \in \text{Hom}_{\mathcal{C}}^e(n^*, Z)$ is a well-defined \mathbb{k} -linear isomorphism. Thus, using the identity $\dim_l(n) = \dim_r(n^*)$, we obtain:

$$\begin{aligned} \sum_{\substack{m \in I_g \\ n \in I_h}} \dim_l(m) \dim_l(n) N_{X \otimes m \otimes Y \otimes n}^{1,e} &= \sum_{\substack{m \in I_g \\ n \in I_h}} \dim_l(m) \dim_r(n^*) N_{X \otimes m \otimes Y}^{n^*,e} \\ &= \sum_{m \in I_g} \dim_l(m) \sum_{i \in I_{h-1}} \dim_r(i) N_{X \otimes m \otimes Y}^{i,e}. \end{aligned}$$

Now, for any $m \in I_g$, since $X \otimes m \otimes Y$ is a 1-direct sum of homogenous objects of degree $|X|g|Y|$, we have $N_{X \otimes m \otimes Y}^{i,e} = 0$ whenever $i \in I$ satisfies $\chi(e)|i| \neq |X|g|Y|$ or equivalently $|i| \neq h^{-1}$ (see Section 4.10) and so Formula (15) gives that

$$\sum_{i \in I_{h-1}} \dim_r(i) N_{X \otimes m \otimes Y}^{i,e} = \sum_{i \in I} \dim_r(i) N_{X \otimes m \otimes Y}^{i,e} = \dim_r(X \otimes m \otimes Y).$$

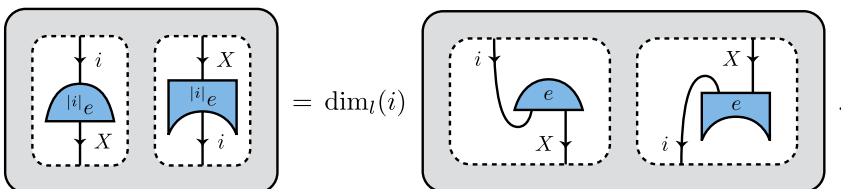
Then, using the \otimes -multiplicativity of the dimensions (which follows from the fact that the unit object $\mathbb{1}$ is simple in \mathcal{C}^1 , see Sections 2.6.2 and 4.2.2 of [20]) and Formula (8), we obtain:

$$\begin{aligned} \sum_{\substack{m \in I_g \\ n \in I_h}} \dim_l(m) \dim_l(n) N_{X \otimes m \otimes Y \otimes n}^{1,e} &= \sum_{m \in I_g} \dim_l(m) \dim_r(X \otimes m \otimes Y) \\ &= \dim_r(X) \dim_r(Y) \sum_{m \in I_g} \dim_l(m) \dim_r(m) = \dim_r(X) \dim_r(Y) \dim(\mathcal{C}_1^1). \end{aligned}$$

A similar proof works when the subscripts l, r are exchanged. \square

In the next lemma, each equality has gray areas which are supposed to be the same on both sides.

Lemma 4.2. *For any object X of \mathcal{C} , any homogeneous simple object i of \mathcal{C}^1 , and any $e \in E$, we have:*



Proof. Set $a = |i|e$. Pick a $(\mathbb{1}, e)$ -partition $(p_\alpha, q_\alpha)_{\alpha \in \Lambda}$ of $i^* \otimes X$. For each $\alpha \in \Lambda$, set

$$P_\alpha = \dim_l(i) \begin{array}{c} i \downarrow \\ \text{---} \downarrow \text{---} \\ \boxed{p_\alpha} \\ \text{---} \downarrow \text{---} \\ X \downarrow \end{array} \in \text{Hom}_{\mathcal{C}}^{|i|e^{-1}}(X, i) = \text{Hom}_{\mathcal{C}}^{a^{-1}}(X, i)$$

and

$$Q_\alpha = \begin{array}{c} X \downarrow \\ \text{---} \downarrow \text{---} \\ \boxed{q_\alpha} \\ \text{---} \downarrow \text{---} \\ i \downarrow \end{array} \in \text{Hom}_{\mathcal{C}}^{|i|e}(i, X) = \text{Hom}_{\mathcal{C}}^a(i, X).$$

Since $(p_\alpha)_{\alpha \in \Lambda}$ is a basis of the \mathbb{k} -module $\text{Hom}_{\mathcal{C}}^{e^{-1}}(i^* \otimes X, \mathbb{1})$ and $\dim_l(i)$ is invertible in \mathbb{k} , the family $(P_\alpha)_{\alpha \in \Lambda}$ is a basis of the \mathbb{k} -module $\text{Hom}_{\mathcal{C}}^{a^{-1}}(X, i)$. Since $(q_\alpha)_{\alpha \in \Lambda}$ is a basis of the \mathbb{k} -module $\text{Hom}_{\mathcal{C}}^e(\mathbb{1}, i^* \otimes X)$, the family $(Q_\alpha)_{\alpha \in \Lambda}$ is a basis of the \mathbb{k} -module $\text{Hom}_{\mathcal{C}}^a(i, X)$. Since the object i is simple in \mathcal{C}^1 , for any $\alpha, \beta \in \Lambda$, we have $P_\alpha Q_\beta \in \text{End}_{\mathcal{C}}^1(i) = \mathbb{k} \text{id}_i$ and so $P_\alpha Q_\beta = \lambda_{\alpha, \beta} \text{id}_i$ where

$$\lambda_{\alpha, \beta} \stackrel{(i)}{=} \frac{\text{tr}_l(P_\alpha Q_\beta)}{\dim_l(i)} \stackrel{(ii)}{=} \begin{array}{c} \boxed{p_\alpha} \\ \text{---} \downarrow \text{---} \\ i \uparrow \downarrow X \\ \text{---} \downarrow \text{---} \\ \boxed{q_\beta} \end{array} \stackrel{(iii)}{=} \delta_{\alpha, \beta}.$$

Here (i) follows from the invertibility of $\dim_l(i)$, (ii) from the definitions of P_α, Q_β and the isotopy invariance of the graphical calculus, and (iii) from the equality $p_\alpha q_\beta = \delta_{\alpha, \beta}$. Thus $(P_\alpha, Q_\alpha)_{\alpha \in \Lambda}$ is an (i, a) -partition of X . This directly implies the lemma. \square

5. Multiplicity modules and χ -graphs

Throughout this section, $\chi: E \rightarrow H$ is a crossed module and \mathcal{C} is a pivotal χ -category. We first introduce \mathcal{C} -colored χ -cyclic sets and their multiplicity modules. Then we define an isotopy invariant of \mathcal{C} -colored χ -graphs.

5.1. Multiplicity modules of colored χ -cyclic sets

A χ -cyclic set is a totally cyclically ordered finite set S endowed with maps $\alpha: S \rightarrow H$, $\beta: S \rightarrow E$, and $\varepsilon: S \rightarrow \{+, -\}$ such that for any $s \in S$,

$$\chi(\beta(s)) = \alpha(s_1)^{\varepsilon(s_1)} \dots \alpha(s_n)^{\varepsilon(s_n)} \tag{16}$$

where n is the number of elements of S and $s = s_1 < s_2 < \dots < s_n$ are the elements of S ordered in the given cyclic order starting from s , and

$$\beta(\text{succ}(s)) = (\alpha(s)^{-\varepsilon(s)})\beta(s) \tag{17}$$

where $\text{succ}(s)$ is the successor of s in the given cyclic order. Note that (17) implies that the map β is fully determined by its value on one element of S , and that if (16) holds for one element of S , then it holds for all elements of S .

A χ -cyclic set S is \mathcal{C} -colored if it is endowed with a map $c: S \rightarrow \text{Ob}(\mathcal{C})$ such that for any $s \in S$, the object $c(s)$ is homogeneous of degree $|c(s)| = \alpha(s)$. For shortness, \mathcal{C} -colored χ -cyclic sets are also called χ -cyclic \mathcal{C} -sets.

Each χ -cyclic \mathcal{C} -set $S = (S, \alpha, \beta, \varepsilon, c)$ determines a \mathbb{k} -module $H(S)$ called the *multiplicity module* of S and defined as follows. For $s \in S$, set

$$H_s = H_s(S) = \text{Hom}_{\mathcal{C}}^{\beta(s)}(\mathbb{1}, c(s_1)^{\varepsilon(s_1)} \otimes \cdots \otimes c(s_n)^{\varepsilon(s_n)}),$$

where $s = s_1 < s_2 < \cdots < s_n$ are the elements of S ordered in the given cyclic order starting from s . Here $X^+ = X$ and $X^- = X^*$ for any object X of \mathcal{C} . If $t \in S \setminus \{s\}$, then $t = s_k$ for some $k \in \{2, \dots, n\}$. Set

$$[st) = c(s_1)^{\varepsilon(s_1)} \otimes \cdots \otimes c(s_{k-1})^{\varepsilon(s_{k-1})} \quad \text{and} \quad [ts) = c(s_k)^{\varepsilon(s_k)} \otimes \cdots \otimes c(s_n)^{\varepsilon(s_n)}.$$

Clearly,

$$H_s = \text{Hom}_{\mathcal{C}}^{\beta(s)}(\mathbb{1}, [st) \otimes [ts)) \quad \text{and} \quad H_t = \text{Hom}_{\mathcal{C}}^{\beta(t)}(\mathbb{1}, [ts) \otimes [st)).$$

Set $p_{s,s} = \text{id}_{H_s}: H_s \rightarrow H_s$. For distinct $s, t \in S$, define $p_{s,t}: H_s \rightarrow H_t$ by

$$v \in H_s \mapsto p_{s,t}(v) = \begin{array}{c} \begin{array}{c} \text{[ts)} \\ \downarrow \\ \text{[st)} \\ \downarrow \\ \boxed{v} \end{array} \\ \text{---} \end{array} = \begin{array}{c} \begin{array}{c} \text{[ts)} \\ \downarrow \\ \boxed{v} \\ \downarrow \\ \text{[st)} \end{array} \\ \text{---} \end{array} \in H_t. \tag{18}$$

It follows from (16), (17), and the axioms of a pivotal χ -category that $p_{s,t}$ is a well-defined \mathbb{k} -linear isomorphism and $p_{s,t} p_{r,s} = p_{r,t}$ for all $r, s, t \in S$. Thus the family $(\{H_s\}_{s \in S}, \{p_{s,t}\}_{s,t \in S})$ is a projective system of \mathbb{k} -modules and \mathbb{k} -linear isomorphisms. The multiplicity module $H(S)$ is the projective limit of this system:

$$H(S) = \varprojlim H_s.$$

The module $H(S)$ depends only on $S = (S, \alpha, \beta, \varepsilon, c)$ and is equipped with a family of \mathbb{k} -linear isomorphisms $\{\tau_s: H(S) \rightarrow H_s\}_{s \in S}$ such that $p_{s,t} \tau_s = \tau_t$ for all $s, t \in S$. We call τ_s the *cone isomorphism* and the family $\{\tau_s\}_{s \in S}$ the *universal cone*. Recall that $H(S)$ may be realized as the following submodule of the \mathbb{k} -module $\prod_{s \in S} H_s$:

$$H(S) = \left\{ (x_s)_{s \in S} \in \prod_{s \in S} H_s \mid p_{s,t}(x_s) = x_t \text{ for all } s, t \in S \right\}.$$

the copies of $[0, 1]$ in G are called *edges* of G . The endpoints of the edges of G (that is, the images of $0, 1 \in [0, 1]$) are called *vertices* of G . Each edge of G connects two (possibly, coinciding) vertices, and each vertex of G is incident to at least one edge. By *half-edges* of G , we mean the images of the closed intervals $[0, 1/2] \subset [0, 1]$ and $[1/2, 1] \subset [0, 1]$ in G . The number of half-edges of G incident to a vertex v of G is greater than or equal to 1 and is called the *valence* of v .

A graph is *oriented* if all its edges are oriented. The empty set is viewed as an oriented graph with no vertices and no edges.

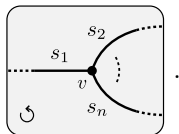
5.3. χ -graphs

Let Σ be an oriented surface. By a *graph* in Σ , we mean a graph embedded in Σ . A vertex v of a graph G in Σ determines a totally cyclically ordered set G_v consisting of the half-edges of G incident to v with cyclic order induced by the opposite orientation of Σ . If G is oriented, then we have a map $\varepsilon_v: G_v \rightarrow \{+, -\}$ assigning $+$ to the half-edges oriented towards v and $-$ to the half-edges oriented away from v .

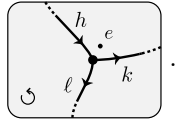
A χ -graph in Σ is an oriented graph G in Σ whose every edge is labeled with an element of H , called *H-label*, and every half-edge is labeled with an element of E , called *E-label*, in such a way that, for every vertex v of G , the cyclically ordered set G_v together with the maps $\alpha_v: G_v \rightarrow H$, $\beta_v: G_v \rightarrow E$ induced by the labels and the map $\varepsilon_v: G_v \rightarrow \{+, -\}$ defined above is a χ -cyclic set. (Here, the *H-label* of a half-edge is the *H-label* of the edge containing it.) Explicitly, this means that for any vertex v of G and any half-edge s_1 of G incident to v ,

$$\chi(\beta_v(s_1)) = \alpha_v(s_1)^{\varepsilon_v(s_1)} \cdots \alpha_v(s_n)^{\varepsilon_v(s_n)} \quad \text{and} \quad \beta_v(s_2) = (\alpha_v(s_1)^{-\varepsilon_v(s_1)})\beta_v(s_1),$$

where $s_1 < s_2 < \cdots < s_n$ are the half-edges of G incident to v ordered by the opposite orientation of Σ starting from s_1 :



When depicting a χ -graph G in Σ , we draw the *H-label* of an edge next to it. By (17), the *E-labels* of the half-edges incident to a vertex v are fully determined by the *E-label* of one half-edge incident to v . Now a dot next to v together with the orientation of Σ determines a half-edge incident to v : it is the first half-edge encountered while traversing from the dot a small loop negatively encircling v . Thus we draw next to each vertex of G a dot and the *E-label* of the half-edge determined by this dot. For example, consider the following trivalent vertex v of a χ -graph G in an oriented surface:



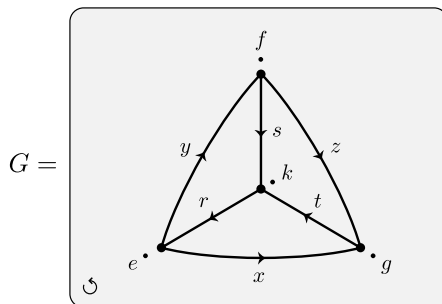
Here h, k, ℓ are the H -labels of the edges incident to v , and e is the E -label of the half-edge determined by the dot, that is, of the half-edge labeled by k . Condition (16) reduces to $\chi(e) = k^{-1}\ell^{-1}h$. By (17), the E -labels of the half-edges labeled by ℓ and h are ${}^k e$ and ${}^{\ell k} e$, respectively.

5.4. Grade of planar χ -graphs

We always orient the plane \mathbb{R}^2 counterclockwise. Let G be a χ -graph in \mathbb{R}^2 . Pick a point m in the unbounded component of $\mathbb{R}^2 \setminus G$. For each vertex v of G , pick a small loop ℓ_v negatively (i.e. clockwise) encircling v and an arc γ_v in \mathbb{R}^2 starting at m , ending in a point of $\ell_v \setminus G$, avoiding the vertices of G , and intersecting the edges of G transversally. We require that the arcs $\{\gamma_v\}_v$ intersect each other only at their initial point m . Denote by $e_v \in E$ the E -label of the first encountered half-edge incident to v while traversing ℓ_v from the endpoint of γ_v . We assign an element $h_v \in H$ to each arc γ_v as follows. Start with $h_v = 1_H$. Go through the arc γ_v starting from m . Each time γ_v intersects an edge s of G at some point c , replace h_v by $h_v x^\varepsilon$, where $\varepsilon = +$ if $(d_{cs}, d_c \gamma_v)$ is a positively oriented basis of \mathbb{R}^2 , $\varepsilon = -$ otherwise, and x is the H -label of the edge s . Denote by v_1, \dots, v_k the vertices of G so that $\gamma_{v_1}, \dots, \gamma_{v_k}$ are the arcs successively encountered while traversing a small loop negatively encircling m (starting from any point on that loop). We define the grade of G by

$$|G| = (h_{v_1})e_{v_1} \dots (h_{v_k})e_{v_k} \in E.$$

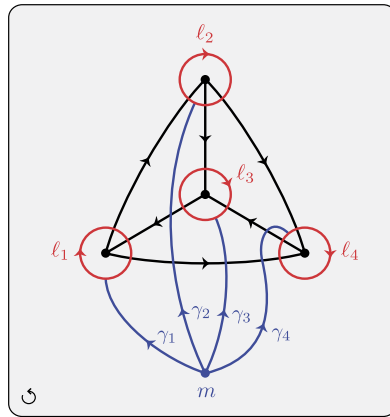
For example, consider the following χ -graph in \mathbb{R}^2 :



Here x, y, z, r, s, t are the H -labels of the edges and e, f, g, k are the E -labels of the half-edges determined by the dots (with the graphical convention of Section 5.3). The conditions for G to be a χ -graph reduce to

$$\chi(e) = y^{-1}rx^{-1}, \quad \chi(f) = z^{-1}s^{-1}y, \quad \chi(g) = xt^{-1}z, \quad \chi(k) = tr^{-1}s.$$

Pick the following point m , loops $\{\ell_i\}_{1 \leq i \leq 4}$, and arcs $\{\gamma_i\}_{1 \leq i \leq 4}$:



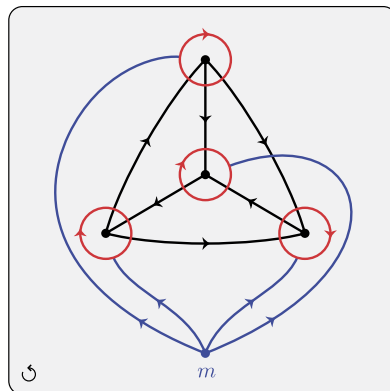
For any $1 \leq i \leq 4$, denote by e_i the element of E associated with the vertex encircled by ℓ_i and by h_i the element of H associated to the arc γ_i as described above. Using (17), we obtain:

$$\begin{aligned}
 e_1 &= e, & e_2 &= yf, & e_3 &= t^{-1}k, & e_4 &= zg, \\
 h_1 &= 1, & h_2 &= xr^{-1}, & h_3 &= x, & h_4 &= xt^{-1}.
 \end{aligned}$$

Then the grade of G is

$$|G| = ({}^1e)(x^{r^{-1}}yf)(x^{t^{-1}}k)(x^{t^{-1}}zg) = e(\chi(e^{-1})f)(\chi(g)z^{-1}k)(\chi(g)g) = fe g(z^{-1}k)$$

where the last equality follows from (2). Note that this final expression for $|G|$ corresponds to the following choice of loops and arcs:



Lemma 5.1. *The grade $|G|$ is well-defined and $|G| \in \text{Ker}(\chi)$.*

In particular, $|G|$ is a central element of E (because $\text{Ker}(\chi)$ is central in E). Also, the grade is multiplicative with respect to the disjoint union: if G and G' are χ -graphs in \mathbb{R}^2 lying in disjoint disks, then $|G \amalg G'| = |G||G'|$.

Proof. We first prove that, for any choice (as above) of the arcs $\gamma_{v_1}, \dots, \gamma_{v_k}$, we have $|G| \in \text{Ker}(\chi)$. Denote by G_0 the set of vertices of G . To any oriented arc λ in $\mathbb{R}^2 \setminus G_0$ intersecting the edges of G transversally, we associate an element $\phi(\lambda) \in H$ as follows. Start with $\phi(\lambda) = 1 \in H$. Go through the arc λ starting from the initial point of λ . Each time λ intersects an edge s of G at some point c , replace $\phi(\lambda)$ by $\phi(\lambda)x^\varepsilon$, where $\varepsilon = +$ if $(d_c s, d_c \gamma)$ is a positively oriented basis of \mathbb{R}^2 , $\varepsilon = -$ otherwise, and x is the H -label of the edge s . In particular $\phi(\lambda) = 1$ if λ does not intersect any edge of G . Clearly, ϕ is multiplicative with respect to the concatenation of arcs in $\mathbb{R}^2 \setminus G_0$: $\phi(\lambda\mu) = \phi(\lambda)\phi(\mu)$ if the endpoint of λ is the starting point of μ . Moreover, if λ and λ' are homotopic (relative endpoints) in $\mathbb{R}^2 \setminus G_0$, then $\phi(\lambda) = \phi(\lambda')$. For example, for any vertex v of G , we have $\phi(\gamma_v) = h_v$ (by definition of h_v). Also, viewing the loop ℓ_v as an oriented arc with starting point the endpoint of γ_v , it follows from (16) that $\phi(\ell_v) = \chi(e_v)$. Then

$$\begin{aligned} \chi(|G|) &\stackrel{(i)}{=} \chi\left((h_{v_1})e_{v_1} \dots (h_{v_k})e_{v_k} \right) \stackrel{(ii)}{=} h_{v_1}\chi(e_{v_1})h_{v_1}^{-1} \dots h_{v_k}\chi(e_{v_k})h_{v_k}^{-1} \\ &\stackrel{(iii)}{=} \phi(\gamma_{v_1})\phi(\ell_{v_1})\phi(\gamma_{v_1})^{-1} \dots \phi(\gamma_{v_k})\phi(\ell_{v_k})\phi(\gamma_{v_k})^{-1} \\ &\stackrel{(iv)}{=} \phi(\gamma_{v_1}\ell_{v_1}\gamma_{v_1}^{-1} \dots \gamma_{v_k}\ell_{v_k}\gamma_{v_k}^{-1}) \stackrel{(v)}{=} 1. \end{aligned}$$

Here (i) follows from the definition of $|G|$, (ii) from the multiplicativity of χ and (1), (iii) from the fact that $h_v = \phi(\gamma_v)$ and $\chi(e_v) = \phi(\ell_v)$ for any vertex v of G , (iv) from the multiplicativity of ϕ with respect to the concatenation of arcs, and (v) from the fact that the concatenated arc $\gamma_{v_1}\ell_{v_1}\gamma_{v_1}^{-1} \dots \gamma_{v_k}\ell_{v_k}\gamma_{v_k}^{-1}$ is homotopic (relative endpoints) in $\mathbb{R}^2 \setminus G_0$ to an arc not intersecting the edges of G (namely, to the boundary of a disk containing m in its boundary and G in its interior).

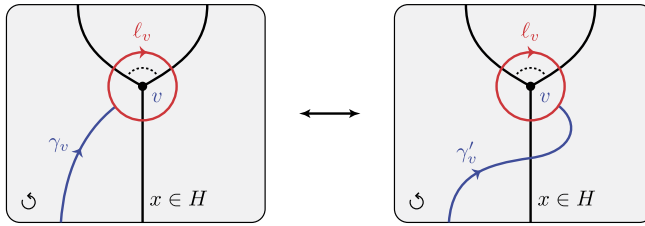
Next, $|G|$ is independent of the choice of the starting point on the small loop negatively encircling m used to enumerate the vertices v_1, \dots, v_k of G . This follows from the fact that for any $1 \leq r \leq k$,

$$(h_{v_r})e_{v_r} \dots (h_{v_k})e_{v_k} (h_{v_1})e_{v_1} \dots (h_{v_{r-1}})e_{v_{r-1}} \stackrel{(i)}{=} w_r^{-1}|G|w_r \stackrel{(ii)}{=} |G|,$$

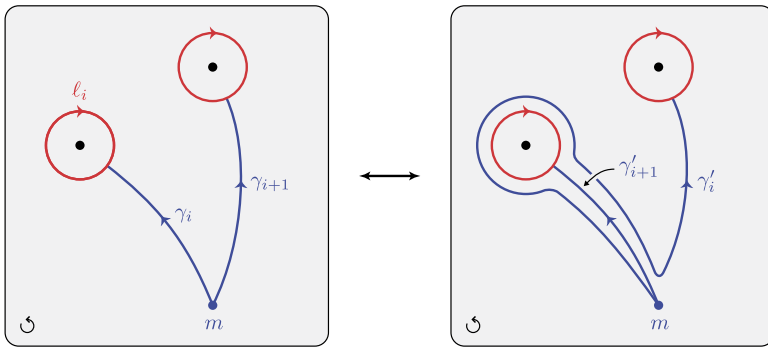
where $w_r = (h_{v_1})e_{v_1} \dots (h_{v_{r-1}})e_{v_{r-1}} \in E$. Here (i) follows from the definition of $|G|$ and (ii) from the fact that $|G|$ is central in E (since $|G| \in \text{Ker}(\chi)$).

Let us prove the invariance of $|G|$ on the choice of the system of arcs $\{\gamma_v\}_{v \in G_0} = (\gamma_1, \dots, \gamma_k)$ where $\gamma_i = \gamma_{v_i}$. Any two such systems of arcs are related by a finite sequence of homotopies (relative endpoints) in $\mathbb{R}^2 \setminus G_0$ and of the following moves:

(a) local move around a vertex v of G :



(b) global exchange move for $1 \leq i < k$:



which replaces the arcs (γ_i, γ_{i+1}) by the arcs $(\gamma'_i, \gamma'_{i+1})$ where $\gamma'_{i+1} = \gamma_i$ and γ'_i is the concatenation $\gamma_i l_i \gamma_i^{-1} \gamma_{i+1}$ (slightly homotoped so that it becomes disjoint from γ_i except at m).

Invariance of $|G|$ under homotopies (relative endpoints) in $\mathbb{R}^2 \setminus G_0$ of the arcs γ_v follows directly from the invariance of the elements $h_v \in H$ and $e_v \in E$ under such homotopies. To prove the invariance under the move (a), denote by $h'_v \in H$ and $e'_v \in E$ the elements associated (as above) to γ'_v . Set $\varepsilon = +$ if the edge labeled with $x \in H$ is oriented towards v and $\varepsilon = -$ otherwise. It follows directly from the definitions that $h'_v = h_v x^{-\varepsilon}$. Also $e_v = x^{-\varepsilon} e'_v$ by (17), and so $e'_v = x^\varepsilon e_v$. Then

$$h'_v e'_v = h_v x^{-\varepsilon} x^\varepsilon e_v = h_v e_v.$$

This proves the invariance of $|G|$ under the move (a). To prove the invariance under the move (b), denote by $h_r \in H$, $e_r \in E$ the elements associated (as above) to γ_r and by $h'_r \in H$, $e'_r \in E$ those associated to γ'_r for $r \in \{i, i + 1\}$. Clearly, $h'_{i+1} = h_i$, $e'_{i+1} = e_i$, and $e'_i = e_{i+1}$. It follows from (16) that $h'_i = h_i \chi(e_i) h_i^{-1} h_{i+1}$, and so $h'_i = \chi(h_i e_i) h_{i+1}$ by (1). Then, using (2), we obtain

$$h'_i e'_i h'_{i+1} e'_{i+1} = \chi(h_i e_i) h_{i+1} e_{i+1} h_i e_i = h_i e_i h_{i+1} e_{i+1}.$$

This proves the invariance of $|G|$ under the move (b).

Finally, the invariance of $|G|$ on the choice of the point m in the unbounded component of $\mathbb{R}^2 \setminus G$ follows from the fact that this component is path connected. \square

5.5. *Colored χ -graphs*

Let Σ be an oriented surface. A χ -graph in Σ is \mathcal{C} -colored if each edge is endowed with a homogeneous object of \mathcal{C} , called its \mathcal{C} -color, whose degree is the H -label of this edge.

Let $G \subset \Sigma$ be a \mathcal{C} -colored χ -graph. The χ -cyclic set G_v associated with a vertex v of G is \mathcal{C} -colored by the map $c_v: G_v \rightarrow \text{Ob}(\mathcal{C})$ assigning to each half-edge the \mathcal{C} -color of the edge containing it. Let

$$H_v(G) = H(G_v)$$

be the multiplicity module of the χ -cyclic \mathcal{C} -set G_v . It can be described as follows. Let $n \geq 1$ be the valence of v and let $s_1 < s_2 < \dots < s_n < s_1$ be the half-edges of G incident to v with cyclic order induced by the opposite orientation of Σ . Let $X_r = c_v(s_r)$, $e_r = \beta(s_r)$, and $\varepsilon_r = \varepsilon_v(s_r)$ be the \mathcal{C} -color, the E -label, and the sign of s_r , respectively. Then we have the cone isomorphism

$$\tau_{s_1}^v: H_v(G) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}}^{\varepsilon_1}(\mathbb{1}, X_1^{\varepsilon_1} \otimes \dots \otimes X_n^{\varepsilon_n}).$$

By the definition of $H_v(G)$, the cone isomorphisms determined by different elements of G_v are related via composition with the permutation maps (18). We set

$$H(G) = \otimes_v H_v(G),$$

where v runs over all vertices of G and \otimes is the unordered tensor product of \mathbb{k} -modules. To emphasize the role of Σ , we sometimes write $H_v(G; \Sigma)$ for $H_v(G)$ and $H(G; \Sigma)$ for $H(G)$. By definition, for $G = \emptyset$, we have $H(G) = \mathbb{k}$.

Any orientation preserving embedding f of the surface Σ into an oriented surface Σ' carries a \mathcal{C} -colored χ -graph $G \subset \Sigma$ into a \mathcal{C} -colored χ -graph $G' = f(G) \subset \Sigma'$ preserving the vertices, the edges, the E -labels of the half-edges, and the orientations, H -labels, and \mathcal{C} -colors of the edges. The map f induces a \mathbb{k} -linear isomorphism $H(f): H(G; \Sigma) \rightarrow H(G'; \Sigma')$ in the obvious way. This applies, in particular, when f is an orientation preserving self-homeomorphism of Σ .

Given \mathcal{C} -colored χ -graphs G and G' in Σ , by an *isotopy* of G to G' , we mean an ambient isotopy of G to G' in Σ preserving the vertices, the edges, the E -labels of the half-edges, and the orientations, H -labels, and \mathcal{C} -colors of the edges. Such an isotopy ι induces an orientation preserving homeomorphism $\Sigma \rightarrow \Sigma$ carrying G to G' . This homeomorphism induces a \mathbb{k} -linear isomorphism $H(G) \rightarrow H(G')$ denoted $H(\iota)$.

Let $-\Sigma$ be Σ with opposite orientation. The *dual* of \mathcal{C} -colored χ -graph G in Σ is the \mathcal{C} -colored χ -graph G^{op} in $-\Sigma$ obtained from G by reversing the orientation of all

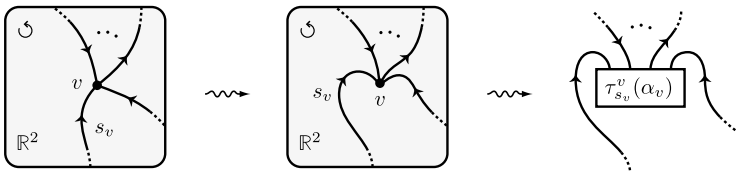
edges, keeping the H -labels and \mathcal{C} -colors of the edges, and defining the E -labels of half-edges of G^{op} as follows: the E -label of a half-edge ℓ of G^{op} incident to a vertex v is the inverse of the E -label of the successor of ℓ in G_v . Observe that $(G^{\text{op}})_v = (G_v)^{\text{op}}$ for any vertex v of G , where $(G_v)^{\text{op}}$ is the dual of G_v in the sense of Section 5.1. If \mathcal{C} is spherical pre-fusion, then the (unordered) tensor product of the non-degenerate \mathbb{k} -bilinear pairings $\omega_{G_v} : H(G_v^{\text{op}}) \otimes H(G_v) \rightarrow \mathbb{k}$ (see Section 5.1) over all vertices v of G yields a non-degenerate \mathbb{k} -bilinear pairing $H(G^{\text{op}}) \otimes H(G) \rightarrow \mathbb{k}$, and so these pairings induce canonical \mathbb{k} -linear isomorphisms

$$H_v(G^{\text{op}}) \simeq H_v(G)^* \quad \text{and} \quad H(G^{\text{op}}) \simeq H(G)^*.$$

Here we denote by $N^* = \text{Hom}_{\mathbb{k}}(N, \mathbb{k})$ the dual of a \mathbb{k} -module N .

5.6. An invariant of planar \mathcal{C} -colored χ -graphs

Let G be a \mathcal{C} -colored χ -graph in \mathbb{R}^2 . For each vertex v of G , pick a half-edge s_v incident to v and deform G near v so that the half-edges incident to v lie above v with respect to the second coordinate of \mathbb{R}^2 and s_v is the leftmost of them. Pick any $\alpha_v \in H_v(G)$ and replace v by a box colored with $\tau_{s_v}^v(\alpha_v)$, where τ^v is the universal cone of $H_v(G)$:



This transforms G into a diagram without free ends. Let $\mathbb{F}_{\mathcal{C}}(G)(\otimes_v \alpha_v) \in \text{End}_{\mathcal{C}}(\mathbb{1})$ be the associated morphism computed via the Penrose graphical calculus. Since the (co)evaluations and the morphisms $\tau_{s_v}^v(\alpha_v)$ are homogeneous morphisms, so is the morphism $\mathbb{F}_{\mathcal{C}}(G)(\otimes_v \alpha_v)$. It follows from the axioms of a pivotal χ -category and of the definition of $H_v(G)$ that the homogeneous morphism $\mathbb{F}_{\mathcal{C}}(G)(\otimes_v \alpha_v)$ is independent of the choice of the half-edges s_v and its degree is the grade $|G| \in \text{Ker}(\chi)$ of G . Then $\mathbb{F}_{\mathcal{C}}(G)(\otimes_v \alpha_v) \in \text{End}_{\mathcal{C}}^{|G|}(\mathbb{1})$. This extends by linearity to a \mathbb{k} -linear homomorphism

$$\mathbb{F}_{\mathcal{C}}(G) : H(G) = \otimes_v H_v(G) \rightarrow \text{End}_{\mathcal{C}}^{|G|}(\mathbb{1}).$$

For $G = \emptyset$, we have $|G| = 1$ and the map $\mathbb{F}_{\mathcal{C}}(G) : H(G) = \mathbb{k} \rightarrow \text{End}_{\mathcal{C}}^1(\mathbb{1})$ is the \mathbb{k} -linear homomorphism carrying $1_{\mathbb{k}}$ to $\text{id}_{\mathbb{1}}$.

The homomorphism $\mathbb{F}_{\mathcal{C}}(G) : H(G) \rightarrow \text{End}_{\mathcal{C}}^{|G|}(\mathbb{1})$ is a well-defined isotopy invariant of the \mathcal{C} -colored χ -graph G in \mathbb{R}^2 . More precisely, for any isotopy ι between \mathcal{C} -colored χ -graphs G and G' in \mathbb{R}^2 , we have $\mathbb{F}_{\mathcal{C}}(G')H(\iota) = \mathbb{F}_{\mathcal{C}}(G)$, where $H(\iota) : H(G) \rightarrow H(G')$ is the \mathbb{k} -linear isomorphism induced by ι . The invariance under isotopies follows from the isotopy invariance of the Penrose graphical calculus.

We state a few simple properties of $\mathbb{F}_{\mathcal{C}}$:

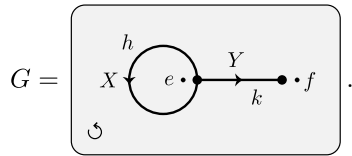
- (i) (Naturality) If a \mathcal{C} -colored χ -graph G' in \mathbb{R}^2 is obtained from a \mathcal{C} -colored χ -graph G in \mathbb{R}^2 through the replacement of the \mathcal{C} -color X of an edge by a 1-isomorphic object X' , then any 1-isomorphism $\phi: X \rightarrow X'$ induces a \mathbb{k} -linear isomorphism $\Phi: H(G') \rightarrow H(G)$ and $\mathbb{F}_{\mathcal{C}}(G') = \mathbb{F}_{\mathcal{C}}(G) \Phi$.
- (ii) If a \mathcal{C} -colored χ -graph G' in \mathbb{R}^2 is obtained from a \mathcal{C} -colored χ -graph G in \mathbb{R}^2 by reversing the orientation of an edge and replacing the \mathcal{C} -color of this edge by the dual object of \mathcal{C} , then the pivotal structure of \mathcal{C} induces a \mathbb{k} -linear isomorphism $\Psi: H(G') \rightarrow H(G)$ and $\mathbb{F}_{\mathcal{C}}(G') = \mathbb{F}_{\mathcal{C}}(G) \Psi$.
- (iii) If an edge e of a \mathcal{C} -colored χ -graph G in \mathbb{R}^2 is colored with $\mathbb{1}$ and the endpoints of e are also endpoints of other edges of G , then $G' = G \setminus \text{Int}(e) \subset \mathbb{R}^2$ inherits from G the structure of a \mathcal{C} -colored χ -graph, there is a canonical \mathbb{k} -linear isomorphism $\Delta: H(G') \rightarrow H(G)$, and $\mathbb{F}_{\mathcal{C}}(G') = \mathbb{F}_{\mathcal{C}}(G) \Delta$.
- (iv) (\otimes -multiplicativity) If G and G' are disjoint \mathcal{C} -colored χ -graphs in \mathbb{R}^2 lying on different sides of a straight line, then there is a canonical \mathbb{k} -linear isomorphism $\Theta: H(G \amalg G') \rightarrow H(G) \otimes H(G')$ and

$$\mathbb{F}_{\mathcal{C}}(G \amalg G') = \mu(\mathbb{F}_{\mathcal{C}}(G) \otimes \mathbb{F}_{\mathcal{C}}(G')) \Theta,$$

where μ is the multiplication of the \mathbb{k} -algebra $\text{End}_{\mathcal{C}}(\mathbb{1})$.

5.7. Example

Consider the following \mathcal{C} -colored χ -graph in \mathbb{R}^2 with two vertices and two edges which are H -labeled with $h, k \in H$ and \mathcal{C} -colored with homogenous objects X, Y :



There are 3 half-edges s_1, s_2, s_3 incident to the leftmost vertex u and one half-edge t incident to rightmost vertex v :

$$s_1 = \curvearrowright, \quad s_2 = \curvearrowleft, \quad s_3 = \bullet \rightarrow, \quad t = \rightarrow \bullet.$$

The cyclic order on $G_u = \{s_1, s_2, s_3\}$ is $s_1 < s_2 < s_3 < s_1$. By the convention of Section 5.4, the elements e and f of E are the E -labels of s_2 and t , respectively. The conditions for G to be a \mathcal{C} -colored χ -graph reduce to

$$\chi(e) = h^{-1}k^{-1}h, \quad \chi(f) = k, \quad |X| = h, \quad \text{and} \quad |Y| = k.$$

The degree of G is $|G| = {}^h e f$. By (17), the E -labels of the half-edges s_1 and s_3 are ${}^h e$ and ${}^{kh} e$, respectively. The cone isomorphisms associated with the half-edges are:

$$\begin{aligned} \tau_{s_1}^u &: H_u(G) \rightarrow \text{Hom}_{\mathcal{C}}^{{}^h e}(\mathbb{1}, X \otimes X^* \otimes Y^*), \\ \tau_{s_2}^u &: H_u(G) \rightarrow \text{Hom}_{\mathcal{C}}^e(\mathbb{1}, X^* \otimes Y^* \otimes X), \\ \tau_{s_3}^u &: H_u(G) \rightarrow \text{Hom}_{\mathcal{C}}^{{}^{kh} e}(\mathbb{1}, Y^* \otimes X \otimes X^*), \\ \tau_t^v &: H_v(G) \rightarrow \text{Hom}_{\mathcal{C}}^f(\mathbb{1}, Y). \end{aligned}$$

The first three isomorphisms are related to each other via composition with the permutation maps (18). For instance, for any $\alpha \in H_u(G)$,

$$\tau_{s_2}^u(\alpha) = \begin{array}{c} \begin{array}{c} \uparrow X \uparrow Y \\ \boxed{\tau_{s_1}^u(\alpha)} \\ \downarrow X \end{array} \end{array} .$$

By definition, $H(G) = H_u(G) \otimes H_v(G)$. For any $\alpha \in H_u(G)$ and $\beta \in H_v(G)$,

$$\mathbb{F}_{\mathcal{C}}(G)(\alpha \otimes \beta) = \begin{array}{c} \begin{array}{c} \uparrow X \\ \boxed{\tau_{s_1}^u(\alpha)} \\ \downarrow X \end{array} \begin{array}{c} \uparrow Y \\ \boxed{\tau_t^v(\beta)} \\ \downarrow Y \end{array} = \begin{array}{c} \begin{array}{c} \uparrow Y \\ \boxed{\tau_{s_2}^u(\alpha)} \\ \downarrow X \end{array} \begin{array}{c} \uparrow Y \\ \boxed{\tau_t^v(\beta)} \\ \downarrow Y \end{array} = \begin{array}{c} \begin{array}{c} \uparrow Y \\ \boxed{\tau_{s_3}^u(\alpha)} \\ \downarrow X \end{array} \begin{array}{c} \uparrow Y \\ \boxed{\tau_t^v(\beta)} \\ \downarrow Y \end{array} . \end{array}$$

5.8. The spherical case

Consider the 2-sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$ endowed with the orientation extending the counterclockwise orientation in \mathbb{R}^2 . We say that a χ -graph G in S^2 is *1-spherical* if when pushing G away from ∞ , we obtain a χ -graph G_0 in \mathbb{R}^2 with trivial grade. Note that this notion does not depend on the way G is pushed away from ∞ . Indeed, if G'_0 is another χ -graph in \mathbb{R}^2 obtained by pushing G away from ∞ , then it follows from Section 5.4 that $|G'_0| = {}^x |G_0|$ for some $x \in H$, and so $|G'_0| = 1$ if and only if $|G_0| = 1$.

Assume \mathcal{C} is spherical (see Section 4.9). Then the invariant $\mathbb{F}_{\mathcal{C}}$ of \mathcal{C} -colored χ -graphs in \mathbb{R}^2 induces an isotopy invariant of 1-spherical \mathcal{C} -colored χ -graphs in S^2 . Indeed, consider a \mathcal{C} -colored χ -graph G in S^2 . Pushing G away from ∞ by an isotopy, we obtain a \mathcal{C} -colored χ -graph G_0 in \mathbb{R}^2 . The isotopy induces a \mathbb{k} -linear isomorphism $H(G; S^2) \simeq H(G_0; \mathbb{R}^2)$. Composing with $\mathbb{F}_{\mathcal{C}}(G_0): H(G_0; \mathbb{R}^2) \rightarrow \text{End}_{\mathcal{C}}^1(\mathbb{1})$ we obtain a \mathbb{k} -linear homomorphism

$$\mathbb{F}_{\mathcal{C}}(G): H(G; S^2) \rightarrow \text{End}_{\mathcal{C}}^1(\mathbb{1}).$$

The sphericity of \mathcal{C}^1 ensures that $\mathbb{F}_{\mathcal{C}}(G)$ is a well-defined isotopy invariant of G . The properties of $\mathbb{F}_{\mathcal{C}}$ formulated in Section 5.6 extend to 1-spherical \mathcal{C} -colored χ -graphs in S^2 in the obvious way. The condition in (iv) involving a straight line may be dropped here because for any pair of disjoint connected graphs in S^2 , there is an isotopy of S^2 in itself carrying these graphs into new positions in \mathbb{R}^2 separated by a line.

The invariant $\mathbb{F}_{\mathcal{C}}$ further extends to 1-spherical \mathcal{C} -colored χ -graphs in an arbitrary oriented surface Σ homeomorphic to S^2 . Given a 1-spherical \mathcal{C} -colored χ -graph G in Σ , pick an orientation preserving homeomorphism $f: \Sigma \rightarrow S^2$ and set

$$\mathbb{F}_{\mathcal{C}}(G) = \mathbb{F}_{\mathcal{C}}(f(G)) H(f): H(G; \Sigma) \rightarrow \text{End}_{\mathcal{C}}^1(1).$$

Since all orientation preserving homeomorphisms $\Sigma \rightarrow S^2$ are isotopic, the homomorphism $\mathbb{F}_{\mathcal{C}}(G)$ does not depend on the choice of f .

5.9. Dual vertices

Consider a \mathcal{C} -colored χ -graph G in an oriented surface Σ and a \mathcal{C} -colored χ -graph G' in an oriented surface Σ' . By a *duality* between a vertex u of G and a vertex v of G' , we mean an isomorphism of χ -cyclic \mathcal{C} -sets $\phi: G_u^{\text{op}} \rightarrow G'_v$, where G_u^{op} is the dual of G_u (see Section 5.1). When \mathcal{C} is spherical pre-fusion, the *contraction vector* of a duality $\phi: G_u^{\text{op}} \rightarrow G'_v$ is the vector

$$*_\phi = (\text{id}_{H(G_u)} \otimes H(\phi))(*_{G_u}) \in H(G_u) \otimes H(G'_v) = H_u(G) \otimes H_v(G'),$$

where

$$H(\phi): H(G_u^{\text{op}}) \rightarrow H(G'_v) \quad \text{and} \quad *__{G_u} \in H(G_u) \otimes H(G_u^{\text{op}})$$

are respectively the \mathbb{k} -linear isomorphism induced by ϕ and the contraction vector of the χ -cyclic \mathcal{C} -set G_u (see Section 5.1). Clearly, $*_\phi = *_{\phi^{-1}}$.

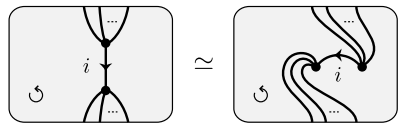
The next lemma formulates a local relation for the invariant $\mathbb{F}_{\mathcal{C}}$ of \mathcal{C} -colored χ -graphs in \mathbb{R}^2 involving dual vertices and contraction vectors.

Lemma 5.2. *Assume that \mathcal{C} is spherical pre-fusion and let I be a χ -representative set for \mathcal{C} . The pictures below stand for pieces of \mathcal{C} -colored χ -graphs in \mathbb{R}^2 which coincide outside these pieces. The orientation of the edges \mathcal{C} -colored by the object X_i should match in both sides of the equality, and let $\varepsilon_i = +$ if this orientation is downwards and $\varepsilon_i = -$ otherwise. Set $h = |X_1|^{\varepsilon_1} \cdots |X_n|^{\varepsilon_n} \in H$. Then, for all $e \in E$,*

$$\mathbb{F}_{\mathcal{C}} \left(\begin{array}{|c|c|c|} \hline X_1 & & X_n \\ \hline \cup & & \\ \hline \end{array} \right) = \sum_{i \in I_{h\chi(e^{-1})}} \dim(i) \mathbb{F}_{\mathcal{C}} \left(\begin{array}{|c|c|c|} \hline X_1 & \dots & X_n \\ \hline i & \bullet & e \\ & \bullet & e^{-1} \\ \hline \cup & & \\ \hline X_1 & \dots & X_n \\ \hline \end{array} \right) (*).$$

Here, $e^{\pm 1}$ are the E -labels of the half-edges determined by the points (with the graphical convention of Section 5.3) and $*$ is the contraction vector provided by the duality between the two vertices induced by the symmetry with respect to a horizontal line.

Proof. We only need to compare the contributions to $\mathbb{F}_{\mathcal{C}}$ of the depicted pieces of \mathcal{C} -colored χ -graphs. The contribution of the left-hand piece is the identity id_X where $X = X_1^{\varepsilon_1} \otimes \cdots \otimes X_n^{\varepsilon_n}$. Using the definitions of $\mathbb{F}_{\mathcal{C}}$ and $*$, Formula (19), and considering the isotopy



we obtain that the contribution of the right-hand piece is equal to

$$\sum_{i \in I_g} \dim(i) \left[\begin{array}{c} \text{Diagram 1: A semi-circular arc labeled } e \text{ with a vertex } i \text{ above it. A vertical arrow labeled } X \text{ points down from } i. \end{array} \right] \stackrel{(i)}{=} \sum_{i \in I_g} \dim(i) \left[\begin{array}{c} \text{Diagram 2: Two semi-circular arcs labeled } e \text{ with a vertex } i \text{ between them. Two vertical arrows labeled } X \text{ point down from } i. \end{array} \right] \stackrel{(ii)}{=} \sum_{i \in I_g} \dim(i) \left[\begin{array}{c} \text{Diagram 3: Two semi-circular arcs labeled } g_e \text{ with a vertex } i \text{ between them. Two vertical arrows labeled } X \text{ point down from } i. \end{array} \right] \stackrel{(iii)}{=} \text{id}_X$$

where $g = h\chi(e^{-1})$. Here (i) follows from the isotopy invariance of the graphical calculus, (ii) from Lemma 4.2, and (iii) from (13) and the fact that $\text{Hom}_{\mathcal{C}}^{g_e}(i, X) = 0$ for all $i \in I \setminus I_g$ (since X is a 1-direct sum of homogenous objects of degree h). \square

6. Skeletons and maps

Throughout this section, $\chi: E \rightarrow H$ is a crossed module and $B\chi$ is the classifying space of χ (see Section 2.4). We first recall the notions of stratified 2-polyhedra and skeletons of 3-manifolds (referring to [20] for details). Then we explain how to encode maps from closed 3-manifolds to $B\chi$ in terms of labelings of skeletons.

6.1. Stratified 2-polyhedra

A 2-polyhedron is a compact topological space that can be triangulated using a finite number of simplices of dimension ≤ 2 so that all 0-simplices and 1-simplices are faces of 2-simplices. The interior $\text{Int}(P)$ of a 2-polyhedron P consists of all points which have a neighborhood homeomorphic to \mathbb{R}^2 . By the definition of a 2-polyhedron, the surface $\text{Int}(P)$ is dense in P .

A stratified polyhedron is a 2-polyhedron P endowed with a graph $P^{(1)}$ embedded in P such that $P \setminus \text{Int}(P) \subset P^{(1)}$. The vertices and edges of $P^{(1)}$ are called respectively the

vertices and edges of P . Any 2-polyhedron P can be stratified. For example, the 0-simplices and the 1-simplices of a triangulation of P form a graph which is a stratification of P . Any (possibly, empty) graph embedded in a compact surface without boundary is a stratification of this surface.

An *orientation* of a stratified 2-polyhedron P is an orientation of the surface $P \setminus P^{(1)}$. A stratified 2-polyhedron is *oriented* if it is endowed with an orientation. Cutting a stratified 2-polyhedron P along the graph $P^{(1)} \subset P$, we obtain a compact surface \tilde{P} with interior $P \setminus P^{(1)}$. The connected components of \tilde{P} are called the *regions* of P . Each component of $P \setminus P^{(1)} \subset \tilde{P}$ is the interior of a unique region. The set $\text{Reg}(P)$ of the regions of P is finite. To orient P , one must orient all its regions.

A *branch* of a stratified 2-polyhedron P at a vertex v of P is a germ at v of an adjacent region. Similarly, a *branch* of P at an edge e of P is a germ at e of an adjacent region. The set of branches of P at e is denoted P_e . This set is finite and non-empty. The number of elements of P_e is called the *valence* of e .

The edges of a stratified 2-polyhedron P of valence 1 together with their vertices form a graph called the *boundary* of P and denoted ∂P . An orientation of P induces an orientation of ∂P : each edge e of ∂P is oriented so that the orientation of the unique region of P adjacent to e is determined by a pair (a vector directed outward at an interior point of e , a positive tangent vector of e).

6.2. Skeletons of closed 3-manifolds

A *skeleton* of a closed oriented (possibly, non-connected) 3-dimensional manifold M is an oriented stratified 2-polyhedron $P \subset M$ such that $\partial P = \emptyset$ and $M \setminus P$ is a disjoint union of open 3-balls. These open 3-balls are called *P-balls*. The condition $\partial P = \emptyset$ ensures that all edges of P have valence ≥ 2 . An example of a skeleton of M is provided by the (oriented) 2-skeleton $t^{(2)}$ of a triangulation t of M , where the edges of $t^{(2)}$ are the edges of t .

Given a skeleton P of M , a *P-ball branch* at an edge e of P is a germ at e of an adjacent P -ball.

Any vertex x of a skeleton P of M has a closed ball neighborhood $B_x \subset M$ such that $\Gamma_x = P \cap \partial B_x$ is a non-empty graph and $P \cap B_x$ is the cone over Γ_x with summit x . The vertices of Γ_x are the intersection points of the 2-sphere ∂B_x with the edges of P incident to x . Each edge of Γ_x is the intersection of ∂B_x with a branch b of P at x , and we endow this edge with the orientation induced by that of b restricted to $b \setminus \text{Int}(B_x)$. We endow $\partial B_x \cong S^2$ with the orientation induced by that of M restricted to $M \setminus \text{Int}(B_x)$. In this way, Γ_x becomes an oriented graph in the oriented 2-sphere ∂B_x . We call B_x a *P-cone neighborhood* of x and call Γ_x the *link graph* of x . The pair (B_x, Γ_x) is determined by the triple (M, P, x) uniquely up to homeomorphism. Since all edges of P have valence ≥ 2 , so do all vertices of Γ_x . An example is given in Fig. 1 where there are 10 branches of P at x , 6 P -ball branches at x , and ∂B_x is identified with $S^2 = \mathbb{R}^2 \cup \{\infty\}$ via the stereographic projection from a pole in the upper left quarter.

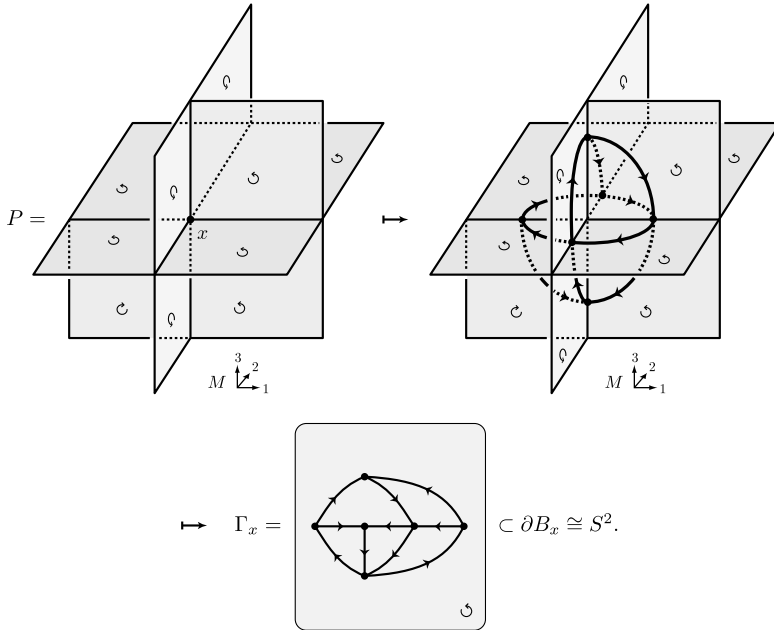


Fig. 1. The link graph of a vertex.

6.3. χ -labelings

Let P be a skeleton of an oriented closed 3-dimensional manifold M . By an *oriented edge* of P , we mean an edge of P endowed with an orientation. For each oriented edge e of P , the orientations of e and M determine a positive direction on a small loop in M encircling e (so that the linking number of this loop with e is $+1$). The resulting oriented loop determines cyclic orders on the set P_e of branches of P at e and on the set P_e^{ball} of P -ball branches at e . Any P -ball branch $B \in P_e^{\text{ball}}$ determines a branch $b_B \in P_e$ which is the first branch of P at e encountered while traversing the oriented loop starting from B . This induces a bijection $P_e^{\text{ball}} \rightarrow P_e$ preserving the cyclic order. In particular, any P -ball branch $B \in P_e^{\text{ball}}$ turns the cyclic order on P_e into a linear order so that the first element is b_B . To each branch $b \in P_e$ we assign a sign equal to $+$ if the orientation of b induces the one of $e \subset \partial b$ (that is, the orientation of b is given by the orientation of e followed by a vector at a point of e directed inside b) and equal to $-$ otherwise. This gives a map $\varepsilon_e: P_e \rightarrow \{+, -\}$. When orientation of e is reversed, the cyclic orders on P_e and P_e^{ball} are reversed and ε_e is multiplied by $-$.

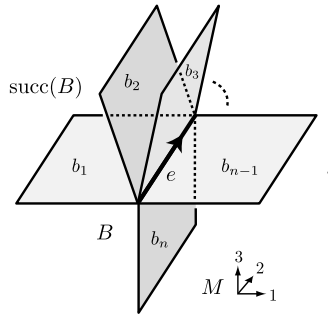
Recall that $\text{Reg}(P)$ denotes the (finite) set of regions of P (see Section 6.1). Let $\text{EB}(P)$ be the set of pairs (e, B) where e is an oriented edge of P and B is a P -ball branch at e . Note that $\text{EB}(P) = \bigcup_e \{e\} \times P_e^{\text{ball}}$ where e runs over all oriented edges of P . A *pre- χ -labeling* of P is a pair $(\alpha: \text{Reg}(P) \rightarrow H, \beta: \text{EB}(P) \rightarrow E)$ of maps such that for any $(e, B) \in \text{EB}(P)$,

$$\chi(\beta(e, B)) = \alpha(b_1)^{\varepsilon_e(b_1)} \dots \alpha(b_n)^{\varepsilon_e(b_n)}, \tag{20}$$

$$\beta(e, \text{succ}(B)) = (\alpha(b_1)^{-\varepsilon_e(b_1)})\beta(e, B), \tag{21}$$

$$\beta(-e, B) = \beta(e, B)^{-1}, \tag{22}$$

where $b_1 < \dots < b_n$ are the elements of P_e enumerated in the linear order determined by B , $\text{succ}(B)$ is the successor of B in the cyclic order on P_e^{ball} , and $-e$ is e with opposite orientation:



Here and in the sequel, for any branch b of P at a vertex or an edge of P , we let $\alpha(b) = \alpha(r_b)$ where r_b is the unique region of P containing b . It follows from (21) that for any oriented edge e of P , the map $B \in P_e^{\text{ball}} \mapsto \beta(e, B) \in E$ is fully determined by its value on one element of P_e^{ball} . Also, if (20) holds for one P -ball branch at e , then it holds for all P -ball branches at e . Similarly, (22) implies that if (20) or (21) holds for $(e, B) \in \text{EB}(P)$, then it holds for $(-e, B)$.

A pre- χ -labeling (α, β) of P turns the set P_e of branches of P at an oriented edge e of P into a χ -cyclic set $P_e = (P_e, \alpha_e, \beta_e, \varepsilon_e)$ as follows. For any branch $b \in P_e$, set $\alpha_e(b) = \alpha(b) \in H$ and $\beta_e(b) = \beta(e, B) \in E$, where B is the P -ball branch at e such that $b = b_B$ (in the above notation).

A pre- χ -labeling (α, β) of P turns the link graph Γ_x of a vertex x of P into a χ -graph in $\partial B_x \cong S^2$ as follows, where B_x is a P -cone neighborhood of x (see Section 6.2). Each edge of Γ_x is the intersection of ∂B_x with a branch b of P at x and we endow this edge with the H -label $\alpha(b) \in H$. Each half-edge h of Γ_x is uniquely determined by an oriented edge e of P incident to x and a P -ball branch B at e . Indeed, the vertex v of Γ_x to which h is incident is the first intersection point of ∂B_x with e while traversing e following its orientation. The orientation of e and M determine as above a branch b_B of P at e . The half-edge h is then the connected component containing v of the intersection of ∂B_x with b_B . We endow the half-edge h with the E -label $\beta(e, B) \in E$. It follows from (20)-(22) that these labels of (half-)edges make Γ_x a χ -graph (see Section 5.3).

A χ -labeling of P is a pre- χ -labeling of P such that, for any vertex x of P , the χ -graph Γ_x in $B_x \cong S^2$ is 1-spherical (see Section 5.8).

Let us define the *gauge group* of \mathcal{G}_P of P as follows. As a set,

$$\mathcal{G}_P = \text{Map}(\text{Ball}(P), H) \times \text{Map}(\text{Reg}(P), E)$$

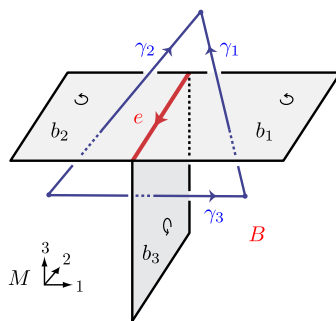
where $\text{Ball}(P)$ is the set of P -balls. For any $r \in \text{Reg}(P)$, let r_{\pm} be the P -balls (possibly coinciding) adjacent to the region r and indexed so that the orientation of r followed by a normal vector pointing from r_- to r_+ yields the given orientation of M . The product in \mathcal{G}_P is defined by $(\lambda', \mu') \cdot (\lambda, \mu) = (\lambda'', \mu'')$ with

$$\lambda''(B) = \lambda'(B)\lambda(B) \quad \text{and} \quad \mu''(r) = \mu(r) \left(\lambda^{(r_+)^{-1}} \mu'(r) \right)$$

for any P -ball B and any region r of P . The group \mathcal{G}_P acts (on the left) on the set of χ -labelings of P . To define this action we need some notations. Pick a point, called the *center*, in every P -ball. For each region r of P , pick an oriented arc γ_r in M connecting the center of r_- to the center of r_+ and intersecting r transversely at a single point. We choose the arcs $\{\gamma_r\}_r$ so that they intersect with each other only at their endpoints. For any branch b of P , we set $\gamma_b = \gamma_{r_b}$ where r_b is the unique region of P containing b . Each $(e, B) \in \text{EB}(P)$ determines a loop $\gamma_{(e,B)}$ in M based at the center of B defined by the concatenation

$$\gamma_{(e,B)} = \gamma_{b_1}^{\varepsilon_e(b_1)} \dots \gamma_{b_n}^{\varepsilon_e(b_n)},$$

where $b_1 < \dots < b_n$ are the elements of P_e enumerated in the linear order determined by B . For example, we have $\gamma_{(e,B)} = \gamma_1 \gamma_2^{-1} \gamma_3$ for the following trivalent oriented edge e and the P -ball branch B :



The action of \mathcal{G}_P on the set of χ -labelings of P is then defined as follows. For any $(\lambda, \mu) \in \mathcal{G}_P$ and any χ -labeling (α, β) of P , the χ -labeling $(\alpha', \beta') = (\lambda, \mu) \cdot (\alpha, \beta)$ of P is defined by

$$\alpha'(r) = \lambda(r_-)\alpha(r)\chi(\mu(r))\lambda(r_+)^{-1}, \tag{23}$$

$$\beta'(e, B) = \lambda^{(B)}(\beta(e, B)\mu_\alpha(\gamma_{(e,B)})), \tag{24}$$

for all $r \in \text{Reg}(P)$ and $(e, B) \in \text{EB}(P)$. Here, μ_α extends μ to the groupoid freely generated by the morphisms $\{\gamma_r: r_- \rightarrow r_+\}_{r \in \text{Reg}(P)}$ via the following rules: $\mu_\alpha(\emptyset) = 1$ and for any region r of P and any morphism ω in this groupoid,

$$\mu_\alpha(\omega * \gamma_r) = \left(\alpha^{(r)^{-1}} \mu_\alpha(\omega) \right) \mu(r) \quad \text{and} \quad \mu_\alpha(\omega * \gamma_r^{-1}) = \alpha^{(r)}(\mu_\alpha(\omega) \mu(r)^{-1}),$$

where the symbol $*$ denotes the concatenation of the edges of a graph (which is opposite to the composition of morphisms in this groupoid). Two χ -labelings of P are *gauge equivalent* if they belong to the same orbit under the action of the gauge group \mathcal{G}_P .

Lemma 6.1. *There is an injective map from the set of gauge equivalence classes of χ -labelings of P to the set of homotopy classes of maps $M \rightarrow B\chi$ which is bijective when the regions of P are disks.*

We prove Lemma 6.1 in Section 6.5. The next example illustrates the lemma for lens spaces.

6.4. Example

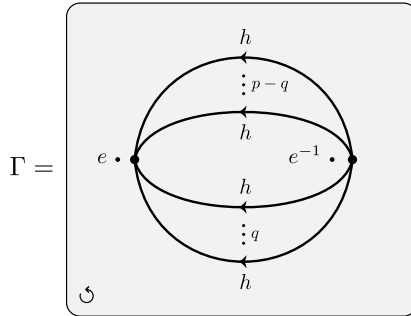
Consider the lens space $L(p, q)$ where p, q are coprime integers with $1 \leq q < p$. It is a closed connected oriented 3-dimensional manifold obtained by gluing two solid tori as follows. Let $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ be the unit disk in \mathbb{C} with counterclockwise orientation. Let $S^1 = \partial D$ be the unit circle with counterclockwise orientation. Let U, V be two copies of the solid torus $S^1 \times D$. The boundaries of U and V are copies of $S^1 \times S^1$. Then $L(p, q) = U \cup_\varphi V$ is obtained by gluing U to V along a homeomorphism $\varphi: \partial U \rightarrow \partial V$ such that $\varphi(1, t) = (t^p, t^q)$ for all $t \in S^1$. We provide $L(p, q)$ with the orientation extending the product orientation of $V = S^1 \times D$.

A skeleton P of $L(p, q)$ is obtained from the disk $\{1\} \times D \subset U$ and the circle $S^1 \times \{0\} \subset V$ by pulling the boundary of this disk towards this circle in V . More precisely,

$$P = (\{1\} \times D) \cup_\varphi \{(t^p, kt^q) \mid t \in S^1, k \in [0, 1]\} \subset U \cup_\varphi V = L(p, q).$$

The polyhedron P contains the circle $S^1 \times \{0\} \subset V$ which, viewed as a loop based at the point $(1, 0)$, determines a stratification of P . The stratified polyhedron P has one vertex, one edge, and one disk region which we orient by extending the orientation of $D = \{1\} \times D \subset U$. In this way, P becomes a skeleton of $L(p, q)$ with a single P -ball. Note that there are p branches of P at the edge of P .

Any pre- χ -labeling of P is determined by a pair $(h, e) \in H \times E$ with $\chi(e) = h^p$. Here h is the H -label of the disk region and e is the E -label of the edge (oriented as $S^1 = S^1 \times \{0\} \subset V$ and with respect to some chosen adjacent P -ball branch). The χ -graph Γ in $S^2 = \mathbb{R}^2 \cup \{\infty\}$ associated with the vertex of P is



with q edges below and $(p - q)$ edges above the horizontal axis. By Section 5.4, the grade of Γ is

$$|\Gamma| = e \left((h^{-q})(e^{-1}) \right).$$

Then Γ is 1-spherical (that is, $|\Gamma| = 1$) if and only if $(h^q)e = e$. Since p, q are coprime and $(h^p)e = \chi(e)e = ee^{-1} = e$, this condition is equivalent to $h^q e = e$. Consequently any χ -labeling of P is fully determined by a pair $(h, e) \in H \times E$ such that

$$\chi(e) = h^p \quad \text{and} \quad h^q e = e. \tag{25}$$

Note that the gauge group of P is $\mathcal{G}_P = H \times E$ with product

$$(\lambda', \mu') * (\lambda, \mu) = \left(\lambda' \lambda, \mu^{(\lambda^{-1})} \mu' \right)$$

and that it acts on χ -labelings as:

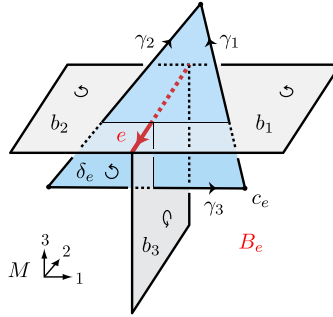
$$(\lambda, \mu) \cdot (h, e) = \left(\lambda h \chi(\mu) \lambda^{-1}, {}^\lambda(e f_p) \right) \quad \text{where} \quad f_p = \left((h^{-(p-1)}) \mu \right) \dots \left((h^{-1}) \mu \right) \mu.$$

By Lemma 6.1, the χ -labeling of P induced by a pair $(h, e) \in H \times E$ satisfying (25) determines a homotopy class $g_{h,e} \in [L(p, q), B\chi]$ which depends only on the \mathcal{G}_P -orbit of (h, e) . Moreover, since the region of P is a disk, any $g \in [L(p, q), B\chi]$ is of the form $g = g_{h,e}$ for some $(h, e) \in H \times E$ satisfying (25).

6.5. Proof of Lemma 6.1

For each (unoriented) edge e of P , pick a P -ball branch B_e at e , an orientation for e , and a disk δ_e embedded in M which bounds $\gamma_{(e, B_e)}$, intersects e transversely at a single point, and intersects P transversely in arcs lying in the regions adjacent to e . We endow δ_e with the basepoint given by the center c_e of B_e and orient δ_e so that its orientation followed by that of e yields the given orientation of M . The condition δ_e bounds $\gamma_{(e, B_e)}$ means that $\partial \delta_e = \gamma_{(e, B_e)}$ as loops in M based at c_e . (Here we endow $\partial \delta_e$

with the boundary orientation using the first outward pointing convention.) For example, we have $\partial\delta_e = \gamma_1\gamma_2^{-1}\gamma_3$ for the following trivalent edge e and P -ball branch B_e :



Each region r of P is a disk with holes and may be collapsed onto a wedge of circles $W_r \subset r \setminus \partial r$ based at the point $\alpha_r \cap r$. (Here, a circle corresponding to a hole h is chosen to be isotopic to $p_h(\partial h)p_h^{-1}$, where p_h is a path from the point $\alpha_r \cap r$ to some vertex of h .) Let S_r be obtained from $W_r \times [-1, 1]$ by contracting the sets $W_r \times \{-1\}$ and $W_r \times \{1\}$ into points which are called the summits of S_r . (If the two P -balls adjacent to r are equal, then we identify the two summits of S_r .) We embed S_r into M as a union of two cones with base W_r and with summits in the centers of the P -balls adjacent to r . We can assume that $\gamma_r \subset S_r$ and that S_r does not meet $S_{r'}$ for $r \neq r'$ except possibly in the summits. Note that $S_r = \gamma_r$ when r is a disk. Set $S = \cup_r S_r$. Then the pair (M, S) has a relative CW-decomposition with only 2-cells and 3-cells given by the disks $\{\delta_e\}_e$ and ball neighborhoods in M of the vertices of P . Let $D = S \cup (\cup_e \delta_e)$ be the relative 2-skeleton of (M, S) and C be the set of centers of P -balls. Consider the following filtration of M :

$$M_* = (C \subset S \subset D \subset M).$$

By Section 2.4, the classifying space $B\chi$ of χ has a canonical filtration

$$(B\chi)_* = (\{x\} \subset BH \subset B\chi)$$

where x is the unique 0-cell of $B\chi$ (serving as basepoint) and BH is a subcomplex of $B\chi$ which is a classifying space of the group H , contains all 1-cells of $B\chi$, and whose associated boundary map $\partial: \pi_2(B\chi, BH; x) \rightarrow \pi_1(BH, x)$ is the crossed module χ .

We denote by $\text{FTop}(M_*, (B\chi)_*)$ the set of filtered maps $M_* \rightarrow (B\chi)_*$, that is, of continuous maps $f: M \rightarrow B\chi$ such that $f(C) = \{x\}$ and $f(S) \subset BH$.

Claim 6.2. *Any $f \in \text{FTop}(M_*, (B\chi)_*)$ defines a χ -labeling (α_f, β_f) of P such that:*

- $\alpha_f(r) = f_*([\gamma_r]) \in \pi_1(BH, x) = H$ for every region r of P ,
- $\beta_f(e, B_e) = f_*([\delta_e]) \in \pi_2(B\chi, BH; x) = E$ for every edge e of P .

The induced map $f \in \text{FTop}(M_*, (B\chi)_*) \mapsto (\alpha_f, \beta_f) \in \{\chi\text{-labelings of } P\}$ has a section $(\alpha, \beta) \mapsto f_{\alpha, \beta}$.

Proof. For each (unoriented) edge e of P , the above chosen orientation for e and P -ball branch B_e at e makes (e, B_e) an element of $\text{EB}(P)$. Denote by $\text{EB}_0(P)$ the subset of $\text{EB}(P)$ made of pairs (e, B_e) as e runs over all edges of P . The definitions in the claim give well defined maps $\alpha_f: \text{Reg}(P) \rightarrow H$ and $\beta_f: \text{EB}_0(P) \rightarrow E$. For any edge e of P , recall that $\partial\delta_e = \gamma_{b_1}^{\varepsilon_e(b_1)} \cdots \gamma_{b_n}^{\varepsilon_e(b_n)}$ where $b_1 < \cdots < b_n$ are the branches of P at e enumerated in the linear order determined by B_e (and the chosen orientation of e), and so

$$\chi(\beta_f(e, B_e)) = \chi(f_*([\delta_e])) = f_*([\partial\delta_e]) = \alpha_f(b_1)^{\varepsilon_e(b_1)} \cdots \alpha_f(b_n)^{\varepsilon_e(b_n)}.$$

Thus (20) is satisfied for all elements in $\text{EB}_0(P)$. Now, one easily verifies that any pair of maps $(\alpha: \text{Reg}(P) \rightarrow H, \beta: \text{EB}_0(P) \rightarrow E)$ satisfying (20) for all elements in $\text{EB}_0(P)$ extends uniquely to a pre- χ -labeling of P (by extending β to $\text{EB}(P)$ using (21) and (22)). Then (α_f, β_f) extends uniquely to a pre- χ -labeling of P also denoted by (α_f, β_f) . This pre- χ -labeling is a χ -labeling since the restriction of f to the boundary of any P -cone neighborhood of a vertex of P (see Section 6.2) is null-homotopic.

Let us prove the second assertion of the claim. For each region r of P , the projection $W_r \times [-1, 1] \rightarrow [-1, 1]$ induces a retraction $S_r \rightarrow \gamma_r$. Composing with loops $\gamma_r \rightarrow BH$ representing the elements $\alpha(r) \in H = \pi_1(BH, x)$, we obtain a map $S = \cup_r S_r \rightarrow BH \subset B\chi$ carrying the centers of the P -balls to the basepoint x of $B\chi$ and carrying each γ_r to a loop in BH representing $\alpha(r)$. Condition (20) ensures that this map extends to the relative 2-skeleton $D = S \cup (\cup_e \delta_e)$ of (M, S) by carrying each pointed disk δ_e to a disk in $B\chi$ representing the element $\beta(e, B_e) \in E = \pi_2(B\chi, BH; x)$. The 1-sphericity of the χ -graphs associated with the vertices of P implies that the restriction of this map to the boundary of each 3-cell is null-homotopic. Therefore, there is a further extension to M producing a filtered map $f_{\alpha, \beta}: M_* \rightarrow (B\chi)_*$. The conditions (21) and (22) ensure that the map $f_{\alpha, \beta}$ does not depend on the choices of the adjacent P -ball branches and orientations for the edges of P . Finally, it follows directly from the definitions that $(\alpha_{f_{\alpha, \beta}}, \beta_{f_{\alpha, \beta}}) = (\alpha, \beta)$. \square

Note that for any χ -labeling (α, β) of P , the homotopy class $[f_{\alpha, \beta}] \in [M, B\chi]$ of the map $f_{\alpha, \beta}$ defined in Claim 6.2 depends only on P and (α, β) since any two systems of arcs $\{\gamma_r\}_r$, disks $\{\delta_e\}_e$, and suspensions $\{S_r\}_r$ are isotopic in M .

Claim 6.3. *Two χ -labelings (α, β) and (α', β') of P are gauge equivalent if and only if their associated maps $f_{\alpha, \beta}$ and $f_{\alpha', \beta'}$ are homotopic.*

Proof. Assume first that $f_{\alpha, \beta}$ and $f_{\alpha', \beta'}$ are homotopic. Set $I = [0, 1]$ with its canonical filtration $I_* = (\{0, 1\} \subset I)$. Let $G: M \times I \rightarrow B\chi$ be a homotopy from $f_{\alpha', \beta'}$ to $f_{\alpha, \beta}$. Note that G is filtered (where $M \times I$ is endowed with the product filtration) if and only

Conversely, assume that (α, β) and (α', β') are gauge equivalent. Pick $(\lambda, \mu) \in \mathcal{G}_P$ such that $(\alpha', \beta') = (\lambda, \mu) \cdot (\alpha, \beta)$. For any $c \in C$, let B_c be the P -ball whose center is c . Pick a loop θ_c in BH representing $\lambda(B_c) \in H = \pi_1(BH, x)$. For any region r of P , $a_r = (f_{\alpha, \beta})|_{\gamma_r}$ and $a'_r = (f_{\alpha', \beta'})|_{\gamma_r}$ are loops in BH whose homotopy classes are $\alpha(r)$ and $\alpha'(r)$, respectively (see Claim 6.2). Using (23), the homotopy class of the loop $a_r^{-1}(\theta_{c_{r_-}})^{-1}a'_r\theta_{c_{r_+}}$ is

$$\alpha(r)^{-1}\lambda(r_-)^{-1}\alpha'(r)\lambda(r_+) = \chi(\mu(r)).$$

Therefore, by viewing the square $\gamma_r \times I$ as a disk pointed at $(c_{r_+}, 1)$, there is a map $u_r: \gamma_r \times I \rightarrow B\chi$ which represents $\mu(r) \in E = \pi_2(B\chi, BH; x)$ and such that

$$(u_r)|_{\{c_{r_+}\} \times I} = \theta_{c_{r_+}}, \quad (u_r)|_{\{c_{r_-}\} \times I} = \theta_{c_{r_-}}, \quad (u_r)|_{\gamma_r \times \{1\}} = a_r, \quad (u_r)|_{\gamma_r \times \{0\}} = a'_r$$

as depicted above. We now construct a (filtered) homotopy $f_{\alpha', \beta'}$ to $f_{\alpha, \beta}$ via successive extensions. First, we define a map $G_1: (M \times \{0, 1\}) \cup (S \times I) \rightarrow B\chi$ as follows: set $(G_1)|_{M \times \{0\}} = f_{\alpha', \beta'}$, $(G_1)|_{M \times \{1\}} = f_{\alpha, \beta}$, and for any region r of P , let $(G_1)|_{S_r \times I}$ be the composition of the map $S_r \times I \rightarrow \gamma_r \times I$ (induced by the canonical retraction $S_r \rightarrow \gamma_r$ and id_I) with the map u_r . For any edge e of P , (24) applied to (e, B_e) implies that the restriction of G_1 to the boundary of the 3-cell $\delta_e \times I$ is null-homotopic, and so G_1 can be extended to $\delta_e \times I$. Pick then an extension $G_2: (M \times \{0, 1\}) \cup (D \times I) \rightarrow B\chi$ of G_1 . Note that G_2 can further be extended to $M \times I$ since the target $B\chi$ is a 2-type. Any such extension $G: M \times I \rightarrow B\chi$ is then a (filtered) homotopy from $f_{\alpha', \beta'}$ to $f_{\alpha, \beta}$. \square

Claims 6.2 and 6.3 imply that the assignment $(\alpha, \beta) \mapsto f_{\alpha, \beta}$ induces a well-defined injective map $\{\chi\text{-labelings of } P\} / \mathcal{G}_P \rightarrow [M, B\chi]$ which maps the gauge equivalence class of a χ -labeling (α, β) to the homotopy class of the map $f_{\alpha, \beta}$.

Finally, assume that the regions of P are disks. Then the above filtration $M_* = (C \subset S \subset D \subset M)$ is the skeletal filtration of the CW-decomposition of M dual to P (since $S = \cup_r \gamma_r$). By the cellular approximation theorem, any map $f: M \rightarrow B\chi$ is homotopic to a cellular map $f': M \rightarrow B\chi$. Any such map f' is then a filtered map $M_* \rightarrow (B\chi)_*$ since $f'(C) = \{x\}$ and $f'(S) \subset \{1\text{-cells of } B\chi\} \subset BH$. Thus any map $M \rightarrow B\chi$ is homotopic to a filtered one $M_* \rightarrow (B\chi)_*$. Also, any filtered map $f: M_* \rightarrow (B\chi)_*$ is homotopic to the map f_{α_f, β_f} (because the χ -labeling (α_f, β_f) encodes the restriction of these maps on the 2-skeleton of M and $B\chi$ is a 2-type). Consequently, the map $\{\chi\text{-labelings of } P\} / \mathcal{G}_P \rightarrow [M, B\chi]$ is surjective.

6.6. Remark

As in the end of the above proof, assume that the regions of P are disks. Let $\Pi(P)$ be the crossed complex associated with the skeletal filtration of the CW-decomposition of M dual to P . Then χ -labelings of P correspond to crossed complex morphisms $\Pi(P) \rightarrow \chi$

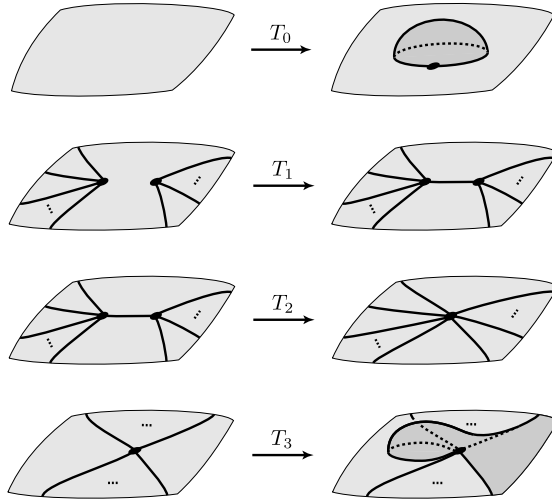


Fig. 2. Local moves on skeletons.

(called formal maps in [11]) and the last claim of Lemma 6.1 follows from the homotopy classification theorem [3, Theorem A].

6.7. Closed χ -manifolds and their χ -skeletons

By a *closed χ -manifold*, we mean a pair (M, g) where M is a closed oriented 3-dimensional manifold and g is a homotopy class of maps $M \rightarrow B\chi$.

A χ -skeleton of a closed χ -manifold (M, g) is a skeleton P of M endowed with a χ -labeling of P representing $g \in [M, B\chi]$. For brevity, χ -skeleton will be often denoted by the same letter P as the underlying skeleton. Lemma 6.1 shows that any closed χ -manifold has a χ -skeleton (since any oriented closed 3-dimensional manifold always has a skeleton with disk regions, such as the 2-skeleton of a triangulation).

6.8. Moves on χ -skeletons

Let M be a closed oriented 3-dimensional manifold. We consider four local moves T_0, T_1, T_2, T_3 on a skeleton P of M transforming P into a new skeleton of M (see [20]). Each of these moves modifies P inside a closed 3-ball in M as in Fig. 2 where the shaded horizontal plane represents (a piece of) P and the upper half-space represents an adjacent P -ball branch B . The edges shown in the horizontal plane before the moves may be adjacent to regions of P lying in the lower half-space (and not shown in the picture).

The *bubble move* T_0 adds to P an embedded disk $D_+ \subset M$ such that $\partial D_+ = D_+ \cap P \subset P \setminus P^{(1)}$, the circle ∂D_+ bounds a disk D_- in $P \setminus P^{(1)}$, and the 2-sphere $D_+ \cup D_-$ bounds a ball in M meeting P precisely along D_- . A point of the circle $\partial D_+ = \partial D_-$ is chosen as a vertex of the skeleton $T_0(P) = P \cup D_+$, and the circle itself is an edge of $T_0(P)$. The

disks D_+ and D_- become regions of $T_0(P)$, and all other regions of $T_0(P)$ correspond bijectively to the regions of P .

The *phantom edge move* T_1 keeps P as a polyhedron and adds a new 2-valent edge meeting $P^{(1)}$ solely at its endpoints which must be distinct vertices of P . The *contraction move* T_2 collapses an edge of P with distinct endpoints into a point. This move is allowed only when at least one endpoint of the collapsing edge is an endpoint of some other edge. (The valence of the collapsing edge may be arbitrary.) The *percolation move* T_3 pushes a branch of P through a vertex x of P . The branch is pushed across a small disk D lying in another branch of P at x so that these branches are adjacent to the same component of $M \setminus P$ and $D \cap P^{(1)} = \partial D \cap P^{(1)} = \{x\}$. The loop ∂D based at x becomes an edge of the resulting 2-polyhedron. The disk D becomes a region of the resulting skeleton $T_3(P)$ and all other regions of $T_3(P)$ correspond bijectively to the regions of P in the obvious way.

In the pictures of $T_0 - T_3$, we distinguish the “small” regions and edges entirely contained in the 3-ball where the move proceeds, from the “big” regions and edges not entirely contained in that 3-ball. Recall that all regions of a skeleton are oriented. The moves $T_0 - T_3$ preserve orientation in the big regions while orientation of the small disk regions created by T_0 and T_3 may be arbitrary.

The moves $T_0 - T_3$ lift to χ -labelings as follows. The H -labels of all big regions and E -labels of all big edges are preserved under the moves. Under T_0 , the H -label of D_+ and the E -label of the edge ∂D_+ (for some orientation of ∂D_+ and with respect to B) may be chosen arbitrarily, and then the H -label of D_- is determined uniquely by (20). One easily verifies that different choices for the labels of D_+ and ∂D_+ give rise to gauge equivalent χ -labelings. Under T_1 , the E -label of the added 2-valent edge should be $1 \in E$. For T_2 , there are no conditions on the E -label of the collapsed edge. Under T_3 , the E -label of the edge ∂D (for some orientation of ∂D and with respect to B) may be chosen arbitrarily and the H -label of D is then determined uniquely by (20). Different choices for the E -label of ∂D give rise to gauge equivalent χ -labelings. Note that the link graphs of the vertices created by these moves are indeed 1-spherical χ -graphs. These moves lifted to χ -labelings are called χ -moves.

The χ -moves $T_0 - T_3$ have inverses. The χ -move T_0^{-1} is obvious from the picture of T_0 . The χ -move T_1^{-1} deletes a 2-valent edge with distinct endpoints which are also endpoints of some other edges. This move is allowed only when the E -label (with respect to B) of the edge is 1 and the orientations of the two regions adjacent to that edge are compatible. The χ -move T_2^{-1} stretches a vertex into an edge and the E -label (with respect to B) of this new edge is uniquely determined by the 1-sphericity of the χ -graph associated to a created vertex. The χ -move T_3^{-1} collapses an embedded disk region whose boundary is formed by a single 3-valent edge with coinciding endpoints.

The χ -moves $T_0^{\pm 1}, T_1^{\pm 1}, T_2^{\pm 1}, T_3^{\pm 1}$ together with the label-preserving ambient isotopies of χ -skeletons in M are called *primary χ -moves*.

Lemma 6.4. *Any primary χ -move transforms a χ -skeleton of a closed χ -manifold (M, g) into a χ -skeleton of (M, g) . Moreover, any two χ -skeletons of (M, g) can be related by a finite sequence of primary χ -moves.*

We prove Lemma 6.4 in the next section.

6.9. Proof of Lemma 6.4

If a skeleton P of M is transformed into a skeleton P' of M via a move T_i with $i \in \{0, 2, 3\}$, then any lift of this move to a χ -move induces a bijection

$$\{\chi\text{-labelings of } P\}_{/G_P} \xrightarrow{\cong} \{\chi\text{-labelings of } P'\}_{/G_{P'}}.$$

Also, if a skeleton P of M is transformed into a skeleton P' of M via the move T_1 by adding an edge e inside a region r of P , then the lift of this move to a χ -move (by labeling e with $1 \in E$) induces a bijection

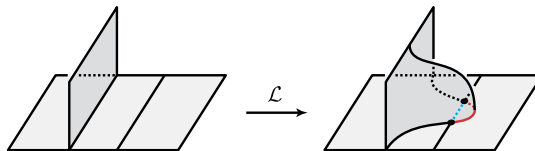
$$\{\chi\text{-labelings of } P\}_{/G_P} \xrightarrow{\cong} \{\chi\text{-labelings of } P' \text{ with } e \text{ labeled by } 1\}_{/G_{P'}}.$$

It is worth to notice that if adding the edge e transforms r into two distinct regions, then

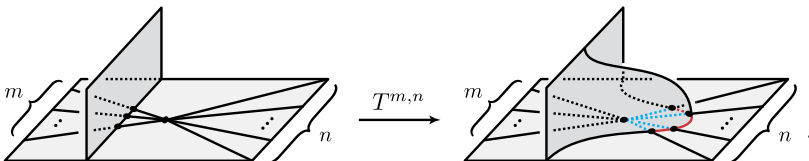
$$\{\chi\text{-labelings of } P' \text{ with } e \text{ labeled by } 1\}_{/G_{P'}} = \{\chi\text{-labelings of } P'\}_{/G_{P'}}.$$

The existence and injectivity of these maps follow from the fact that two χ -labelings of P give rise to gauge equivalent χ -labelings of P' if and only if they are gauge equivalent. The surjectivity of these maps follow from the existence of inverse χ -moves. Moreover, these bijections commute with the maps to the set of homotopy classes of maps from M to $B\chi$ given by Lemma 6.1. This implies the first assertion of the lemma.

To prove the second assertion of the lemma, we need further χ -moves on χ -skeletons. The lune χ -move \mathcal{L} is defined as:

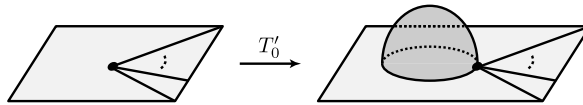


and, for any non-negative integers m, n with $m + n \geq 1$, the χ -move $T^{m,n}$ is defined as:



The χ -move \mathcal{L} creates a small disk region. The χ -move $T^{m,n}$ destroys $\max(m - 1, 0)$ small disk regions and creates $\max(n - 1, 0)$ small disk regions. These χ -moves preserve orientation in the big regions. The orientation of a small created disk is induced by the orientation of the corresponding big region before the move. As above, the H -labels of all big regions and E -labels of all big edges are preserved under the move. The E -labels of the created edges depicted in red are all equal to $1 \in E$ (to distinguish the colors in the pictures, the reader is referred to the web version of this article). This uniquely determines the H -labels of the small created disk regions (using (20)) as well as the E -labels of the created edges depicted in blue (using the 1-sphericity of the χ -graphs associated to the created vertices). The χ -move \mathcal{L} has an obvious inverse \mathcal{L}^{-1} . It is allowed only when the orientations and H -labels of two regions united under this move as well as the E -labels of two edges united under this move are compatible. The χ -moves $\mathcal{L}^{\pm 1}$ and $T^{m,n}$ transform a χ -skeleton of (M, g) into a χ -skeleton of (M, g) . Indeed, they are compositions of the primary χ -moves $T_1^{\pm 1}$, $T_2^{\pm 1}$, and T_3 for which the E -label of the created edge is $1 \in E$. (These compositions lift the decomposition of the corresponding moves on skeletons given in [20, Section 11.3.2].)

We also need the χ -move T'_0 defined as:



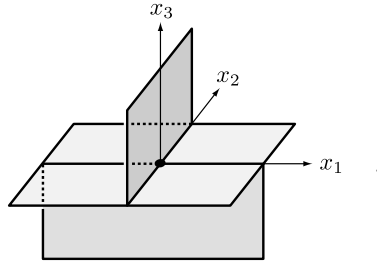
and viewed as the composition of the χ -moves T_0 , T_1 , and then T_2 (first apply a bubble move to the left region, then connect the two vertices with an edge, and lastly contract this edge). The inverse of T'_0 always exists. This follows from the fact that the 1-sphericity of the χ -graph associated to the vertex of the right-hand side of the picture always implies the 1-sphericity of the χ -graph associated to the vertex of the left-hand side. As above, we will denote the moves on skeletons underlying the χ -moves $T^{m,n}$ and $(T'_0)^{\pm 1}$ by the same letters.

Claim 6.5. *Two skeletons of M with disk regions can be related by a finite sequence of the moves $(T'_0)^{\pm 1}$, $T_2^{\pm 1}$, and $T_3^{\pm 1}$.*

Proof. Recall from [20, Section 11.1.5] that an s -skeleton of M is a skeleton P of M such that any point of P has an open neighborhood homeomorphic to an open subset of the set

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 0, \text{ or } x_1 = 0 \text{ and } x_3 > 0, \text{ or } x_2 = 0 \text{ and } x_3 < 0\}$$

depicted as



The proof of [20, Lemma 11.4] implies that any skeleton of M with disk regions can be turned into an s -skeleton of M with disk regions by finitely many $(T_0')^{\pm 1}, T_2^{\pm 1}, T_3^{\pm 1}$ -moves. Next, by [10, Theorem 1.2.30], any two s -skeletons of M with disk regions can be related by finitely many $T^{1,2}$ and $T^{2,1}$ -moves which can be expressed using $T_2^{\pm 1}$ and $T_3^{\pm 1}$ -moves. \square

Claim 6.6. *Two χ -skeletons of (M, g) with the same underlying skeleton of M are related by a finite sequence of primary χ -moves.*

Proof. Let $(P, (\alpha, \beta))$ and $(P, (\alpha', \beta'))$ be χ -skeletons of (M, g) with the same underlying skeleton P of M . By Lemma 6.1, the χ -labelings (α, β) and (α', β') are gauge equivalent (since they represent the same homotopy class g). Moreover, the gauge group \mathcal{G}_P is generated by the set

$$\{(\lambda_{B,h}, \mu_{r,e}) \mid B \text{ is a } P\text{-ball, } r \text{ is a region adjacent to } B, h \in H, e \in E\},$$

where $\lambda_{B,h}: \pi_0(M \setminus P) \rightarrow H$ carries B to h and carries all other P -balls to 1, while $\mu_{r,e}: \text{Reg}(P) \rightarrow E$ carries r to e and carries all other regions to 1. Thus, it is enough to show that there is a finite sequence of primary χ -moves between $(P, (\alpha, \beta))$ and $(P, (\alpha', \beta'))$ when $(\alpha', \beta') = (\lambda_{B,h}, \mu_{r,e}) \cdot (\alpha, \beta)$. To prove this, we first apply a bubble move which adds to P a 2-disk $D_+ \subset \overline{B}$ such that the circle ∂D_+ lies in r and bounds a 2-disk $D_- \subset r$. We endow D_- with the orientation induced by that of r . We orient D_+ so that the orientation of D_+ followed by a normal vector pointing towards B yields the given orientation of M . We endow the small region D_+ with the H -label h . We endow the oriented edge ∂D_- (relative to B) with E -label e if $r_+ = B$ and with the E -label $\alpha^{(r)}e$ otherwise. The H -label of D_- , uniquely determined by (20), is then equal to $\alpha^{(r)}$. Next, we isotope the disk D_+ in B so that it sweeps B almost entirely while its boundary slides along $\partial \overline{B}$. We arrange that in the terminal position D'_+ of the moving disk, its boundary circle lies in $r \setminus D_-$ and bounds there a 2-disk. This isotopy of D_+ can be performed using a finite sequence of the χ -moves $\mathcal{L}^{\pm 1}$ and $T^{m,n}$. Next we remove the disk D'_+ using the inverse bubble χ -move. This sequence transforms the χ -skeleton $(P, (\alpha, \beta))$ into a χ -skeleton $(P, (\alpha'', \beta''))$ of (M, g) . Finally, it follows from the choices of labelings of the first bubble χ -move that $(\alpha'', \beta'') = (\lambda_{B,h}, \mu_{r,e}) \cdot (\alpha, \beta) = (\alpha', \beta')$. \square

Let us prove the second assertion of the lemma. Let $(P, (\alpha, \beta))$ and $(P', (\alpha', \beta'))$ be two χ -skeletons of (M, g) . If necessary, apply finitely many T_1 -moves to transform non-disk regions of P and P' into disk regions, obtaining then skeletons Q and Q' of M with disk regions. These T_1 -moves lift to χ -moves by labeling added edges with $1 \in E$ and we obtain χ -skeletons $(Q, (\gamma, \delta))$ and $(Q', (\gamma', \delta'))$ of (M, g) . Note that these χ -moves have inverses. In particular, there are finite sequences of primary χ -moves from $(P, (\alpha, \beta))$ to $(Q, (\gamma, \delta))$ and from $(Q', (\gamma', \delta'))$ to $(P', (\alpha', \beta'))$. By Claim 6.5, there is a finite sequence of moves $(T_0')^{\pm 1}$, $T_2^{\pm 1}$, and $T_3^{\pm 1}$ transforming Q into Q' . Since all these moves can always be lifted to χ -moves (because they transform a skeleton with disk regions into a skeleton with disk regions), the χ -labeling (γ, δ) of Q can be transported under this sequence (by lifting successively the moves to χ -moves) to a χ -labeling (γ'', δ'') of Q' . In particular, there is a sequence of primary χ -moves between $(Q, (\gamma, \delta))$ and $(Q', (\gamma'', \delta''))$. By the first assertion of the lemma, $(Q', (\gamma'', \delta''))$ and $(Q', (\gamma', \delta'))$ are χ -skeletons of (M, g) . Then Claim 6.6 implies that there is a finite sequence of primary χ -moves between $(Q', (\gamma'', \delta''))$ and $(Q', (\gamma', \delta'))$. Hence, by composing the above sequences, we obtain a finite sequence of primary χ -moves between $(P, (\alpha, \beta))$ and $(P', (\alpha', \beta'))$.

7. State-sum invariants of closed χ -manifolds

We fix throughout this section a crossed module $\chi: E \rightarrow H$, a spherical χ -fusion category \mathcal{C} (over \mathbb{k}), and a χ -representative set $I = \coprod_{h \in H} I_h$ for \mathcal{C} . We assume that the dimension $\dim(\mathcal{C}_1^1)$ of the neutral component \mathcal{C}_1^1 of the 1-subcategory \mathcal{C}^1 of \mathcal{C} is invertible in the ground ring \mathbb{k} (see Section 4.10). We derive from this data a scalar topological invariant of closed χ -manifolds (see Section 6.7) defined as a state sum on χ -skeletons.

7.1. The state sum invariant

Let (M, g) be a closed χ -manifold. Pick a χ -skeleton P of M (see Section 6.7). Denote by (α, β) the χ -labeling of P and recall from Section 6.1 that $\text{Reg}(P)$ is the (finite) set of regions of P .

A coloring of P is a map $c: \text{Reg}(P) \rightarrow I$ such that $c(r) \in I_{\alpha(r)}$ for all $r \in \text{Reg}(P)$. The object $c(r) \in I$ assigned to a region r of P is called the c -color of r . Note that there are finitely many colorings of P (because the sets I_h with $h \in H$ and $\text{Reg}(P)$ are finite). For a coloring c of P , we set

$$\dim(c) = \prod_{r \in \text{Reg}(P)} (\dim c(r))^{\text{Eul}(r)} \in \mathbb{k}, \tag{26}$$

where $\text{Eul}(r)$ is the Euler characteristic of the region r . Note that $\dim(c)$ is well-defined since the dimension of a simple object in the 1-subcategory \mathcal{C}^1 of \mathcal{C} is invertible (see Section 4.10).

Next, we define a scalar $|c| \in \mathbb{k}$ for any coloring c of P . For each oriented edge e of P , the coloring c of P turns the χ -cyclic set P_e of branches of P at e (see Section 6.3)

into a χ -cyclic \mathcal{C} -set by assigning to each branch $b \in P_e$ the c -color of the region of P containing b . Let $H_c(e) = H(P_e)$ be its multiplicity module (see Section 5.1). It is a free \mathbb{k} -module of finite rank (by Lemma 3.1). Let

$$H_c = \bigotimes_e H_c(e)$$

be the unordered tensor product of the \mathbb{k} -modules $H_c(e)$ over all oriented edges e of P .

Each unoriented edge e of P gives rise to two opposite oriented edges e_1, e_2 of P . The χ -cyclic \mathcal{C} -set P_{e_1} and P_{e_2} are dual: $P_{e_2} = (P_{e_1})^{\text{op}}$. This yields a vector

$$*_e = *_{P_{e_1}} \in H_c(e_1) \otimes H_c(e_2)$$

independent of the numeration of e_1, e_2 (see Section 5.1). Set

$$*_c = \otimes_e *_e \in H_c$$

where \otimes_e is the unordered tensor product over all unoriented edges e of P .

For a vertex x of P , consider the link graph $\Gamma_x \subset \partial B_x$ where $B_x \subset M$ is a P -cone neighborhood of x (see Section 6.2). Recall from Section 6.3 that the χ -labeling of P turns Γ_x into a 1-spherical χ -graph in $\partial B_x \cong S^2$. The coloring c of P makes it \mathcal{C} -colored: every edge of Γ_x lies in a region r of P and is colored with $c(r) \in I$. In this way, Γ_x becomes a 1-spherical \mathcal{C} -colored χ -graph in ∂B_x . It is denoted Γ_x^c . Since $\text{End}_{\mathcal{C}}^1(\mathbb{1}) = \mathbb{k}$, Section 5.8 yields a vector

$$\mathbb{F}_{\mathcal{C}}(\Gamma_x^c) \in H(\Gamma_x^c)^*$$

where we denote by $U^* = \text{Hom}_{\mathbb{k}}(U, \mathbb{k})$ the dual of a \mathbb{k} -module U . Note that the χ -cyclic \mathcal{C} -set associated with any vertex v of Γ_x^c is canonically isomorphic to the χ -cyclic \mathcal{C} -set P_e , where e is the edge of P containing v and oriented away from x . Therefore, there are canonical isomorphisms

$$H(\Gamma_x^c) \simeq \otimes_{e_x} H_c(e_x) \quad \text{and} \quad H(\Gamma_x^c)^* \simeq \otimes_{e_x} H_c(e_x)^*,$$

where e_x runs over all edges of P incident to x and oriented away from x . (An edge with both endpoints in x appears in each of these tensor products twice with opposite orientations.) The tensor product of the latter isomorphisms over all vertices x of P yields a \mathbb{k} -linear isomorphism

$$\bigotimes_x H(\Gamma_x^c)^* \simeq \bigotimes_x \bigotimes_{e_x} H_c(e_x)^* \simeq \bigotimes_e H_c(e)^* \simeq H_c^*$$

where e runs over all oriented edges of P . The image under this isomorphism of the unordered tensor product $\bigotimes_x \mathbb{F}_{\mathcal{C}}(\Gamma_x^c)$, where x runs over all vertices of P , is a vector $V_c \in H_c^*$. We evaluate V_c on $*_c$ and set

$$|c| = V_c(*_c) \in \mathbb{k}.$$

Finally, set

$$|M, g|_c = \dim(\mathcal{C}_1^1)^{-|M \setminus P|} \sum_c \dim(c) |c| \in \mathbb{k}, \tag{27}$$

where $|M \setminus P|$ is the number of components of $M \setminus P$ and c runs over all colorings of P . The right-hand side of (27) is well-defined because there are finitely many colorings of P .

Theorem 7.1. *$|M, g|_c$ is a topological invariant of the closed χ -manifold (M, g) independent of the choice of P and I .*

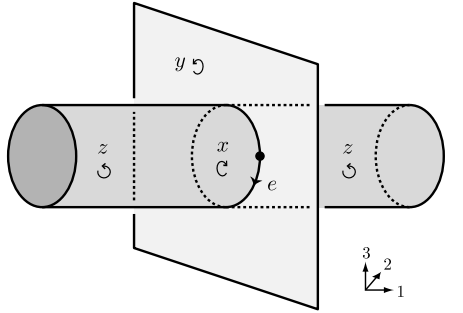
We prove Theorem 7.1 in Section 7.6 using moves on χ -skeletons introduced in Section 6.8. The invariant of Theorem 7.1 is nontrivial and may even distinguish homotopy classes of phantom maps (i.e., of maps inducing trivial homomorphisms on homotopy groups), see for instance Example 7.10 below.

Note that if g is the homotopy class of a constant map, then $|M, g|_c = |M|_{\mathcal{C}_1^1}$ where the right hand side is the Turaev-Viro-Barrett-Westbury state sum invariant of the oriented closed 3-dimensional manifold M derived from the spherical fusion category \mathcal{C}_1^1 (see [1,20]). Moreover Theorem 7.1 generalizes the construction of [17] which produces an invariant of closed oriented 3-dimensional manifolds endowed with a homotopy class of maps to a $K(G, 1)$ space, where G is a group, from any spherical G -fusion category (over \mathbb{k}) whose neutral component has invertible dimension in \mathbb{k} . Indeed, the classifying space $B\chi$ is aspherical if and only if χ is injective. And if $\chi: E \rightarrow H$ is injective (for example $E = 1$), then $B\chi$ is a $K(G, 1)$ space with $G = \text{Coker}(\chi) = H/\text{Im}(\chi)$, the category \mathcal{C} is a G -fusion category (in the sense of [17]) whose neutral component has invertible dimension $\dim(\mathcal{C}_1^1)$ (see Remark 4.11), and the invariant of closed χ -manifolds from [17] defined using this G -fusion category is equal to the invariant $|\cdot|_c$ of Theorem 7.1.

We illustrate Theorem 7.1 with the following two examples.

7.2. Example

Consider the closed 3-dimensional manifold $S^1 \times S^2$ with the product orientation. Pick a point $p \in S^1$ and an embedded loop $\gamma \subset S^2$. The 2-polyhedron $P = (\{p\} \times S^2) \cup (S^1 \times \gamma)$ has two disk regions and one annulus region. Stratified by the based loop $\{p\} \times \gamma$, the polyhedron P is a skeleton of $S^1 \times S^2$ with one vertex, one edge, and two P -balls. Any χ -labeling of P is fully determined by a tuple $\ell = (x, y, z, e) \in H^3 \times E$ such that $\chi(e) = z^{-1}xzy$. Here x, y are the H -labels of the disk regions, z is the H -label of the annulus region, and e is the E -label of the edge of P with respect to the front P -ball branch and for the choices of orientations as in the next picture which represents a neighborhood of the vertex of P :



By Lemma 6.1, this χ -labeling determines a homotopy class $g_\ell \in [S^1 \times S^2, B\chi]$. Then

$$|S^1 \times S^2, g_\ell|_{\mathcal{C}} \stackrel{(i)}{=} \dim(\mathcal{C}_1^1)^{-2} \sum_{i \in I_x, j \in I_y, k \in I_z} \dim(i) \dim(j) N_{k^* \otimes i \otimes k \otimes j}^{1,e}$$

$$\stackrel{(ii)}{=} \dim(\mathcal{C}_1^1)^{-1} \sum_{k \in I_z} \dim(k)^2 \stackrel{(iii)}{=} 1_{\mathbb{k}}.$$

Here (i) follows from (27), (19), and (12), (ii) from Lemma 4.1 (with $g = x, h = y, X = k^*, Y = k, m = i, n = j$), and (iii) from (8).

7.3. Example

Consider the lens space $L(p, q)$ where p, q are coprime integers with $1 \leq q < p$. Recall from Example 6.4 that any $g \in [L(p, q), B\chi]$ is of the form $g = g_{h,e}$ for some $(h, e) \in H \times E$ satisfying $\chi(e) = h^p$ and ${}^h e = e$. By applying Formula (27) to the χ -skeleton P of $(L(p, q), g)$ induced by (h, e) as in Example 6.4, we obtain

$$|L(p, q), g|_{\mathcal{C}} = \dim(\mathcal{C}_1^1)^{-1} \sum_{i \in I_h} \dim(i) \text{Tr}((\sigma_i)^q) \in \mathbb{k},$$

where σ_i is the \mathbb{k} -linear endomorphism of $\text{Hom}_{\mathcal{C}}^e(\mathbb{1}, i^{\otimes p})$ defined by

$$u \mapsto \sigma_i(u) = \begin{array}{c} i^{\otimes(p-1)} \downarrow \\ \text{---} \downarrow \text{---} \\ \boxed{u} \\ \text{---} \uparrow \text{---} \\ i \end{array}$$

and Tr is the standard trace of \mathbb{k} -linear endomorphisms. This follows from the description of the χ -graph Γ associated with the single vertex of P given in Example 6.4, the definition of the invariant $\mathbb{F}_{\mathcal{C}}(\Gamma)$, and Formula (19) which provides a computation of the contraction vector associated with the single edge of P .

7.4. A conjecture

Consider the spherical χ -fusion category $\mathbb{k}\mathcal{G}_\chi$ of Example 4.12. It follows directly from the definitions that for any closed χ -manifold (M, g) ,

$$|M, g|_{\mathbb{k}\mathcal{G}_\chi} = 1_{\mathbb{k}}. \tag{28}$$

More generally, consider the spherical χ -fusion category $\mathbb{k}\mathcal{G}_\chi^\omega$ obtained by twisting the χ -category $\mathbb{k}\mathcal{G}_\chi$ with a normalized 3-cocycle ω for χ with values in \mathbb{k}^* (see Section 4.16). Let $[\omega] \in H^3(B\chi, \mathbb{k}^*)$ be the cohomology class induced by ω . We conjecture that for any closed χ -manifold (M, g) ,

$$|M, g|_{\mathbb{k}\mathcal{G}_\chi^\omega} = \langle g^*([\omega]), [M] \rangle, \tag{29}$$

where $g^*([\omega]) \in H^3(M, \mathbb{k}^*)$ is the pullback of $[\omega]$ along g , $[M] \in H_3(M, \mathbb{Z})$ is the fundamental class of M , and $\langle \cdot, \cdot \rangle: H^3(M, \mathbb{k}^*) \times H_3(M, \mathbb{Z}) \rightarrow \mathbb{k}$ is the Kronecker pairing. Note that the computations in [20, Appendix H] imply that this conjecture is true when $E = 1$ (and so ω is a 3-cocycle for H), that is, for the crossed modules of the form $1 \rightarrow H$ with H a group.

7.5. Invariants of 3-manifolds

Let M be a closed oriented 3-dimensional manifold. It follows from Theorem 7.1 that if $\mathbf{a} = (\mathbf{a}_g)_{g \in [M, B\chi]}$ is a family of elements of \mathbb{k} such that $\mathbf{a}_g = 0$ for all but finitely many g , then the linear combination

$$I_{\mathbb{C}, \mathbf{a}}(M) = \sum_{g \in [M, B\chi]} \mathbf{a}_g |M, g|_{\mathbb{C}} \in \mathbb{k}$$

is a topological invariant of M .

Examples of such families of coefficients can for instance be derived from the mapping space $\text{TOP}(M, B\chi)$ of continuous maps from M to $B\chi$ (endowed with the compact-open topology). In particular, assume that the crossed module $\chi: E \rightarrow H$ is finite (meaning that both E and H are finite), so that the set $[M, B\chi] = \pi_0(\text{TOP}(M, B\chi))$ and the homotopy groups of $\text{TOP}(M, B\chi)$ are finite. Then, assuming also that \mathbb{k} is a field of characteristic zero, we can define the family of coefficients $\eta = (\eta_g)_{g \in [M, B\chi]}$ by setting

$$\eta_g = \frac{\text{card}(\pi_2(\text{TOP}(M, B\chi), g))}{\text{card}(\pi_1(\text{TOP}(M, B\chi), g))} \in \mathbb{k}.$$

For the spherical χ -fusion category $\mathbb{k}\mathcal{G}_\chi$ of Example 4.12, we obtain

$$I_{\mathbb{k}\mathcal{G}_\chi, \eta}(M) = \text{Yetter}_\chi(M),$$

where $\text{Yetter}_\chi(M)$ is the Yetter invariant of M associated with the finite crossed module χ (see [21]). This follows from (28) and the interpretation of the Yetter invariant given in [6, Theorem 2.25]. More generally, for the spherical χ -fusion category $\mathbb{k}\mathcal{G}_\chi^\omega$ obtained by twisting the χ -category $\mathbb{k}\mathcal{G}_\chi$ with a normalized 3-cocycle ω for χ with values in \mathbb{k}^* (see Section 4.16) and provided Conjecture (29) is true, we obtain

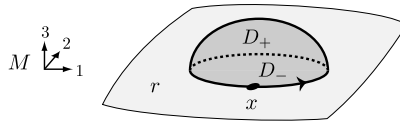
$$I_{\mathbb{k}\mathcal{G}_\chi^\omega, \eta}(M) = \text{Yetter}_{\chi, \omega}(M),$$

where $\text{Yetter}_{\chi, \omega}(M)$ is the ω -twisted Yetter invariant of M defined by Faria Martins and Porter in [6].

7.6. Proof of Theorem 7.1

The state sum $|M|_c$ does not depend on the choice of the χ -representative set I by the naturality of \mathbb{F}_c and of the contraction vectors. Denote the right hand side of (27) by $|P|_c$. Any χ -homeomorphism of χ -manifolds $M \rightarrow M'$ carries a χ -skeleton P of M to a χ -skeleton P' of M' and, clearly, $|P|_c = |P'|_c$. Thus, by Lemma 6.4, we need only to prove the invariance of $|P|_c$ under the primary χ -moves $P \mapsto T_s(P)$ for all $s \in \{0, 1, 2, 3\}$. This follows from the ‘‘local invariance’’ which says that the contribution of any coloring c of P to the state sum is equal to the sum of the contributions of all colorings of $T_s(P)$ which are equal to c on all big regions.

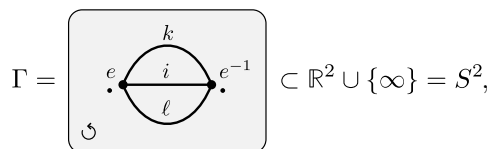
Let us verify the local invariance for the bubble χ -move T_0 . Denote by x the created vertex, by r the region of P where the bubble is attached, by D_\pm the disks created by the move:



Let $\alpha, \beta, \gamma \in H$ be the H -labels and $i, k, \ell \in I$ be the colors of r, D_+, D_- , respectively. Let $e \in E$ be the E -label of the created edge endowed with the depicted orientation and with respect to the (small) P -ball bounded by $D_+ \cup D_-$. Condition (20) becomes

$$\chi(e) = \beta^\nu \alpha^\varepsilon \gamma^\mu,$$

where ε, ν, μ are signs depending on the orientations of the regions r, D_+, D_- , respectively. The \mathcal{C} -colored χ -graph in S^2 associated with x is



where the edge colored by k (respectively, i, ℓ) is oriented towards the left vertex if the sign ν (respectively, ε, μ) is positive and towards the right vertex otherwise. The contraction vector associated to the edge created by the move is the contraction vector $*$ determined by the duality between the two vertices of Γ induced by the symmetry with respect to a vertical great circle. Now, using (19) and (12), we obtain:

$$\mathbb{F}_{\mathcal{C}}(\Gamma)(*) = \text{Diagram 1} = \text{Diagram 2} = N_{k^\nu \otimes i^\varepsilon \otimes \ell^\mu}^{\mathbb{1}, e}$$

Then the contribution of x to the state sum is

$$\sum_{\substack{k \in I_\beta \\ \ell \in I_\gamma}} \dim(k) \dim(\ell) \mathbb{F}_{\mathcal{C}}(\Gamma)(*) \stackrel{(i)}{=} \sum_{\substack{k \in I_\beta \\ \ell \in I_\gamma}} \dim(k) \dim(\ell) N_{k^\nu \otimes i^\varepsilon \otimes \ell^\mu}^{\mathbb{1}, e}$$

$$\stackrel{(ii)}{=} \sum_{\substack{m \in I_{\beta^\nu} \\ n \in I_{\gamma^\mu}}} \dim(m) \dim(n) N_{m \otimes i^\varepsilon \otimes n}^{\mathbb{1}, e} \stackrel{(iii)}{=} \dim(i^\varepsilon) \dim(\mathcal{C}_1^1) \stackrel{(iv)}{=} \dim(i) \dim(\mathcal{C}_1^1).$$

Here (i) follows from the previous computation, (ii) and (iv) from the equality $\dim(X^*) = \dim(X)$ for any $X \in \mathcal{C}$, and (iii) from Lemma 4.1 applied with $X = \mathbb{1}$ and $Y = i^\varepsilon$. The factor $\dim(i) \dim(\mathcal{C}_1^1)$ is compensated by the change in the Euler characteristic of the big region and in the number of components of $M \setminus P$.

Let us verify the local invariance for the phantom edge χ -move T_1 . This move adds a new edge e connecting two distinct vertices of P . This modifies the link graphs of these vertices by adding a new vertex u (respectively, v) inside an edge. The E -label of e is $1 \in E$ and so, by (20), the (possibly coinciding) regions r and r' of $T_1(P)$ lying on the two sides of e have the same H -label $h \in H$. The colorings of $T_1(P)$ assigning different colors to the regions r and r' contribute zero to the state sum (because $\text{Hom}_{\mathcal{C}}^1(i, j) = 0$ for distinct $i, j \in I$). The colorings of $T_1(P)$ assigning the same color $i \in I$ to the regions r and r' contribute the same as the colorings of P assigning i to the region of P containing e . Indeed,

$$\mathbb{F}_{\mathcal{C}} \left(\text{Diagram 3} \right) \otimes \mathbb{F}_{\mathcal{C}} \left(\text{Diagram 4} \right) (*_e)$$

$$= \dim(i)^{-1} \mathbb{F}_{\mathcal{C}} \left(\text{Diagram 5} \right) \otimes \mathbb{F}_{\mathcal{C}} \left(\text{Diagram 6} \right)$$

and the factor $\dim(i)^{-1}$ is compensated by the change in the Euler characteristics of the regions. The latter equality follows from the fact that $*_e$ is induced by the inverse Ω of the pairing

$$\omega: H_i \otimes_{\mathbb{k}} H_i \rightarrow \mathbb{k}, \quad u \otimes_{\mathbb{k}} v \mapsto \begin{array}{c} \text{---} i \text{---} \\ \curvearrowright \quad \curvearrowleft \\ \boxed{u} \quad \boxed{v} \end{array}$$

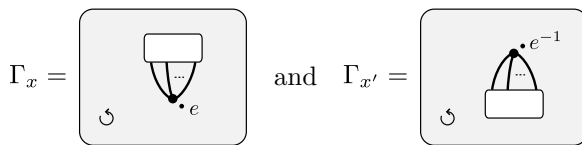
where $H_i = \text{Hom}_{\mathcal{C}}^1(\mathbb{1}, i^* \otimes i)$, and that $\Omega: \mathbb{k} \rightarrow H_i \otimes_{\mathbb{k}} H_i$ is computed by

$$\Omega(\mathbb{1}_{\mathbb{k}}) = \dim(i)^{-1} \widetilde{\text{coev}}_i \otimes_{\mathbb{k}} \widetilde{\text{coev}}_i$$

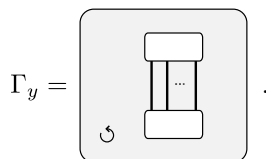
because $H_i \simeq \text{End}_{\mathcal{C}}^1(i)$ is free \mathbb{k} -module of rank 1 and

$$\omega(\widetilde{\text{coev}}_i \otimes_{\mathbb{k}} \widetilde{\text{coev}}_i) = \begin{array}{c} \text{---} i \text{---} \\ \curvearrowright \quad \curvearrowleft \\ \text{---} i \text{---} \end{array} = \begin{array}{c} \text{---} i \text{---} \\ \circlearrowright \\ \text{---} i \text{---} \end{array} = \dim(i).$$

Let us verify the local invariance for the contraction χ -move T_2 . This move collapses an edge of P between distinct vertices x, x' of P into a vertex y of $T_2(P)$. We can assume that x is an endpoint of some other edge. Let $e \in E$ be the E -label of the collapsed edge oriented from x to x' and with respect to the P -ball branch represented by the upper half-space. Note that the colorings of P and $T_2(P)$ are the same. Given such a coloring, the \mathcal{C} -colored χ -graphs $\Gamma_x, \Gamma_{x'}$ in S^2 associated with x, x' are of the following form:



where white boxes stand for a piece of a \mathcal{C} -colored χ -graph and the depicted vertices are in duality (induced by the collapsed edge). Denote by $*$ the contraction vector determined by this duality. The \mathcal{C} -colored χ -graph Γ_y in S^2 associated with y is obtained from Γ_x and $\Gamma_{x'}$ by joining their dual vertices:

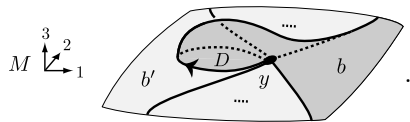


Then the contribution of x, x' to the state sum $|P|_{\mathcal{C}}$ is

$$\begin{aligned}
 & (\mathbb{F}_{\mathcal{C}}(\Gamma_x) \otimes \mathbb{F}_{\mathcal{C}}(\Gamma_{x'})) (*) \stackrel{(i)}{=} \mathbb{F}_{\mathcal{C}} \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right) (*) \\
 & \stackrel{(ii)}{=} \sum_{i \in I_1} \dim(i) \mathbb{F}_{\mathcal{C}} \left(\begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right) (*) \stackrel{(iii)}{=} \mathbb{F}_{\mathcal{C}}(\Gamma_y)
 \end{aligned}$$

and thus is equal to the contribution of y to the state sum $|T_3(P)|_{\mathcal{C}}$. Here (i) follows from the \otimes -multiplicativity of $\mathbb{F}_{\mathcal{C}}$ (see Section 5.6), (ii) from the facts that the χ -graph Γ_x is 1-spherical, $\text{Hom}_{\mathcal{C}}^1(i, \mathbb{1}) = 0$ for all $i \in I_1$ distinct from $\mathbb{1}$, and $\dim(\mathbb{1}) = 1_{\mathbb{k}}$, and (iii) from Lemma 5.2 since the element h in that lemma is equal to $\chi(e)$ by (20).

Finally, let us verify the local invariance for the percolation χ -move T_3 . This move pushes a branch b of P through a vertex x of P and across a small disk D lying in another branch b' of P at x . Denote by y the vertex of $T_3(P)$ corresponding to x :



Let $\alpha, \beta, \gamma \in H$ be the H -labels and $m, k, \ell \in I$ be the colors of D, b, b' , respectively. Let $e \in E$ be the E -label of ∂D endowed with the depicted orientation and with respect to the P -ball branch represented by the upper half-space above D . Condition (20) becomes

$$\chi(e) = \alpha^{-\varepsilon} \gamma^{\mu} \beta^{\nu},$$

where ε, ν, μ are signs depending on the orientations of the regions D, b, b' , respectively. The \mathcal{C} -colored χ -graph in S^2 associated with y is obtained from \mathcal{C} -colored χ -graph in S^2 associated with x by locally replacing



where the edge colored by k (respectively, ℓ, m) is oriented downwards if the sign ν (respectively, μ, ε) is positive and upwards otherwise. The contraction vector associated

to the edge ∂D is the contraction vector $*$ determined by the duality between the two vertices (in the above right picture) induced by the symmetry with respect to a horizontal great circle. Then the contribution of y to the state sum $|T_3(P)|_{\mathcal{C}}$ is

$$\sum_{m \in I_{\alpha}} \dim(m) \mathbb{F}_{\mathcal{C}} \left(\begin{array}{c} \ell \quad k \\ \cdot \quad \cdot \\ m^{\varepsilon} \quad e \\ \cdot \quad \cdot \\ \ell \quad k \end{array} \right) (*)$$

$$\stackrel{(i)}{=} \sum_{i \in I_{\alpha^{\varepsilon}}} \dim(i) \mathbb{F}_{\mathcal{C}} \left(\begin{array}{c} \ell \quad k \\ \cdot \quad \cdot \\ i \quad e \\ \cdot \quad \cdot \\ \ell \quad k \end{array} \right) (*) \stackrel{(ii)}{=} \mathbb{F}_{\mathcal{C}} \left(\begin{array}{|c|c|} \hline \ell & k \\ \hline \end{array} \right)$$

and thus is equal to the contribution of x to the state sum $|P|_{\mathcal{C}}$. Here (i) follows from the equality $\dim(X^*) = \dim(X)$ for any $X \in \mathcal{C}$ and (ii) follows from Lemma 5.2 since $\alpha^{\varepsilon} = \gamma^{\mu} \beta^{\nu} \chi(e^{-1})$.

7.7. Remark

The right-hand side of Formula (27) is the product of $\dim(\mathcal{C}_1^1)^{-|M \setminus P|}$ and a certain sum which we denote $\Sigma_{\mathcal{C}}(P)$. The definition of $\Sigma_{\mathcal{C}}(P) \in \mathbb{k}$ does not use the assumption that $\dim(\mathcal{C}_1^1)$ is invertible in \mathbb{k} and applies to an arbitrary spherical χ -fusion category \mathcal{C} . This allows us to generalize the invariant $|M, g|_{\mathcal{C}}$ of a closed χ -manifold (M, g) to any such \mathcal{C} . We use the theory of spines, see [10]. By a spine of M , we mean an oriented stratified 2-polyhedron $P \subset M$ such that P has at least 2 vertices, P is locally homeomorphic to the cone over the 1-skeleton of a tetrahedron, and $M \setminus P$ is an open ball. By [10], M has a spine P and any two spines of M can be related by the moves $T^{1,2}, T^{2,1}$ in the class of spines. These moves lift to χ -moves and the arguments of Section 7.6 imply that $\Sigma_{\mathcal{C}}(P)$ is preserved under the χ -moves $T^{1,2}, T^{2,1}$. Therefore $\|M, g\|_{\mathcal{C}} = \Sigma_{\mathcal{C}}(P)$ is a topological invariant of (M, g) . If $\dim(\mathcal{C}_1^1)$ is invertible, then $\|M, g\|_{\mathcal{C}} = \dim(\mathcal{C}_1^1) |M, g|_{\mathcal{C}}$.

7.8. The case of pointed χ -manifolds

For pointed topological spaces X, Y , we denote by $[X, Y]_*$ the set of pointed homotopy classes of pointed maps $X \rightarrow Y$. Note that if X is a connected CW-complex with basepoint a 0-cell and if Y is path-connected, then $\pi_1(Y)$ acts (on the right) on $[X, Y]_*$ and the canonical map $[X, Y]_* \rightarrow [X, Y]$ induces a bijection

$$[X, Y]_* / \pi_1(Y) \simeq [X, Y].$$

By a *pointed closed χ -manifold*, we mean a pair (M, g) where M is a pointed closed oriented 3-dimensional manifold and $g \in [M, B\chi]_*$. Here $B\chi$ is pointed by its unique 0-cell (see Section 2.4). Then the invariant $|\cdot|_{\mathcal{C}}$ of closed χ -manifolds from Theorem 7.1 extends trivially to pointed closed χ -manifolds by setting

$$|M, g|_{\mathcal{C}} = |M, v(g)|_{\mathcal{C}}$$

for any pointed closed χ -manifold (M, g) , where $v(g)$ is the image of g under the canonical map $[M, B\chi]_* \rightarrow [M, B\chi]$.

7.9. *The case of push-forwards*

Let $\phi = (\psi: E \rightarrow E', \varphi: H \rightarrow H')$ be a morphism from a crossed module $\chi: E \rightarrow H$ to a crossed module $\chi': E' \rightarrow H'$ such that ψ and φ are surjective, $\text{Ker}(\psi) \cap \text{Ker}(\chi) = 1$, and $\text{Ker}(\varphi)$ is finite. Let \mathcal{C} be a spherical χ -fusion category. By Section 4.14, the push-forward $\phi_*(\mathcal{C})$ of \mathcal{C} is a χ' -fusion category such that

$$\dim(\phi_*(\mathcal{C})_1^1) = d_\phi \dim(\mathcal{C}_1^1) \quad \text{with} \quad d_\phi = \frac{\text{card}(\text{Ker}(\varphi))}{\text{card}(\text{Ker}(\psi))} \in \mathbb{Z}_+.$$

Assume that $\text{Ker}(\chi')$ is finite and that $\text{card}(\text{Ker}(\varphi))$, $\text{card}(\text{Ker}(\chi'))$, and $\dim(\mathcal{C}_1^1)$ are invertible in \mathbb{k} . Then d_ϕ and $\dim(\phi_*(\mathcal{C})_1^1)$ are also invertible. Moreover, since ψ induces an injective group homomorphism $\text{Ker}(\chi) \rightarrow \text{Ker}(\chi')$ (because $\chi'\psi = \varphi\chi$ and $\text{Ker}(\psi) \cap \text{Ker}(\chi) = 1$), we get that $\text{card}(\text{Ker}(\chi))$ divides $\text{card}(\text{Ker}(\chi'))$ and so is finite and invertible in \mathbb{k} .

By Section 7.8, the category $\phi_*(\mathcal{C})$ defines the invariant $|\cdot|_{\phi_*(\mathcal{C})}$ of pointed closed χ' -manifolds and the category \mathcal{C} defines the invariant $|\cdot|_{\mathcal{C}}$ of pointed closed χ -manifolds. The next theorem computes the former from the latter. A (pointed) closed χ -manifold is *connected* if the underlying manifold is connected.

Theorem 7.2. *Let (M, g') be a connected pointed closed χ' -manifold. Then*

$$|M, g'|_{\phi_*(\mathcal{C})} = d_\phi^{-1} \sum_{\substack{g \in [M, B\chi]_* \\ B\phi \circ g = g'}} \frac{\text{card}(\pi_1(\text{TOP}_*(M, B\chi'), g'))}{\text{card}(\pi_1(\text{TOP}_*(M, B\chi), g))} |M, g|_{\mathcal{C}},$$

where $\text{TOP}_*(X, Y)$ denotes the space of pointed continuous maps from X to Y (endowed with the compact-open topology) and $B\phi: B\chi \rightarrow B\chi'$ is the (pointed) map induced by ϕ .

Note that the finiteness of the above sum follows from the fact that $\text{Ker}(\varphi)$ is finite. Also, the involved fundamental groups are finite with invertible cardinal in \mathbb{k} (since M is compact and $\text{Ker}(\chi) = \pi_2(B\chi)$ and $\text{Ker}(\chi') = \pi_2(B\chi')$ are finite with invertible cardinals, see Claim 7.16 below). We prove Theorem 7.2 in Section 7.12.

In the case of groups, that is, when E and E' are trivial, the fundamental groups $\pi_1(\text{TOP}(M, B\chi), g)$ and $\pi_1(\text{TOP}(M, B\chi'), g')$ are trivial, $d_\phi = \text{card}(\text{Ker}(\varphi))$, and Theorem 7.2 gives back the relationship given in the appendix of [17].

If ϕ is an equivalence of crossed modules (meaning that ψ and φ respectively induce isomorphisms $\text{Ker}(\chi) \cong \text{Ker}(\chi')$ and $\text{Coker}(\chi) \cong \text{Coker}(\chi')$), then $B\phi$ is a homotopy equivalence (by the Whitehead theorem), $d_\phi = 1$, and Theorem 7.2 gives

$$|M, g'|_{\phi_*(\mathcal{C})} = |M, (B\phi)^{-1} \circ g'|_{\mathcal{C}}$$

where $(B\phi)^{-1}$ is a homotopy inverse of $B\phi$.

7.10. Example

Consider the spherical χ -fusion category $\mathcal{C} = \mathbb{k}\mathcal{G}_\chi$ of Example 4.12. Then Theorem 7.2 together with (28) implies that for any connected pointed closed χ' -manifold (M, g') ,

$$|M, g'|_{\phi_*(\mathbb{k}\mathcal{G}_\chi)} = d_\phi^{-1} \sum_{\substack{g \in [M, B\chi]_* \\ B\phi \circ g = g'}} \frac{\text{card}(\pi_1(\text{TOP}_*(M, B\chi'), g'))}{\text{card}(\pi_1(\text{TOP}_*(M, B\chi), g))} 1_{\mathbb{k}}. \tag{30}$$

Let us illustrate this with the following explicit morphism of crossed modules: consider finite abelian groups A, G with invertible cardinal in \mathbb{k} and such that $H^2(G, A) = 0$. Let

$$\phi = \left(\begin{array}{ccc} A & \xrightarrow{\chi} & A \times G \\ \psi \downarrow & & \downarrow \varphi \\ A & \xrightarrow{\chi'} & 1 \end{array} \right)$$

where $\chi(a) = (a, 1)$ for all $a \in A$, the action of $A \times G$ on A is trivial, and $\psi = \text{id}_A$. Note that $\text{Ker}(\chi) = 1$ and so $\text{Ker}(\psi) \cap \text{Ker}(\chi) = 1$. Also $\text{Ker}(\varphi) = A \times G$ and $\text{Ker}(\chi') = A$ are finite with invertible cardinal in \mathbb{k} . Using (i) of Section 2.4, we obtain that the classifying spaces $B\chi$ and $B\chi'$ are Eilenberg-MacLane spaces of type $K(G, 1)$ and $K(A, 2)$, respectively. Recall that $[X, K(B, n)]_* \simeq H^n(X, B)$ for any CW-complex X , abelian group B , and $n \geq 1$. Then the induced map $B\phi: B\chi \rightarrow B\chi'$ is (pointed) null-homotopic because

$$[B\chi, B\chi']_* \simeq H^2(B\chi, A) \simeq H^2(G, A) = 0.$$

Let M be a connected pointed closed oriented 3-dimensional manifold. For any $g' \in [M, B\chi']_*$ which is not null-homotopic, there are no $g \in [M, B\chi]_*$ such that $B\phi \circ g = g'$ (since $B\phi$ is null-homotopic) and so Formula (30) gives that

$$|M, g'|_{\phi_*(\mathbb{k}\mathcal{G}_\chi)} = 0.$$

Let $c' \in [M, B\chi']_*$ be the null-homotopy class. For any $g \in [M, B\chi]_*$, we have $B\phi \circ g = c'$ (because $B\phi$ is null-homotopic) and $\pi_1(\text{TOP}_*(M, B\chi), g) = 1$ (because $B\chi$ is a $K(G, 1)$ space). Since $d_\phi = \text{card}(A)\text{card}(G)$, $[M, B\chi]_* \simeq H^1(M, G)$, and

$$\begin{aligned} \pi_1(\text{TOP}_*(M, B\chi'), c') &\simeq [S^1, (\text{TOP}_*(M, B\chi'), c')]_* \\ &\simeq [M \wedge S^1, B\chi']_* \simeq H^2(M \wedge S^1, A) \simeq H^1(M, A), \end{aligned}$$

Formula (30) gives that

$$|M, c'|_{\phi_*(\mathbb{k}\mathcal{G}_\chi)} = \frac{\text{card}(H^1(M, A))}{\text{card}(A)} \frac{\text{card}(H^1(M, G))}{\text{card}(G)} 1_{\mathbb{k}}.$$

For example, assume that $\mathbb{k} = \mathbb{Q}$, $A = \mathbb{Z}/2\mathbb{Z}$, $G = \mathbb{Z}/3\mathbb{Z}$, and $M = \mathbb{R}P^3$. (Note that we do have $H^2(G, A) = H^2(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = 0$.) Since

$$[\mathbb{R}P^3, B\chi']_* = H^2(\mathbb{R}P^3, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z},$$

there are two distinct pointed homotopy classes from $\mathbb{R}P^3$ to $B\chi'$, namely the null-homotopy class c' and another class $g' \neq c'$. Using that $H^1(\mathbb{R}P^3, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ and $H^1(\mathbb{R}P^3, \mathbb{Z}/3\mathbb{Z}) = 0$, the above computations give:

$$|\mathbb{R}P^3, c'|_{\phi_*(\mathbb{Q}\mathcal{G}_\chi)} = \frac{1}{3} \quad \text{and} \quad |\mathbb{R}P^3, g'|_{\phi_*(\mathbb{Q}\mathcal{G}_\chi)} = 0.$$

In particular, $|\mathbb{R}P^3, c'|_{\phi_*(\mathbb{Q}\mathcal{G}_\chi)} \neq |\mathbb{R}P^3, g'|_{\phi_*(\mathbb{Q}\mathcal{G}_\chi)}$. Since $\pi_1(B\chi') = 1$, we have:

$$[\mathbb{R}P^3, B\chi'] \simeq [\mathbb{R}P^3, B\chi']_*.$$

Also, all the maps $\mathbb{R}P^3 \rightarrow B\chi'$ are phantom maps (meaning that they induce trivial homomorphisms on homotopy groups) because $\pi_2(\mathbb{R}P^3) = 1$ and $B\chi'$ is a $K(\mathbb{Z}/2\mathbb{Z}, 2)$ space. This example shows that the invariant $|\cdot|_C$ of Theorem 7.1 is nontrivial and may distinguish distinct homotopy classes of phantom maps.

7.11. Example

Let $\chi: E \rightarrow H$ be a crossed module such that E is finite and assume that \mathbb{k} is a field of characteristic zero. Then the category $\chi\text{-vect}_{\mathbb{k}}$ of Example 4.15 is a spherical χ -fusion category with $\dim((\chi\text{-vect}_{\mathbb{k}})_1^1) = \text{card}(E)1_{\mathbb{k}} \neq 0$. Recall from Section 2.4 the canonical inclusion $BH \hookrightarrow B\chi$. Then, for any connected pointed closed χ -manifold (M, g) , we have:

$$|M, g|_{\chi\text{-vect}_{\mathbb{k}}} \neq 0 \iff g \text{ factorizes through } BH \hookrightarrow B\chi. \tag{31}$$

Indeed, recall that $\chi\text{-vect}_{\mathbb{k}} = \phi_*((E, H \rtimes E)\text{-vect}_{\mathbb{k}})$ where

$$\phi = \left(\begin{array}{ccc} E & \xrightarrow{\iota} & H \rtimes E \\ \text{id}_E \downarrow & & \downarrow \varphi \\ E & \xrightarrow{\chi} & H \end{array} \right)$$

with $\iota(e) = (1, e)$ and $\varphi(h, e) = \chi(e)h$. Note that $B\iota$ is a $K(H, 1)$ space (since ι is injective and $\text{Coker}(\iota) \simeq H$). For any $\tilde{g} \in [M, B\iota]_*$, $\pi_1(\text{TOP}_*(M, B\iota), \tilde{g}) = 1$ (because $B\iota$ is a $K(H, 1)$ space). Also,

$$|M, \tilde{g}|_{(E, H \rtimes E) - \text{vect}_{\mathbb{k}}} \stackrel{(i)}{=} |M, \tilde{g}|_{H - \text{vect}_{\mathbb{k}}} \stackrel{(ii)}{=} 1_{\mathbb{k}}.$$

Here (i) follows from the remark after Theorem 7.1 relating the invariants of Theorem 7.1 and [17] (using the injectivity of ι) and the fact that $(E, H \rtimes E) - \text{vect}_{\mathbb{k}}$ is isomorphic (as a H -fusion category) to the category of finite-dimensional H -graded \mathbb{k} -vectors spaces, and (ii) follows from [17, Example 8.6] (with $\theta = 1$). Then, since $d_\phi = \text{card}(E)$, Theorem 7.2 implies that

$$|M, g|_{\chi - \text{vect}_{\mathbb{k}}} = \frac{\text{card}(\pi_1(\text{TOP}_*(M, B\chi), g))}{\text{card}(E)} \text{card}(\{\tilde{g} \in [M, B\iota]_* \mid B\phi \circ \tilde{g} = g\}) 1_{\mathbb{k}}.$$

Since \mathbb{k} is a field of characteristic zero, we obtain that $|M, g|_{\chi - \text{vect}_{\mathbb{k}}} \neq 0$ if and only if there is $\tilde{g} \in [M, B\iota]_*$ such that $B\phi \circ \tilde{g} = g$. This proves (31) using that $B\phi$ is pointed homotopic to the canonical inclusion $BH \hookrightarrow B\chi$ (up to a pointed homotopy equivalence $B\iota \simeq BH$).

7.12. Proof of Theorem 7.2

For $X \in \mathcal{C}_{\text{hom}} = \phi_*(\mathcal{C})_{\text{hom}}$, the degree of X in \mathcal{C} is denoted by $|X|$ and the degree of X in $\phi_*(\mathcal{C})$ is then $\varphi(|X|)$. Let $I = \coprod_{h \in H} I_h$ be a χ -representative set for \mathcal{C} . Consider the left action $E \times I \rightarrow I$, $(e, i) \mapsto e \cdot i$ given by Lemma 3.2(iii). Recall that $j = e \cdot i$ if and only if $i \cong_e j$, and so $|e \cdot i| = \chi(e)|i|$. This together with the equality $\varphi\chi = \chi'\psi$ implies that for all $e \in \text{Ker}(\psi)$ and $i \in I$,

$$\varphi(|e \cdot i|) = \varphi(|i|). \tag{32}$$

Claim 7.3. *The subgroup $\text{Ker}(\psi)$ of E acts freely on I .*

Proof. Let $e \in \text{Ker}(\psi)$ and $i \in I$ such that $i = e \cdot i$. Then $|i| = |e \cdot i| = \chi(e)|i|$ and so $\chi(e) = 1$. Therefore $e \in \text{Ker}(\psi) \cap \text{Ker}(\chi) = 1$ and $e = 1$. \square

Pick a complete set $J \subset I$ of representatives of $I/\text{Ker}(\psi)$ such that $1 \in J$.

Claim 7.4. *We have:*

- (i) *For all $i \in I$, there is a unique $(e, j) \in \text{Ker}(\psi) \times J$ such that $i = e \cdot j$.*
- (ii) *J is a χ' -representative set for $\phi_*(\mathcal{C})$ splitting as $J = \coprod_{h' \in H'} J_{h'}$ with*

$$J_{h'} = \coprod_{h \in \varphi^{-1}(h')} J \cap I_h.$$

(iii) For any $h' \in H'$, the map

$$\begin{cases} \text{Ker}(\psi) \times J_{h'} & \rightarrow \coprod_{h \in \varphi^{-1}(h')} I_h \\ (e, j) & \mapsto e \cdot j \end{cases}$$

is a bijection.

Proof. (i) follows directly from the definition of J and Claim 7.3. Let us prove (ii). By definition, $J \subset I \subset \mathcal{C}_{\text{hom}} = \phi_*(\mathcal{C})_{\text{hom}}$ and $1 \in J$. Recall from Section 4.14 that an object is simple in $\phi_*(\mathcal{C})^1$ if and only if it is simple in \mathcal{C}^1 . In particular $J \subset I$ is made of simple objects of $\phi_*(\mathcal{C})^1$. Next, let X be a simple object of $\phi_*(\mathcal{C})^1$. By definition of I and since X is simple in \mathcal{C}^1 , there is $i \in I$ such that X is 1-isomorphic in \mathcal{C} to i . By (i), there is $(e, j) \in \text{Ker}(\psi) \times J$ such that $i = e \cdot j$, that is, i is e -isomorphic in \mathcal{C} to j . Then, by (3), X is e -isomorphic in \mathcal{C} to j . Since $\psi(e) = 1$, we deduce that X is 1-isomorphic in $\phi_*(\mathcal{C})$ to j . To prove the uniqueness of such a $j \in J$, consider another $k \in J$ such that X is 1-isomorphic in $\phi_*(\mathcal{C})$ to k . Then k is 1-isomorphic in $\phi_*(\mathcal{C})$ to j and so, by Lemma 3.1(ii), the \mathbb{k} -module $\text{Hom}_{\phi_*(\mathcal{C})}^1(k, j)$ is free of rank 1. Since

$$\text{Hom}_{\phi_*(\mathcal{C})}^1(k, j) = \bigoplus_{a \in \text{Ker}(\psi)} \text{Hom}_{\mathcal{C}}^a(k, j),$$

there is $a \in \text{Ker}(\psi)$ such that $\text{Hom}_{\mathcal{C}}^a(k, j) \neq 0$. Lemma 3.1(ii) implies that k is a -isomorphic in \mathcal{C} to j , and so $j = a \cdot k$. Then j and k belong to the same class in $I/\text{Ker}(\psi)$ and so are equal (by definition of J). Then X is 1-isomorphic in $\phi_*(\mathcal{C})$ to a unique element of J . Hence J is a χ' -representative set for $\phi_*(\mathcal{C})$. The splitting of J is derived directly from the splitting of I and the fact that the degree in $\phi_*(\mathcal{C})$ of $j \in J$ is $\varphi(|j|)$.

Let us prove (iii). The map is well-defined since for any $(e, j) \in \text{Ker}(\psi) \times J_{h'}$, (32) gives that $\varphi(|e \cdot j|) = \varphi(|j|) = h'$ and so $|e \cdot j| \in \varphi^{-1}(h')$. Let $i \in I_h$ with $h \in \varphi^{-1}(h')$. By (i), there is a unique $(e, j) \in \text{Ker}(\psi) \times J$ such that $i = e \cdot j$. Now $j \in J_{h'}$ since $j = e^{-1} \cdot i$ and $\varphi(|e^{-1} \cdot i|) = \varphi(|i|) = \varphi(h) = h'$ by (32). Thus there is a unique $(e, j) \in \text{Ker}(\psi) \times J_{h'}$ such that $i = e \cdot j$. Hence the map is bijective. \square

Claim 7.5. $\dim(\phi_*(\mathcal{C})_1^1) = d_\phi \dim(\mathcal{C}_1^1)$.

Proof. The bijection of Claim 7.4(iii) for $h' = 1$ gives that

$$\sum_{(e,j) \in \text{Ker}(\psi) \times J_1} \dim(e \cdot j)^2 = \sum_{i \in \coprod_{h \in \text{Ker}(\varphi)} I_h} \dim(i)^2.$$

Note that $\dim(e \cdot j) = \dim(j)$ because $e \cdot j$ and j are isomorphic in \mathcal{C} , the left-hand side is equal to $\text{card}(\text{Ker}(\psi)) \dim(\phi_*(\mathcal{C})_1^1)$. By (8), the right-hand side is equal to $\text{card}(\text{Ker}(\varphi)) \dim(\mathcal{C}_1^1)$. We conclude by using the definition of d_ϕ . \square

Let M be a connected pointed oriented closed 3-dimensional manifold. Since M is connected, there is a skeleton P of M with disk regions and with a single P -ball containing the basepoint of M . (An example of such a skeleton is provided by a spine of $M \setminus B$ where B is an open 3-ball in M containing the basepoint of M .) Denote by $\mathcal{G}_{P,\chi}$ the gauge group of χ -colorings of P (see Section 6.3) and consider its subgroup

$$\mathcal{G}_{P,\chi}^* = \{1\} \times \text{Map}(\text{Reg}(P), E),$$

where 1 is the trivial map (sending the unique P -ball to $1 \in H$). Two χ -labelings of P are *pointed gauge equivalent* if they belong to the same $\mathcal{G}_{P,\chi}^*$ -orbit. The next claim is a version of Lemma 6.1 for pointed homotopy classes.

Claim 7.6. *There is a bijection from the set of pointed gauge equivalence classes of χ -labelings of P to the set of pointed homotopy classes of pointed maps $M \rightarrow B\chi$.*

Proof. The proof is similar to that of Lemma 6.1 by picking the basepoint of M as the center of the unique P -ball and picking arcs $\{\gamma_r\}_r$ and disks $\{\delta_e\}_e$ as in Section 6.5. For any χ -labeling (α, β) of P , the pointed homotopy class of the map $f_{\alpha,\beta}$ defined in Claim 6.2 depends only on P and (α, β) since any two systems of arcs $\{\gamma_r\}_r$ and disks $\{\delta_e\}_e$ are isotopic in M via an isotopy preserving the basepoint of M . Also, the fact that two χ -labelings (α, β) and (α', β') of P are pointed gauge equivalent if and only if their associated maps $f_{\alpha,\beta}$ and $f_{\alpha',\beta'}$ are pointed homotopic is shown similarly to Claim 6.3. Thus the assignment $(\alpha, \beta) \mapsto f_{\alpha,\beta}$ induces a well-defined injective map $\{\chi\text{-labelings of } P\} / \mathcal{G}_{P,\chi}^* \rightarrow [M, B\chi]_*$ which maps the pointed gauge equivalence class of a χ -labeling (α, β) to the pointed homotopy class of the map $f_{\alpha,\beta}$. Since regions of P are disks, this map is surjective by the same arguments of the end of Section 6.5. \square

Let $g' \in [M, B\chi']_*$. By Claim 7.6, the pointed homotopy class g' is encoded by a χ' -labeling (α', β') of P . Note that it follows from the definition of a crossed module morphism that if (α, β) is a χ -labeling of P , then $(\varphi\alpha, \psi\beta)$ is a χ' -labeling of P . Consider the set

$$\mathcal{L} = \{(\alpha, \beta) \text{ } \chi\text{-labeling of } P \mid (\varphi\alpha, \psi\beta) = (\alpha', \beta')\}.$$

For a χ -labeling (α, β) of P , denote by $g_{\alpha,\beta} \in [M, B\chi]_*$ the pointed homotopy class determined by the pointed gauge equivalence class of (α, β) as in Claim 7.6.

Claim 7.7. *For any $(\alpha, \beta) \in \mathcal{L}$, we have $B\phi \circ g_{\alpha,\beta} = g'$. Conversely, for any $g \in [M, B\chi]_*$ such that $B\phi \circ g = g'$, there is $(\alpha, \beta) \in \mathcal{L}$ such that $g = g_{\alpha,\beta}$.*

Proof. Notice first that if (α, β) is a χ -labeling of P , then $B\phi \circ g_{\alpha,\beta} = g_{\varphi\alpha, \psi\beta}$. In particular, for any $(\alpha, \beta) \in \mathcal{L}$, we have: $B\phi \circ g_{\alpha,\beta} = g_{\varphi\alpha, \psi\beta} = g_{\alpha', \beta'} = g'$. Conversely, let

$g \in [M, B\chi]_*$ such that $B\phi \circ g = g'$. By Claim 7.6, there is a χ -labeling (α_0, β_0) of P such that $g = g_{\alpha_0, \beta_0}$. Since

$$g_{\varphi\alpha_0, \psi\beta_0} = B\phi \circ g_{\alpha_0, \beta_0} = B\phi \circ g = g' = g_{\alpha', \beta'},$$

Claim 7.6 implies that $(\varphi\alpha_0, \psi\beta_0)$ is pointed gauge equivalent to (α', β') , and so there is a map $\mu' : \text{Reg}(P) \rightarrow E'$ such that $(\alpha', \beta') = (1, \mu') \cdot (\varphi\alpha_0, \psi\beta_0)$. Since ψ is surjective, there is a map $\mu : \text{Reg}(P) \rightarrow E$ such that $\mu' = \psi\mu$. Consider the χ -labeling $(\alpha, \beta) = (1, \mu) \cdot (\alpha_0, \beta_0)$ of P . By Claim 7.6, $g_{\alpha, \beta} = g_{\alpha_0, \beta_0} = g$. Also, for any $r \in \text{Reg}(P)$,

$$\varphi\alpha(r) \stackrel{(i)}{=} \varphi(\alpha_0(r)\chi(\mu(r))) \stackrel{(ii)}{=} \varphi\alpha_0(r)\varphi\chi(\mu(r)) \stackrel{(iii)}{=} \varphi\alpha_0(r)\chi'(\mu'(r)) \stackrel{(iv)}{=} \alpha'(r).$$

Here (i) and (iv) follow from (23), (ii) from the multiplicativity of φ , and (iii) from the facts that $\varphi\chi = \chi'\psi$ and $\psi\mu = \mu'$. Moreover, for any $(e, B) \in \text{EB}(P)$,

$$\begin{aligned} \psi\beta(e, B) &\stackrel{(i)}{=} \psi(\beta_0(e, B)\mu_{\alpha_0}(\gamma_{(e, B)})) \stackrel{(ii)}{=} \psi\beta_0(e, B)\psi(\mu_{\alpha_0}(\gamma_{(e, B)})) \\ &\stackrel{(iii)}{=} \psi\beta_0(e, B)\mu'_{\varphi\alpha_0}(\gamma_{(e, B)}) \stackrel{(iv)}{=} \beta'(e, B). \end{aligned}$$

Here (i) and (iv) follow from (24), (ii) from the multiplicativity of ψ , and (iii) from the facts that $\psi(\mu_{\alpha_0}) = (\psi\mu)_{\varphi\alpha_0}$ (see Section 6.3) and $\psi\mu = \mu'$. Thus we get that $(\varphi\alpha, \psi\beta) = (\alpha', \beta')$. Hence $(\alpha, \beta) \in \mathcal{L}$. \square

For $\mathcal{A} \subset \phi_*(\mathcal{C})_{\text{hom}} = \mathcal{C}_{\text{hom}}$, let

$$\text{Col}_{\mathcal{A}} = \{c : \text{Reg}(P) \rightarrow \mathcal{A} \mid \varphi(|c(r)|) = \alpha'(r) \text{ for all } r \in \text{Reg}(P)\}.$$

For $c \in \text{Col}_{\mathcal{A}}$, we define the \mathbb{k} -module $H_{c, \phi_*(c)} = H_c$ and the scalars $\dim(c)$ and $|c|_{\phi_*(c)} = V_c(*_c)$ as in Section 7.1 but using the χ' -labeling (α', β') and the spherical χ' -fusion category $\phi_*(\mathcal{C})$. Clearly, $|c|_{\phi_*(c)} = 0$ whenever $H_{c, \phi_*(c)} = 0$. Set

$$\text{Col}_{\mathcal{A}}^* = \{c \in \text{Col}_{\mathcal{A}} \mid H_{c, \phi_*(c)} \neq 0\}.$$

To maps $\eta : \text{Reg}(P) \rightarrow \text{Ker}(\psi)$ and $d : \text{Reg}(P) \rightarrow J$, we associate the map

$$\eta \cdot d : \begin{cases} \text{Reg}(P) & \rightarrow I \\ r & \mapsto \eta(r) \cdot d(r). \end{cases}$$

Denote by \mathcal{M} the set of maps from $\text{Reg}(P)$ to $\text{Ker}(\psi)$.

Claim 7.8. *The map $\mathcal{M} \times \text{Col}_J \rightarrow \text{Col}_I$, defined by $(\eta, d) \mapsto \eta \cdot d$, is bijective.*

Proof. Note that $\eta \cdot d \in \text{Col}_I$ for any $(\eta, d) \in \mathcal{M} \times \text{Col}_J$ since, by using (32),

$$\varphi(|(\eta \cdot d)(r)|) = \varphi(|\eta(r) \cdot d(r)|) = \varphi(|d(r)|) = \alpha'(r)$$

for all $r \in \text{Reg}(P)$. Hence the map is well-defined. Let $c \in \text{Col}_I$. Claim 7.4(i) implies that there is a unique pair of maps $(\eta: \text{Reg}(P) \rightarrow \text{Ker}(\psi), d: \text{Reg}(P) \rightarrow J)$ such that $c(r) = \eta(r) \cdot d(r)$ for all $r \in \text{Reg}(P)$. Note that $d \in \text{Col}_J$ since, by using (32),

$$\varphi(|d(r)|) = \varphi(|\eta(r)^{-1} \cdot c(r)|) = \varphi(|c(r)|) = \alpha'(r)$$

for all $r \in \text{Reg}(P)$. Thus there is a unique $(\eta, d) \in \mathcal{M} \times \text{Col}_J$ such that $c = \eta \cdot d$. Hence the map is bijective. \square

Claim 7.9. For all $\eta \in \mathcal{M}$ and $d \in \text{Col}_J$,

$$\dim(\eta \cdot d) = \dim(d) \quad \text{and} \quad |\eta \cdot d|_{\phi_*(\mathcal{C})} = |d|_{\phi_*(\mathcal{C})}.$$

Proof. For any $r \in \text{Reg}(P)$, the objects $\eta(r) \cdot d(r)$ and $d(r)$ are isomorphic in \mathcal{C} and so have same dimension. Then $\dim(\eta \cdot d) = \dim(d)$.

For each $j \in J$ and $e \in \text{Ker}(\psi)$, pick an e -isomorphism $\theta_{e,j}: j \rightarrow e \cdot j$ in \mathcal{C} . Note that different choices of such an isomorphism differ by multiplication of an invertible scalar since $\text{Hom}_{\mathcal{C}}^e(j, e \cdot j)$ is free of rank 1 (see Lemma 3.1(ii)). For any $r \in \text{Reg}(P)$, define the isomorphism $\theta_{r,\pm}: d(r)^\pm \rightarrow (\eta(r) \cdot d(r))^\pm$ by

$$\theta_{r,+} = \theta_{\eta(r),d(r)} \quad \text{and} \quad \theta_{r,-} = (\theta_{\eta(r),d(r)}^*)^{-1}.$$

Note that $\theta_{r,+}$ has degree $\psi(\eta(r)) = 1$ in $\phi_*(\mathcal{C})$, as well as $\theta_{r,-} = (\theta_{r,+}^*)^{-1}$. For any oriented edge e of P , the above isomorphisms induce a \mathbb{k} -linear isomorphism

$$\theta_e: H_d(e) \xrightarrow{\cong} H_{\eta \cdot d}(e)$$

such that for any P -ball branch B at e , the following diagram commutes:

$$\begin{CD} H_d(e) @>\cong>> \text{Hom}_{\phi_*(\mathcal{C})}^{\beta'(e,B)}(\mathbb{1}, d(b_1)^{\varepsilon_1} \otimes \cdots \otimes d(b_n)^{\varepsilon_n}) \\ @V\theta_eVV @VV\theta_{e,B}V \\ H_{\eta \cdot d}(e) @>\cong>> \text{Hom}_{\phi_*(\mathcal{C})}^{\beta'(e,B)}(\mathbb{1}, (\eta(b_1) \cdot d(b_1))^{\varepsilon_1} \otimes \cdots \otimes (\eta(b_n) \cdot d(b_n))^{\varepsilon_n}). \end{CD}$$

Here, the horizontal maps are the cone isomorphisms, $b_1 < \cdots < b_n$ are the branches of P at e enumerated in the linear order determined by B , $\varepsilon_i = \varepsilon_e(b_i) \in \{+, -\}$, and $\theta_{e,B}$ is defined by

$$\theta_{e,B}(\alpha) = (\theta_{b_1,\varepsilon_1} \otimes \cdots \otimes \theta_{b_n,\varepsilon_n}) \circ \alpha.$$

For any vertex x of P , the unordered tensor product $\otimes_{e_x} \theta_{e_x}$, where e_x runs over all edges of P incident to x and oriented away from x , induces a \mathbb{k} -linear isomorphism $\theta_x : H(\Gamma_x^d) \rightarrow H(\Gamma_x^{\eta \cdot d})$ satisfying

$$\mathbb{F}_{\phi_*(c)}(\Gamma_x^d) = \mathbb{F}_{\phi_*(c)}(\Gamma_x^{\eta \cdot d}) \circ \theta_x. \tag{33}$$

This follows from the naturality of $\mathbb{F}_{\phi_*(c)}$ (see Section 5.6). Also, as in Section 7.1, each unoriented edge e of P gives rise to two opposite oriented edges e_1, e_2 of P and to the vectors $*_e^d \in H_d(e_1) \otimes H_d(e_2)$ and $*_e^{\eta \cdot d} \in H_{\eta \cdot d}(e_1) \otimes H_{\eta \cdot d}(e_2)$ satisfying

$$(\theta_{e_1} \otimes \theta_{e_2})(*_e^d) = *_e^{\eta \cdot d}. \tag{34}$$

Then the equality $|\eta \cdot d|_{\phi_*(c)} = |d|_{\phi_*(c)}$ follows directly from (33) and (34). \square

For a χ -labeling (α, β) of P , let

$$\text{Col}(\alpha, \beta) = \{c : \text{Reg}(P) \rightarrow I \mid |c(r)| = \alpha(r) \text{ for all } r \in \text{Reg}(P)\}.$$

For $c \in \text{Col}(\alpha, \beta)$, we define the \mathbb{k} -module $H_{c, \mathcal{C}} = H_c$ and the scalar $|c|_{\mathcal{C}} = V_c(*_c)$ as in Section 7.1 by using the χ -labeling (α, β) and the spherical χ -fusion category \mathcal{C} . Clearly, $|c|_{\mathcal{C}} = 0$ whenever $H_{c, \mathcal{C}} = 0$. Set

$$\text{Col}^*(\alpha, \beta) = \{c \in \text{Col}(\alpha, \beta) \mid H_{c, \mathcal{C}} \neq 0\}.$$

The next claim is instrumental to define a map $\text{Col}_I^* \rightarrow \mathcal{L}$ in Claim 7.11.

Claim 7.10. *Let X be a 1-direct sum in \mathcal{C} of homogeneous objects of degree $h \in H$. Let $e' \in E'$ with $\text{Hom}_{\phi_*(c)}^{e'}(\mathbb{1}, X) \neq 0$. Then there is a unique $e \in \psi^{-1}(e')$ such that*

$$\text{Hom}_{\phi_*(c)}^{e'}(\mathbb{1}, X) = \text{Hom}_{\mathcal{C}}^e(\mathbb{1}, X).$$

Moreover $\chi(e) = h$.

Proof. Assume that X is a 1-direct sum of a finite family $(X_\alpha)_{\alpha \in A}$ of homogeneous objects of \mathcal{C} of degree h . Since

$$\bigoplus_{e \in \psi^{-1}(e')} \text{Hom}_{\mathcal{C}}^e(\mathbb{1}, X) = \text{Hom}_{\phi_*(c)}^{e'}(\mathbb{1}, X) \neq 0,$$

there is $e \in \psi^{-1}(e')$ with $\text{Hom}_{\mathcal{C}}^e(\mathbb{1}, X) \neq 0$. Then $\chi(e) = h$. Indeed, since by (5)

$$\bigoplus_{\alpha \in A} \text{Hom}_{\mathcal{C}}^e(\mathbb{1}, X_\alpha) \simeq \text{Hom}_{\mathcal{C}}^e(\mathbb{1}, X) \neq 0,$$

there is $\alpha \in A$ such that $\text{Hom}_{\mathcal{C}}^e(\mathbb{1}, X_\alpha) \neq 0$, and so $\chi(e) = |X_\alpha| = h$. If $f \in \psi^{-1}(e')$ is such that $\text{Hom}_{\mathcal{C}}^f(\mathbb{1}, X) \neq 0$, then $\chi(f) = h$ (as above) and so $f = e$ because $fe^{-1} \in \text{Ker}(\psi) \cap \text{Ker}(\chi) = 1$. Thus $\text{Hom}_{\mathcal{C}}^f(\mathbb{1}, X) = 0$ for all $f \in \psi^{-1}(e')$ with $f \neq e$. Hence $\text{Hom}_{\phi_*(\mathcal{C})}^{e'}(\mathbb{1}, X) = \bigoplus_{f \in \psi^{-1}(e')} \text{Hom}_{\mathcal{C}}^f(\mathbb{1}, X) = \text{Hom}_{\mathcal{C}}^e(\mathbb{1}, X)$. \square

Claim 7.11. *Let $c \in \text{Col}_I^*$. Then there is a unique $(\alpha_c, \beta_c) \in \mathcal{L}$ with $c \in \text{Col}^*(\alpha_c, \beta_c)$. Moreover $|c|_{\phi_*(\mathcal{C})} = |c|_c$.*

Proof. The condition $c \in \text{Col}^*(\alpha_c, \beta_c)$ implies that $\alpha_c(r) = |c(r)|$ for all $r \in \text{Reg}(P)$. Let $(e, B) \in \text{EB}(P)$. The cone isomorphism gives a \mathbb{k} -linear isomorphism

$$H_{c, \phi_*(\mathcal{C})}(e) \simeq \text{Hom}_{\phi_*(\mathcal{C})}^{\beta'(e, B)}(\mathbb{1}, c(b_1)^{\varepsilon_1} \otimes \dots \otimes c(b_n)^{\varepsilon_n}),$$

where $b_1 < \dots < b_n$ are the branches of P at e enumerated in the linear order determined by B and $\varepsilon_i = \varepsilon_e(b_i) \in \{+, -\}$. Since $H_{c, \phi_*(\mathcal{C})}(e) \neq 0$ and the object $c(b_1)^{\varepsilon_1} \otimes \dots \otimes c(b_n)^{\varepsilon_n}$ is a 1-direct sum in \mathcal{C} of homogeneous objects of degree $|c(b_1)|^{\varepsilon_1} \dots |c(b_n)|^{\varepsilon_n}$ by (c) of Section 4.1, Claim 7.10 gives that there is a unique $\beta_c(e, B) \in \psi^{-1}(\beta'(e, B))$ such that

$$\text{Hom}_{\phi_*(\mathcal{C})}^{\beta'(e, B)}(\mathbb{1}, c(b_1)^{\varepsilon_1} \otimes \dots \otimes c(b_n)^{\varepsilon_n}) = \text{Hom}_{\mathcal{C}}^{\beta_c(e, B)}(\mathbb{1}, c(b_1)^{\varepsilon_1} \otimes \dots \otimes c(b_n)^{\varepsilon_n}), \quad (35)$$

and moreover

$$\chi(\beta_c(e, B)) = |c(b_1)|^{\varepsilon_1} \dots |c(b_n)|^{\varepsilon_n} = \alpha_c(b_1)^{\varepsilon_1} \dots \alpha_c(b_n)^{\varepsilon_n}.$$

Let $\text{succ}(B)$ be as in Section 6.3. Since the cone isomorphisms are compatible with the map p_{b_1, b_2} as in (18), we obtain that

$$\begin{aligned} &\text{Hom}_{\phi_*(\mathcal{C})}^{\beta'(e, \text{succ}(B))}(\mathbb{1}, c(b_2)^{\varepsilon_2} \otimes \dots \otimes c(b_n)^{\varepsilon_n} \otimes c(b_1)^{\varepsilon_1}) \\ &= \text{Hom}_{\mathcal{C}}^f(\mathbb{1}, c(b_2)^{\varepsilon_2} \otimes \dots \otimes c(b_n)^{\varepsilon_n} \otimes c(b_1)^{\varepsilon_1}) \end{aligned}$$

with $f = (|c(b_1)|^{-\varepsilon_1})\beta_c(e, B)$ by (e) of Section 4.1. The uniqueness in Claim 7.10 implies that

$$\beta_c(e, \text{succ}(B)) = f = (\alpha_c(b_1)^{-\varepsilon_1})\beta_c(e, B).$$

Therefore (α_c, β_c) is a χ -labeling of P . By construction, $(\varphi\alpha_c, \psi\beta_c) = (\alpha', \beta')$, so that $(\alpha_c, \beta_c) \in \mathcal{L}$, and $c \in \text{Col}(\alpha_c, \beta_c)$. It follows from (35) that there is a canonical \mathbb{k} -linear isomorphism $H_{c, \phi_*(\mathcal{C})}(e) \simeq H_{c, \mathcal{C}}(e)$ for each oriented edge e of P , and so $H_{c, \mathcal{C}} \simeq H_{c, \phi_*(\mathcal{C})} \neq 0$. Hence $c \in \text{Col}^*(\alpha_c, \beta_c)$. The uniqueness of (α_c, β_c) follows from the fact that α_c is fully determined by the condition $c \in \text{Col}^*(\alpha_c, \beta_c)$ and from the uniqueness in Claim 7.10.

The fact that $\phi_*(\mathcal{C}) = \mathcal{C}$ as pivotal categories implies that, using $\phi_*(\mathcal{C})$ and $c \in \text{Col}_I^*$ on the one hand and using \mathcal{C} and $c \in \text{Col}^*(\alpha_c, \beta_c)$ on the other, the contraction vectors

associated with the unoriented edges of P are the same and that the vectors associated with the vertices of P are the same (up to the canonical isomorphism $H_{c,\phi_*(c)} \simeq H_{c,c}$). Consequently, $|c|_{\phi_*(c)} = |c|_c$. \square

Claim 7.12. *The map*

$$\text{Col}_I^* \rightarrow \coprod_{(\alpha,\beta) \in \mathcal{L}} \text{Col}^*(\alpha, \beta), \quad c \mapsto c \in \text{Col}^*(\alpha_c, \beta_c)$$

is bijective.

Proof. The uniqueness in Claim 7.11 implies that if $(\alpha, \beta) \in \mathcal{L}$ and $c \in \text{Col}^*(\alpha, \beta)$, then $(\alpha_c, \beta_c) = (\alpha, \beta)$. In particular, this gives that $\bigcup_{(\alpha,\beta) \in \mathcal{L}} \text{Col}^*(\alpha, \beta)$ is a disjoint union. Then the map is well-defined and injective. Let us prove that this map is surjective. Let $(\alpha, \beta) \in \mathcal{L}$ and $c \in \text{Col}^*(\alpha, \beta)$. We need to prove that $c \in \text{Col}_I^*$. Since $\alpha'(r) = \varphi(\alpha(r)) = \varphi(|c(r)|)$ for all $r \in \text{Reg}(P)$, we get that $c \in \text{Col}_I$. Let e be an oriented edge of P . Pick a P -ball branch B at e and set $X = c(b_1)^{\varepsilon_1} \otimes \cdots \otimes c(b_n)^{\varepsilon_n}$, where $b_1 < \cdots < b_n$ are the branches of P at e enumerated in the linear order determined by B and $\varepsilon_i = \varepsilon_e(b_i) \in \{+, -\}$. Using that $\beta(e, B) \in \psi^{-1}(\beta'(e, B))$ and $c \in \text{Col}^*(\alpha, \beta)$, we obtain

$$H_{c,\phi_*(c)}(e) \simeq \text{Hom}_{\phi_*(c)}^{\beta'(e,B)}(\mathbb{1}, X) \supset \text{Hom}_c^{\beta(e,B)}(\mathbb{1}, X) \simeq H_{c,c}(e) \neq 0.$$

Thus $H_{c,\phi_*(c)} = \bigotimes_e H_{c,\phi_*(c)}(e) \neq 0$ and so $c \in \text{Col}_I^*$. \square

By definition, since $|M \setminus P| = 1$,

$$|M, g'|_{\phi_*(c)} = \dim(\phi_*(c)_1^1)^{-1} \sum_{d \in \text{Col}_J} \dim(d) |d|_{\phi_*(c)}.$$

Now, we have:

$$\begin{aligned} \sum_{\substack{(\alpha,\beta) \in \mathcal{L} \\ c \in \text{Col}(\alpha,\beta)}} \dim(c) |c|_c &\stackrel{(i)}{=} \sum_{\substack{(\alpha,\beta) \in \mathcal{L} \\ c \in \text{Col}^*(\alpha,\beta)}} \dim(c) |c|_c \stackrel{(ii)}{=} \sum_{c \in \text{Col}_I^*} \dim(c) |c|_{\phi_*(c)} \\ &\stackrel{(iii)}{=} \sum_{c \in \text{Col}_I} \dim(c) |c|_{\phi_*(c)} \stackrel{(iv)}{=} \sum_{(\eta,d) \in \mathcal{M} \times \text{Col}_J} \dim(\eta \cdot d) |\eta \cdot d|_{\phi_*(c)} \\ &\stackrel{(v)}{=} \text{card}(\mathcal{M}) \sum_{d \in \text{Col}_J} \dim(d) |d|_{\phi_*(c)}. \end{aligned}$$

Here (i) follows from the fact that $|c|_c = 0$ for all $c \in \text{Col}(\alpha, \beta)$ with $H_{c,c} = 0$, (ii) from the bijection of Claim 7.12 and the last assertion of Claim 7.11, (iii) from the fact that

$|c|_{\phi_*(c)} = 0$ for all $c \in \text{Col}_I$ with $H_{c, \phi_*(c)} = 0$, (iv) from the bijection in Claim 7.8, and (v) from Claim 7.9. Therefore

$$|M, g'|_{\phi_*(c)} = \dim(\phi_*(\mathcal{C})_1^1)^{-1} \text{card}(\mathcal{M})^{-1} \sum_{\substack{(\alpha, \beta) \in \mathcal{L} \\ c \in \text{Col}(\alpha, \beta)}} \dim(c) |c|_c$$

and so, using Claim 7.5, we obtain

$$|M, g'|_{\phi_*(c)} = \sum_{(\alpha, \beta) \in \mathcal{L}} d_\phi^{-1} \text{card}(\mathcal{M})^{-1} \dim(\mathcal{C}_1^1)^{-1} \sum_{c \in \text{Col}(\alpha, \beta)} \dim(c) |c|_c.$$

Then, it follows from the definition of $|\cdot|_c$ that

$$|M, g'|_{\phi_*(c)} = \sum_{(\alpha, \beta) \in \mathcal{L}} d_\phi^{-1} \text{card}(\mathcal{M})^{-1} |M, g_{\alpha, \beta}|_c.$$

Now the first assertion of Claim 7.7 gives that for any $(\alpha, \beta) \in \mathcal{L}$, the pointed homotopy class $g_{\alpha, \beta} \in [M, B\chi]_*$ satisfies $B\phi \circ g_{\alpha, \beta} = g'$. Thus

$$|M, g'|_{\phi_*(c)} = d_\phi^{-1} \sum_{\substack{g \in [M, B\chi]_* \\ B\phi \circ g = g'}} \lambda_g |M, g|_c \quad \text{where} \quad \lambda_g = \sum_{\substack{(\alpha, \beta) \in \mathcal{L} \\ g_{\alpha, \beta} = g}} \text{card}(\mathcal{M})^{-1}. \tag{36}$$

Let $g \in [M, B\chi]_*$ such that $B\phi \circ g = g'$. By the second assertion of Claim 7.7, there is $(\alpha_0, \beta_0) \in \mathcal{L}$ such that $g = g_{\alpha_0, \beta_0}$. Set

$$\mathcal{L}_{(\alpha_0, \beta_0)} = \{(\alpha, \beta) \in \mathcal{L} \mid (\alpha, \beta) \text{ is pointed gauge equivalent to } (\alpha_0, \beta_0)\}.$$

For any $(\alpha, \beta) \in \mathcal{L}_{(\alpha_0, \beta_0)}$, Claim 7.6 gives that $g_{\alpha, \beta} = g$ if and only if $(\alpha, \beta) \in \mathcal{L}_{(\alpha_0, \beta_0)}$. Thus

$$\lambda_g = \frac{\text{card}(\mathcal{L}_{(\alpha_0, \beta_0)})}{\text{card}(\mathcal{M})}. \tag{37}$$

Consider the sets

$$\mathcal{E} = \left\{ \mu: \text{Reg}(P) \rightarrow E \left| \begin{array}{l} \text{Im}(\mu) \subset \text{Ker}(\varphi\chi), \\ (\psi\mu)_{\alpha'}(\gamma_{(e, B)}) = 1_{E'} \text{ for all } (e, B) \in \text{EB}(P) \end{array} \right. \right\}$$

and

$$\mathcal{S} = \left\{ \mu: \text{Reg}(P) \rightarrow E \left| \begin{array}{l} \text{Im}(\mu) \subset \text{Ker}(\chi), \\ \mu_{\alpha_0}(\gamma_{(e, B)}) = 1_E \text{ for all } (e, B) \in \text{EB}(P) \end{array} \right. \right\}.$$

Claim 7.13. *Let $\mu: \text{Reg}(P) \rightarrow E$ be a map and let $(\alpha, \beta) \in \mathcal{L}$. Then $\mu \in \mathcal{E}$ if and only if $(1, \mu) \cdot (\alpha, \beta) \in \mathcal{L}$.*

Proof. Consider the χ -labeling $(\delta, \varepsilon) = (1, \mu) \cdot (\alpha, \beta)$ of P . As in the proof of Claim 7.7, we get that for any $r \in \text{Reg}(P)$ and $(e, B) \in \text{EB}(P)$,

$$\varphi\delta(r) = \alpha'(r)\varphi\chi(\mu(r)) \quad \text{and} \quad \psi\varepsilon(e, B) = \beta'(e, B) (\psi\mu)_{\alpha'}(\gamma_{(e, B)}).$$

Then $(\delta, \varepsilon) \in \mathcal{L} \Leftrightarrow (\varphi\delta, \psi\varepsilon) = (\alpha', \beta') \Leftrightarrow \mu \in \mathcal{E}$. \square

Claim 7.14. *The set \mathcal{E} is a group (with opposite pointwise product) left acting transitively on $\mathcal{L}_{(\alpha_0, \beta_0)}$ and \mathcal{S} is the stabilizer of (α_0, β_0) under this action.*

Proof. Let $\eta, \mu \in \mathcal{E}$. By Claim 7.13, $(\alpha, \beta) = (1, \mu) \cdot (\alpha_0, \beta_0) \in \mathcal{L}$ and $(1, \eta) \cdot (\alpha, \beta) \in \mathcal{L}$. Using the definition of the action of $\mathcal{G}_{P, \chi}^* \subset \mathcal{G}_{P, \chi}$ on χ -labelings, we obtain:

$$(1, \mu\eta) \cdot (\alpha_0, \beta_0) = ((1, \eta)(1, \mu)) \cdot (\alpha_0, \beta_0) = (1, \eta) \cdot (\alpha, \beta) \in \mathcal{L}.$$

Thus $\mu\eta \in \mathcal{E}$ by Claim 7.13. We deduce from this that \mathcal{E} is a group.

Next, let $(\alpha, \beta) \in \mathcal{L}_{(\alpha_0, \beta_0)}$. Since (α, β) is pointed gauge equivalent to (α_0, β_0) , there is a map $\mu: \text{Reg}(P) \rightarrow E$ such that $(\alpha, \beta) = (1, \mu) \cdot (\alpha_0, \beta_0)$. Since (α_0, β_0) and (α, β) are both in \mathcal{L} , Claim 7.13 implies that $\mu \in \mathcal{E}$. Thus \mathcal{E} acts transitively on $\mathcal{L}_{(\alpha_0, \beta_0)}$. Finally, let $\mu \in \mathcal{E}$. Set $(\alpha, \beta) = (1, \mu) \cdot (\alpha_0, \beta_0)$. For any $r \in \text{Reg}(P)$ and $(e, B) \in \text{EB}(P)$,

$$\alpha(r) = \alpha_0(r)\chi(\mu(r)) \quad \text{and} \quad \beta(e, B) = \beta_0(e, B) \mu_{\alpha_0}(\gamma_{(e, B)}).$$

Then $(\alpha, \beta) = (\alpha_0, \beta_0) \Leftrightarrow \mu \in \mathcal{S}$. Hence \mathcal{S} is the stabilizer of (α_0, β_0) . \square

Note that Claim 7.14 implies in particular that \mathcal{S} is a group (for the opposite pointwise product). Similarly, the set

$$\mathcal{S}' = \left\{ \mu': \text{Reg}(P) \rightarrow E' \left| \begin{array}{l} \text{Im}(\mu') \subset \text{Ker}(\chi'), \\ \mu'_{\alpha'}(\gamma_{(e, B)}) = 1_{E'} \text{ for all } (e, B) \in \text{EB}(P) \end{array} \right. \right\}$$

is a group (with opposite pointwise product).

Claim 7.15. *The map $\mathcal{E} \rightarrow \mathcal{S}'$, $\mu \mapsto \psi\mu$, is a surjective group homomorphism with kernel \mathcal{M} .*

Proof. The map is well defined since for any $\mu \in \mathcal{E}$,

$$\chi'(\text{Im}(\psi\mu)) = \chi'\psi(\text{Im}(\mu)) = \varphi\chi(\text{Im}(\mu)) = \{1_{H'}\}.$$

The fact that it is a group homomorphism follows from the multiplicativity of ψ . Its surjectivity follows from the surjectivity of ψ . Using that a map $\mu: \text{Reg}(P) \rightarrow E$ verifies $\psi\mu = 1$ if and only if $\text{Im}(\mu) \subset \text{Ker}(\psi)$, we obtain that the Kernel of the map is \mathcal{M} (identified with the set of maps $\text{Reg}(P) \rightarrow E$ with image in $\text{Ker}(\psi)$). \square

Claim 7.16. *The group \mathcal{S} is isomorphic to $\pi_1(\text{TOP}_*(M, B\chi), g)$. Similarly, the group \mathcal{S}' is isomorphic to $\pi_1(\text{TOP}_*(M, B\chi'), g')$.*

Proof. Using the notation of the proof of Claim 7.6, consider the pointed map $f = f_{\alpha_0, \beta_0}: M \rightarrow B\chi$ whose pointed homotopy class is $g_{\alpha_0, \beta_0} = g$. As in the proof of Claim 6.3, any $\mu \in \mathcal{S}$ defines a pointed homotopy G_μ from f to itself and so a loop in $\text{TOP}_*(M, B\chi)$ based at f whose homotopy class δ_μ depends only on μ . The assignment $\mu \mapsto \delta_\mu$ is a group homomorphism (by definition of the product of $\mathcal{G}_{P, \chi}$). The injectivity of this assignment follows from the facts that $B\chi$ is a 2-type and the filtration of M used in the construction of G_μ has a unique 0-cell. The surjectivity of the map follows from the fact that any pointed homotopy from f to itself is homotopic to G_μ for some $\mu \in \mathcal{S}$ (as in the proof of Claim 6.3). Therefore \mathcal{S} is isomorphic to $\pi_1(\text{TOP}_*(M, B\chi), f)$. Note that the isomorphism type of $\pi_1(\text{TOP}_*(M, B\chi), f)$ only depends on the path component of f in $\text{TOP}_*(M, B\chi)$, which is g , and is denoted $\pi_1(\text{TOP}_*(M, B\chi), g)$.

The proof of the second statement of the claim is similar (by exchanging χ with χ' and (α_0, β_0) with (α', β')). \square

We can now compute:

$$\lambda_g \stackrel{(i)}{=} \frac{\text{card}(\mathcal{L}_{(\alpha_0, \beta_0)})}{\text{card}(\mathcal{M})} \stackrel{(ii)}{=} \frac{\text{card}(\mathcal{E})}{\text{card}(\mathcal{S}) \text{card}(\mathcal{M})} \stackrel{(iii)}{=} \frac{\text{card}(\mathcal{S}')}{\text{card}(\mathcal{S})} \stackrel{(iv)}{=} \frac{\text{card}(\pi_1(\text{TOP}_*(M, B\chi'), g'))}{\text{card}(\pi_1(\text{TOP}_*(M, B\chi), g))}.$$

Here (i) follows from (37), (ii) from Claim 7.14, (iii) from Claim 7.15, and (iv) from Claim 7.16. This together with (36) concludes the proof of Theorem 7.2.

8. 3-dimensional HQFTs with target $B\chi$

In this section, given a crossed module $\chi: E \rightarrow H$, we define 3-dimensional HQFTs with target the classifying space $B\chi$. We adapt here the definition given in [16,17] by replacing pointed homotopies (which are used for aspherical targets) by σ -homotopies where σ is a graph.

Recall from Section 2.4 that the space $B\chi$ has a canonical filtration

$$(B\chi)_* = (\{x\} \subset BH \subset B\chi)$$

where x is the unique 0-cell of $B\chi$ (serving as a basepoint) and BH is a subcomplex of $B\chi$ which is a classifying space of the group H and whose associated boundary crossed module $\pi_2(B\chi, BH; x) \rightarrow \pi_1(BH, x)$ is $\chi: E \rightarrow H$.

8.1. σ -homotopies

Any graph σ embedded in a topological space X yields a filtration $X_\sigma = (\sigma_0 \subset \sigma \subset X)$ of X , where σ_0 is the set of vertices of σ . Recall that a filtered map $u: X_\sigma \rightarrow (B\chi)_*$ is a continuous map $u: X \rightarrow B\chi$ such that

$$u(\sigma_0) = \{x\} \quad \text{and} \quad u(\sigma) \subset BH.$$

By a σ -homotopy between filtered maps $u, v: X_\sigma \rightarrow (B\chi)_*$, we mean a homotopy from u to v which is filtered at each time, i.e., a continuous map $H: X \times [0, 1] \rightarrow B\chi$ such that $H(\cdot, 0) = u$, $H(\cdot, 1) = v$, and for all $t \in [0, 1]$,

$$H(\sigma_0, t) = \{x\} \quad \text{and} \quad H(\sigma, t) \subset BH.$$

We denote by $[X, B\chi]_\sigma$ the set of σ -homotopy classes of filtered maps $X_\sigma \rightarrow (B\chi)_*$. If A is a subspace of X containing σ , then the restriction to A induces a canonical map $[X, B\chi]_\sigma \rightarrow [A, B\chi]_\sigma$.

Note that if $\sigma = \emptyset$, then σ -homotopy classes of filtered maps $X_\sigma \rightarrow (B\chi)_*$ correspond to (standard) homotopy classes of maps $X \rightarrow B\chi$. Also, if σ has no edges (that is, $\sigma = \sigma_0$ is a set of points of X), then σ -homotopy classes of filtered maps $X_\sigma \rightarrow (B\chi)_*$ correspond to pointed homotopy classes of pointed maps $(X, \sigma_0) \rightarrow (B\chi, x)$.

8.2. Skeletons of surfaces and their duals

Let Σ be a closed oriented surface. A *skeleton* of Σ is an oriented graph $A \subset \Sigma$ such that all components of $\Sigma \setminus A$ are open disks and all vertices of A have valence ≥ 2 .

An oriented graph σ in Σ is *dual* to a skeleton A of Σ if

- (a) the vertices of σ lie in $\Sigma \setminus A$ and each connected component of $\Sigma \setminus A$ contains exactly one vertex of σ ;
- (b) each edge s of σ intersects A transversely at a single point in the interior of a (unique) edge s^* of A and the map $s \mapsto s^*$ induces a bijection between the edges of σ and those of A ;
- (c) for each edge s of σ , the orientation of s^* followed by that of s yields the orientation of Σ .

Note that the skeleton A of Σ gives a cellular decomposition of Σ (with 0-cells the vertices of A , 1-cells the edges of A , and 2-cells corresponding to the connected components of $\Sigma \setminus A$) and that σ is then the 1-skeleton of a cellular decomposition dual to the one given by A . In particular the number of connected components of $\Sigma \setminus A$ is equal to the number of vertices of σ .

For example, the 1-skeleton of a triangulation of Σ is a graph in Σ dual to a skeleton of Σ . Two dual graphs of a skeleton of Σ are isotopic, and two skeletons of Σ having isotopic duals are isotopic.

8.3. Preliminaries on χ -surfaces and χ -manifolds

By a χ -surface, we mean a triple (Σ, σ, f) consisting of a closed oriented surface Σ , an oriented graph σ in Σ which is dual to some skeleton of Σ , and $f \in [\Sigma, B\chi]_\sigma$, see Section 8.1. (The graph σ will play the role of basepoints in [16,17].) The *opposite* of a χ -surface (Σ, σ, f) is the χ -surface $(-\Sigma, \sigma, f)$ where $-\Sigma$ is Σ with the opposite orientation.

A χ -manifold is a triple (M, σ, g) where M is a compact oriented 3-dimensional manifold, σ is an oriented graph in the boundary ∂M of M which is dual to a skeleton of ∂M , and $g \in [M, B\chi]_\sigma$. The *boundary* of such a χ -manifold is the χ -surface $\partial M = (\partial M, \sigma, \partial g)$, where $\partial g \in [\partial M, B\chi]_\sigma$ is the restriction of g to ∂M (see Section 8.1). Here we use the ‘outward vector first’ convention for the induced orientation of the boundary: at any point of ∂M , the given orientation of M is determined by the tuple (a tangent vector directed outward, a basis in the tangent space of ∂M positive with respect to the induced orientation). A χ -manifold (M, σ, g) is *closed* if $\partial M = \emptyset$. In this case $\sigma = \emptyset$, $g \in [M, B\chi]_\emptyset = [M, B\chi]$, and we denote $(M, \sigma, g) = (M, \emptyset, g)$ by (M, g) as in Section 6.7.

Any χ -surface (Σ, σ, f) determines the *cylinder χ -manifold*

$$(\Sigma \times [0, 1], \bar{\sigma} = (\sigma \times \{0\}) \sqcup (\sigma \times \{1\}), \bar{f})$$

where $\Sigma \times [0, 1]$ is endowed with the product orientation and \bar{f} is the image of f under the map $[\Sigma, B\chi]_\sigma \rightarrow [\Sigma \times [0, 1], B\chi]_{\bar{\sigma}}$ induced by pre-composing with the projection $\Sigma \times [0, 1] \rightarrow \Sigma$.

Disjoint union of χ -surfaces (χ -manifolds) are χ -surfaces (χ -manifolds) in the obvious way. A χ -homeomorphism $(\Sigma, \sigma, f) \rightarrow (\Sigma', \sigma', f')$ between χ -surfaces is an orientation preserving diffeomorphism $\phi: \Sigma \rightarrow \Sigma'$ such that $\phi(\sigma) = \sigma'$ (as oriented graphs) and $f = f' \circ \phi$ (in $[\Sigma, B\chi]_\sigma$). A χ -homeomorphism $(M, \sigma, g) \rightarrow (M', \sigma', g')$ between χ -manifolds is an orientation preserving diffeomorphism $\psi: M \rightarrow M'$ such that $\psi(\sigma) = \sigma'$ (as oriented graphs) and $g = g' \circ \psi$ (in $[M, B\chi]_\sigma$). In particular, observe that any χ -homeomorphism $(M, \sigma, g) \rightarrow (M', \sigma', g')$ between χ -manifolds restricts to a χ -homeomorphism $(\partial M, \sigma, \partial g) \rightarrow (\partial M', \sigma', \partial g')$ between the boundary χ -surfaces.

For brevity, we shall sometimes omit the dual graphs σ and the maps to $B\chi$ from the notation for χ -surfaces and χ -manifolds.

8.4. The category of χ -cobordisms

A χ -cobordism from a χ -surface Σ_0 to a χ -surface Σ_1 is a pair (M, h) where M is a χ -manifold and $h: (-\Sigma_0) \sqcup \Sigma_1 \rightarrow \partial M$ is a χ -homeomorphism. Here, $-\Sigma_0$ is the

opposite of Σ_0 (see section 8.3). Two χ -cobordisms (M, h) and (M', h') from Σ_0 to Σ_1 are *equivalent* if there is a χ -homeomorphism $G: M \rightarrow M'$ such that $h' = G \circ h$.

We denote by Cob_3^χ the category of 3-dimensional χ -cobordisms. Objects of Cob_3^χ are χ -surfaces. A morphism $\Sigma_0 \rightarrow \Sigma_1$ in Cob_3^χ is an equivalence class of χ -cobordisms from Σ_0 to Σ_1 . The identity morphism of an χ -surface Σ is represented by the cylinder χ -manifold $\Sigma \times [0, 1]$ with tautological identification of the boundary with $(-\Sigma) \sqcup \Sigma$. Composition of morphisms in Cob_3^χ is the gluing of χ -cobordisms. More precisely, the composition of two morphisms represented by the χ -cobordisms $(M_0 = (M_0, \sigma_0, g_0), h_0): \Sigma_0 \rightarrow \Sigma_1$ and $(M_1 = (M_1, \sigma_1, g_1), h_1): \Sigma_1 \rightarrow \Sigma_2$ is represented by the χ -cobordism $(M = (M, \sigma, g), h)$ defined as follows. The 3-dimensional manifold M is obtained by gluing M_0 and M_1 along $h_1 h_0^{-1}: h_0(\Sigma_1) \rightarrow h_1(\Sigma_1)$. The graph σ in $\partial M = h_0(\Sigma_0) \cup h_1(\Sigma_2)$ is

$$\sigma = (\sigma_0 \cap h_0(\Sigma_0)) \cup (\sigma_1 \cap h_1(\Sigma_2)).$$

Pick representatives $(M_0)_{\sigma_0} \rightarrow (B\chi)_*$ of g_0 and $(M_1)_{\sigma_1} \rightarrow (B\chi)_*$ of g_1 which agree on $h_0(\Sigma_1) \simeq h_1(\Sigma_1)$. They define thus a filtered map $M_\sigma \rightarrow (B\chi)_*$. The fact that $B\chi$ is a 2-type ensures that the σ -homotopy class $g \in [M, B\chi]_\sigma$ of this filtered map is well defined. Finally,

$$h = h_0|_{\Sigma_0} \sqcup h_1|_{\Sigma_2}: (-\Sigma_0) \sqcup \Sigma_2 \simeq \partial M.$$

The disjoint union operation on χ -surfaces and χ -manifolds turns the category Cob_3^χ into a symmetric monoidal category whose unit object is the empty χ -surface.

8.5. Homotopy quantum field theories

Let $\text{Mod}_\mathbb{k}$ be the category of \mathbb{k} -modules and \mathbb{k} -linear homomorphisms. It is a symmetric monoidal category with the standard tensor product and the unit object \mathbb{k} .

A 3-dimensional *homotopy quantum field theory (HQFT) with target $B\chi$* is a symmetric strong monoidal functor $Z: \text{Cob}_3^\chi \rightarrow \text{Mod}_\mathbb{k}$. In particular, $Z(\Sigma \sqcup \Sigma') \simeq Z(\Sigma) \otimes Z(\Sigma')$ for any χ -surfaces Σ, Σ' , and similarly for morphisms. Also, $Z(\emptyset) \simeq \mathbb{k}$. We refer to [8] for a detailed definition of a strong monoidal functor.

8.6. Representations of mapping class groups of χ -surfaces

The category Cob_3^χ of χ -cobordisms includes as a subcategory the category Homeo^χ of χ -surfaces and their χ -homeomorphisms considered up to isotopy (in the class of χ -homeomorphisms). Indeed, a χ -homeomorphism of χ -surfaces $\phi: \Sigma \rightarrow \Sigma'$ determines a morphism $\Sigma \rightarrow \Sigma'$ in Cob_3^χ represented by $(C, h: (-\Sigma) \sqcup \Sigma' \rightarrow \partial C)$, where C is the cylinder χ -manifold associated with Σ' and

$$h(x) = \begin{cases} (\phi(x), 0) & \text{if } x \in \Sigma, \\ (x, 0) & \text{if } x \in \Sigma'. \end{cases}$$

Isotopic χ -homeomorphisms give rise to the same morphism in Cob^χ . The category Homeo^χ inherits a structure of a symmetric monoidal category from that of Cob_3^χ . Restricting a 3-dimensional HQFT $Z: \text{Cob}^\chi \rightarrow \text{Mod}_{\mathbb{k}}$ to Homeo^χ , we obtain a symmetric monoidal functor $\text{Homeo}^\chi \rightarrow \text{Mod}_{\mathbb{k}}$. In particular, Z induces a \mathbb{k} -linear representation of the *mapping class group of a χ -surface* Σ defined as the group of isotopy classes of χ -homeomorphisms $\Sigma \rightarrow \Sigma$.

8.7. *Remark*

Consider two χ -surfaces (Σ, σ_0, f_0) and (Σ, σ_1, f_1) having the same underlying surface Σ . The oriented graph $\bar{\sigma} = (\sigma_0 \times \{0\}) \sqcup (\sigma_1 \times \{1\})$ in $\partial(\Sigma \times [0, 1])$ is dual to a skeleton of $\partial(\Sigma \times [0, 1])$. Then any $g \in [\Sigma \times [0, 1], B\chi]_{\bar{\sigma}}$ restricting to f_i on $\Sigma \times \{i\}$ for $i \in \{0, 1\}$ determines a χ -manifold $(\Sigma \times [0, 1], \bar{\sigma}, g)$ which, endowed with tautological identification of its boundary with $(-\Sigma) \sqcup \Sigma$, represents a χ -cobordism $C_g: (\Sigma, \sigma_0, f_0) \rightarrow (\Sigma, \sigma_1, f_1)$. This χ -cobordism is an isomorphism in the category Cob_3^χ . Consequently, the image of C_g under a 3-dimensional HQFT $Z: \text{Cob}_3^\chi \rightarrow \text{Mod}_{\mathbb{k}}$ induces a \mathbb{k} -linear isomorphism $Z(\Sigma, \sigma_0, f_0) \simeq Z(\Sigma, \sigma_1, f_1)$.

8.8. *The case of aspherical targets*

Let X be an aspherical pointed connected CW-complex. Recall that X is homotopy equivalent to BG with $G = \pi_1(X)$. The trivial group homomorphism $1 \rightarrow G$ is a crossed module such that

$$B(1 \rightarrow G) = BG \quad \text{and} \quad (B(1 \rightarrow G))_* = (\{x\} \subset BG \subset BG).$$

Let Σ be a closed oriented surface. Assume that Σ is pointed (meaning that every connected component of Σ is endowed with a basepoint). Pick an oriented graph σ in Σ which is dual to a skeleton of Σ and whose set of vertices is the set of basepoints of Σ . Note that $[\Sigma, B(1 \rightarrow G)]_\sigma = [\Sigma, BG]_*$, where the right-hand side denotes the set of pointed homotopy classes of pointed maps $\Sigma \rightarrow BG$. Then any homotopy equivalence $X \approx BG$ induces a bijection

$$[\Sigma, B(1 \rightarrow G)]_\sigma \simeq [\Sigma, X]_*.$$

This implies that the category Cob_3^G defined in [17] (whose objects are pointed closed oriented surfaces with a pointed homotopy class of pointed maps to X) is equivalent to the category $\text{Cob}_3^{(1 \rightarrow G)}$ defined in Section 8.5. Consequently, HQFTs with target X in the sense of [17] correspond to HQFTs with target $B(1 \rightarrow G)$ in the sense of Section 8.5.

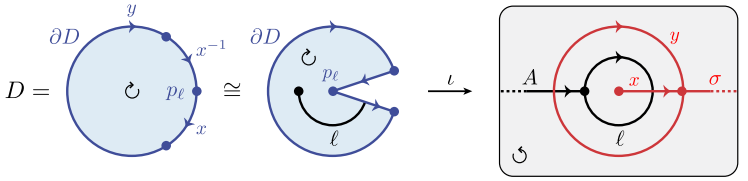


Fig. 3. The pointed disk $(D, \partial D, p_\ell)$ in Σ associated with a half-edge ℓ of A .

9. The state-sum HQFT with target $B\chi$

We fix throughout this section a crossed module $\chi: E \rightarrow H$, a spherical χ -fusion category \mathcal{C} (over \mathbb{k}) such that $\dim(\mathcal{C}_1^1) \in \mathbb{k}$ is invertible, and a χ -representative set $I = \coprod_{h \in H} I_h$ for \mathcal{C} . We extend the numerical state-sum invariant $|\cdot|_{\mathcal{C}}$ of closed χ -manifolds (defined in Section 7) to a 3-dimensional HQFT with target $B\chi$ (in the sense of Section 8).

9.1. Colored χ -surfaces

Let $\Sigma = (\Sigma, \sigma, f)$ be a χ -surface (see Section 8.3). The H -label of an edge s of σ is the element of $H = \pi_1(BH, x)$ represented by the loop $u(s)$ in BH , where $u: \Sigma_\sigma \rightarrow (B\chi)_*$ is any representative of $f \in [\Sigma, B\chi]_\sigma$.

Any skeleton A of Σ dual to σ (see Section 8.2) becomes a χ -graph in Σ as follows. The H -label of an edge a of A is the H -label of the edge of σ dual to a . Let ℓ be a half-edge of A . The vertex of A adjacent to ℓ belongs to a connected component of $\Sigma \setminus \sigma$ which is the interior of a 2-cell D attached to σ along a map $\rho: \partial D \rightarrow \sigma$. We orient D so that the induced orientation of its interior is opposite to that of Σ , and we orient ∂D using the ‘outward vector first’ convention. Let $S = \rho^{-1}(\sigma_0) \subset \partial D$, where σ_0 is the set of vertices of σ . Denote by $\iota: D \rightarrow \Sigma$ the canonical map obtained by attaching D on σ along ρ . By traversing (from its adjacent vertex) the inverse image of ℓ under ι , one reaches a point in $\partial D \setminus S$ whose predecessor in $S \subset \partial D$ (with respect to the orientation of ∂D) is denoted p_ℓ . (For an example see Fig. 3 where $x, y \in H$ are H -labels of edges of σ and blue dots are elements of S .) The E -label of ℓ is then the element of $E = \pi_2(B\chi, BH; x)$ represented by the map $u \circ \iota: (D, \partial D, p_\ell) \rightarrow (B\chi, BH, x)$, where $u: \Sigma_\sigma \rightarrow (B\chi)_*$ is any representative of $f \in [\Sigma, B\chi]_\sigma$. With these labels, A becomes a χ -graph in Σ . This follows from the fact that the boundary crossed module $\partial: \pi_2(B\chi, BH; x) \rightarrow \pi_1(BH, x)$ is $\chi: E \rightarrow H$. In particular, (16) follows from the definition of the boundary map ∂ and (17) from the definition of the action of $\pi_1(BH, x)$ on $\pi_2(B\chi, BH; x)$. For instance, in the above example, the E -label e of ℓ does satisfy the expected relation $\chi(e) = xyx^{-1}$ since both sides of this relation are equal to $\partial([f \circ i])$.

A \mathcal{C} -coloring of a χ -surface $\Sigma = (\Sigma, \sigma, f)$ is a map $d: \{\text{edges of } \sigma\} \rightarrow \text{Ob}(\mathcal{C})$ such that for each edge s of σ , the object $d(s)$ is homogeneous of degree the H -label of s . A \mathcal{C} -coloring d of Σ turns the χ -graph in Σ associated with a skeleton A of Σ dual to σ (as above) into a \mathcal{C} -colored χ -graph denoted A_d by assigning to each edge a of A the object $d(a^*)$, where a^* is the edge of σ dual to a . Section 5.5 yields then the \mathbb{k} -module $H(A, d)$.

If A and B are two skeletons of Σ dual to σ , then the canonical bijection between their half-edges (coming from the fact they are both dual to σ) induces a \mathbb{k} -linear isomorphism $\rho_{A,B}: H(A_d) \rightarrow H(B_d)$. The family $(\{H(A_d)\}_A, \{\rho_{A,B}\}_{A,B})$, where A, B run over all skeletons of Σ dual to σ , is a projective system of \mathbb{k} -modules and \mathbb{k} -linear isomorphisms. Consider the projective limit of this system:

$$H(\Sigma, d) = \varprojlim H(A_d).$$

The module $H(\Sigma, d)$ depends only on $(\Sigma, d) = (\Sigma, \sigma, f, d)$ and comes equipped with a family $\{H(\Sigma, d) \simeq H(A_d)\}_A$ of \mathbb{k} -linear isomorphisms, called the *cone isomorphisms*, which commute with the $\rho_{A,B}$.

Let $-\Sigma = (-\Sigma, \sigma, f)$ be the opposite of Σ (see Section 8.3). Any \mathcal{C} -coloring d of Σ is a \mathcal{C} -coloring of $-\Sigma$ in the obvious way. For any skeleton A of Σ dual to σ , the oriented graph A^{op} (which is A with the orientation of all edges reversed) is a skeleton of $-\Sigma$, the \mathcal{C} -colored χ -graph $A_d^{\text{op}} = (A^{\text{op}})_d \subset -\Sigma$ is opposite to $A_d \subset \Sigma$, and the canonical isomorphism $H(A_d^{\text{op}}) \simeq H(A_d)^*$ (see Section 5.5) yields a canonical isomorphism

$$H(-\Sigma, d) \simeq H(\Sigma, d)^*.$$

9.2. Skeletons of 3-manifolds

By a *skeleton* of an oriented compact 3-dimensional manifold M , we mean an oriented stratified 2-polyhedron $P \subset M$ such that $P \cap \partial M = \partial P$ and

- (i) the oriented graph ∂P is a skeleton of ∂M ;
- (ii) for every vertex v of ∂P , there is a unique edge e_v of P such that v is an endpoint of e_v and $e_v \not\subset \partial M$; the edge e_v is not a loop and $e_v \cap \partial M = \{v\}$;
- (iii) for every edge a of ∂P , the only region r_a of P adjacent to a is a closed 2-disk meeting ∂M precisely along a ;
- (iv) all connected components of $M \setminus P$ are open or half-open 3-balls.

Conditions (i)–(iii) imply that the intersection of P with a tubular neighborhood of ∂M in M is homeomorphic to $\partial P \times [0, 1]$. Also, an edge of P with both endpoints in ∂M is an edge of ∂P .

Components of $M \setminus P$ are called *P-balls*. Note that the boundary disks of the half-open P -balls are precisely the components of $\partial M \setminus \partial P$. A *P-ball branch* at a vertex or an edge of P is a germ (at the vertex or edge) of an adjacent P -ball.

An edge or vertex of P is *interior* if it is not included in ∂P . The *link graph* of an interior vertex of P is defined as in Section 6.2.

9.3. χ -labelings

We extend the notion of a χ -labeling of a skeleton of an oriented closed 3-dimensional manifold given in Section 6.3 to the case of oriented compact 3-dimensional manifolds.

Let P be a skeleton of an oriented compact 3-dimensional manifold M . As in Section 6.3, for each oriented interior edge e of P , the set P_e of branches of P at e inherits a cyclic order from the orientations of e and M , and any P -ball branch B at e turns the cyclic order on P_e into a linear order (so that the first element is the first branch b_B of P at e encountered while traversing a small loop in M positively encircling e starting from B). We define the map $\varepsilon_e: P_e \rightarrow \{+, -\}$ as in Section 6.3. Note that when the orientation of e is reversed, the cyclic order on P_e is reversed and ε_e is multiplied by $-$.

Recall that $\text{Reg}(P)$ denotes the (finite) set of regions of P (see Section 6.1). Let $\text{EB}(P)$ be the set of pairs (e, B) where e is an interior oriented edge of P and B is a P -ball branch at e . A *pre- χ -labeling* of P is a pair

$$(\alpha: \text{Reg}(P) \rightarrow H, \beta: \text{EB}(P) \rightarrow E)$$

of maps verifying the conditions (20), (21), (22) of Section 6.3.

A pre- χ -labeling (α, β) of P turns the set P_e of branches of P at an oriented interior edge e of P into a χ -cyclic set $P_e = (P_e, \alpha_e, \beta_e, \varepsilon_e)$ as follows: for any branch $b \in P_e$, set $\alpha_e(b) = \alpha(b) \in H$ and $\beta_e(b) = \beta(e, B) \in E$, where B is the P -ball branch at e such that $b = b_B$ (in the above notation).

Also, a pre- χ -labeling (α, β) of P turns the oriented graph ∂P into a χ -graph in ∂M as follows. The H -label of an edge a of ∂P is $\alpha(r_a) \in H$, where r_a is the region of P adjacent to a . Let ℓ be a half-edge of ∂P . It is adjacent to a vertex v of ∂P . The vertex v is adjacent to a (unique) interior edge e of P . Orient e so that it originates at v . Let B be the P -ball branch at e so that ℓ is the first half-edge of ∂P encountered while traversing a small loop negatively encircling v (with respect to the orientation of ∂M) from any point of $B \cap \partial M$. Then the E -label of ℓ is $\beta(e, B) \in E$.

As in Section 6.3, a pre- χ -labeling (α, β) of P turns the link graph Γ_v of an interior vertex v of P into a χ -graph in $\partial B_v \cong S^2$, where B_v is a P -cone neighborhood of v (see Section 6.2). A *χ -labeling* of P is a pre- χ -labeling of P such that, for any interior vertex v of P , the χ -graph Γ_v in $B_v \cong S^2$ is 1-spherical (see Section 5.8).

Lemma 9.1. *Let M be an oriented compact 3-dimensional manifold such that its boundary is endowed with a structure of a χ -surface $(\partial M, \sigma, f)$. Let P be a skeleton of M such that ∂P is dual to σ (see Section 8.2). Let (α, β) be a χ -labeling of P compatible with f in the sense that the two χ -graph structures on ∂P induced by (α, β) (as above) and by f (as in Section 9.1) coincide. Then this data determines a class $g_{\alpha, \beta, f} \in [M, B\chi]_\sigma$ such that $\partial g_{\alpha, \beta, f} = f$.*

Proof. Pick a center in every P -ball so that the centers of half-open P -balls are vertices of σ . Let \mathcal{R} be the set of all regions of P contained in $\text{Int}(M)$. For each $r \in \mathcal{R}$, pick an

oriented arc γ_r as in Section 6.3 and let S_r be as in Section 6.5. Let \mathcal{E} be the set of all (unoriented) interior edges of P with both endpoints in $\text{Int}(M)$. For each $e \in \mathcal{E}$, pick a P -ball branch B_e at e , an orientation for e , and a disk δ_e embedded in M as in Section 6.5. Set $S = (\cup_{r \in \mathcal{R}} S_r) \cup \sigma$. Then the pair (M, S) has a relative CW-decomposition with only 2-cells and 3-cells. The 2-cells are the disks $\{\delta_e\}_{e \in \mathcal{E}}$ together with the 2-cells of the cellular decomposition of ∂M induced by σ (i.e., the cellular decomposition dual to that induced by the skeleton ∂P of ∂M , see Section 8.2). The 3-cells are ball neighborhoods in M of the interior vertices of P . Define the map $\cup_{r \in \mathcal{R}} S_r \rightarrow B\chi$ as in the proof of Claim 6.2 and extend it to $S \cup \partial M$ using a representative of $f \in [\partial M, B\chi]_\sigma$. As in the proof of Claim 6.2, we further extend this map to the relative 2-skeleton $S \cup \partial M \cup (\cup_{e \in \mathcal{E}} \delta_e)$ by using Condition (20), and then to all 3-cells by using the 1-sphericity of the χ -graphs associated with the interior vertices of P and the compatibility of (α, β) and f on ∂P . The conditions (21) and (22) ensure that the obtained filtered map $M_\sigma \rightarrow (B\chi)_*$ does not depend on the choices of the adjacent P -ball branches and orientations for the edges $e \in \mathcal{E}$. Its σ -homotopy class $g_{\alpha, \beta, \partial g}$ satisfies $\partial g_{\alpha, \beta, f} = f$ and depends only on P , (α, β) , and f since any two systems of arcs $\{\gamma_r\}_{r \in \mathcal{R}}$, disks $\{\delta_e\}_{e \in \mathcal{E}}$, and suspensions $\{S_r\}_{r \in \mathcal{R}}$ as above are isotopic in M . \square

9.4. χ -skeletons of χ -manifolds

A χ -skeleton of a χ -manifold (M, σ, g) is a skeleton P of M together with a χ -labeling (α, β) such that ∂P is dual to σ , the χ -labeling (α, β) is compatible with ∂g (as in Lemma 9.1), and g is equal to the class $g_{\alpha, \beta, \partial g} \in [M, B\chi]_\sigma$ defined in Lemma 9.1.

The primary χ -moves $T_0^{\pm 1} - T_3^{\pm 1}$ defined in Section 6.8 for χ -skeletons of closed χ -manifolds extend to χ -skeletons of χ -manifolds in the obvious way. All these moves proceed inside 3-balls in the interior $\text{Int}(M)$ of M and do not modify the boundary of the skeletons. In particular, all vertices/edges/regions created or destroyed by the moves lie in $\text{Int}(M)$. The action of the moves on the χ -labelings is determined by the requirement that the labels of the big regions are preserved under the moves. Under $T_0^{\pm 1}, T_1^{\pm 1}, T_2^{\pm 1}, T_3^{\pm 1}$, the labels of the regions and edges lying in $\text{Int}(M)$ are transformed as in Section 6.8 and the labels of the regions and edges meeting ∂M remain unchanged. These moves as well as label-preserving ambient isotopies of χ -skeletons of (M, σ, g) are called *primary χ -moves*. The following lemma extends Lemma 6.4.

Lemma 9.2. *Any primary χ -move transforms a χ -skeleton of χ -manifold (M, σ, g) into a χ -skeleton of (M, σ, g) . Moreover, any two χ -skeletons of (M, σ, g) can be related by a finite sequence of primary χ -moves.*

Proof. The proof is parallel to the proof of Lemma 6.4 with the following changes. Given a skeleton P of an oriented compact 3-dimensional manifold M , denote by $\text{Ball}(P)$ the set of P -balls and by $\text{Reg}(P)$ the set of regions of P . The gauge group \mathcal{G}_P associated with P is the set of pairs $(\lambda, \mu) \in \text{Map}(\text{Ball}(P), H) \times \text{Map}(\text{Reg}(P), E)$ such that $\lambda(B) = 1_H$

whenever B is adjacent to ∂M (i.e., B is a half-open 3-ball) and $\mu(r) = 1_E$ whenever r is adjacent to ∂M (i.e., $r \cap \partial M \neq \emptyset$). The product in \mathcal{G}_P and the action of \mathcal{G}_P on χ -labelings of P are defined as in Section 6.3. Also, we use that any two skeletons of M whose boundaries are isotopic in ∂M can be related by a finite sequence of primary moves (see [20, Theorem 11.5]). \square

9.5. An invariant of χ -manifolds

By an I -coloring of a χ -surface Σ , we mean a \mathcal{C} -coloring of Σ (see Section 9.1) taking values in I .

Let (M, σ, g) be a χ -manifold and d be an I -coloring of the χ -surface $\partial M = (\partial M, \sigma, \partial g)$. We define a vector $|M, \sigma, g, d|_{\mathcal{C}} \in H(\partial M, d)$ where $H(\partial M, d)$ is the \mathbb{k} -module introduced in Section 9.1.

Pick a χ -skeleton $P = (P, (\alpha, \beta))$ of (M, σ, g) , see Section 9.4. An I -coloring of P is a map $c: \text{Reg}(P) \rightarrow I$ such that $c(r) \in I_{\alpha(r)}$ for all region r of P . Such an I -coloring c is said to *extend* d if $c(r_s) = d(s)$ for all edges s of σ , where r_s is the region of P adjacent to the edge of ∂P dual to s .

Given an I -coloring c of P extending d , we define $\dim(c) \in \mathbb{k}$ by (26). We also associate with c a vector $|c| \in H(\partial M, d)$ as follows. Let \mathcal{E} be the set of all oriented interior edges of P . For each $e \in \mathcal{E}$, the coloring c of P turns the χ -cyclic set P_e of branches of P at e (see Section 9.3) into a χ -cyclic \mathcal{C} -set by assigning to each branch $b \in P_e$ the c -color of the region of P containing b . Let $H_c(e) = H(P_e)$ be its multiplicity module (see Section 5.1). Each oriented edge e of P originates at a vertex of P called the *tail* of e . Let $\mathcal{E}_0 \subset \mathcal{E}$ be the set of all oriented edges of P with tail in the interior of M and $\mathcal{E}_{\partial} = \mathcal{E} \setminus \mathcal{E}_0$. Set

$$H_c = \bigotimes_{e \in \mathcal{E}_0} H_c(e) \quad \text{and} \quad H_c^{\partial} = \bigotimes_{e \in \mathcal{E}_{\partial}} H_c(e),$$

where \otimes is the unordered tensor product of \mathbb{k} -modules. Observe that the tail $v(e)$ of any $e \in \mathcal{E}_{\partial}$ is a vertex of ∂P and there is a canonical \mathbb{k} -linear isomorphism $H_c(e) \simeq H_{v(e)}((\partial P)_d)$. Here $(\partial P)_d$ is the χ -graph in ∂M induced by the χ -labeling of P (see Section 9.3) and is \mathcal{C} -colored using d (see Section 9.1). The formula $e \mapsto v(e)$ establishes a bijective correspondence between the edges $e \in \mathcal{E}_{\partial}$ and the vertices v of ∂P . This together with the cone isomorphism (see Section 9.1) gives a canonical \mathbb{k} -linear isomorphism

$$H_c^{\partial} \simeq \bigotimes_v H_v((\partial P)_d) = H((\partial P)_d) \simeq H(\partial M, d).$$

As in Section 7.1, an unoriented interior edge e of P gives rise to two opposite oriented edges $e_1, e_2 \in \mathcal{E}$ and a vector $*_e \in H_c(e_1) \otimes H_c(e_2)$ independent of the numeration of e_1, e_2 . The unordered tensor product of these vectors over all unoriented interior edges is a vector

$$\otimes_e *_e \in \bigotimes_{e \in \mathcal{E}} H_c(e).$$

We let $*_c \in H_c \otimes H(\partial M, d)$ be the image of $\otimes_e *_e$ under the isomorphism

$$\bigotimes_{e \in \mathcal{E}} H_c(e) \simeq H_c \otimes H_c^\partial \simeq H_c \otimes H(\partial M, d).$$

As in Section 7.1, an interior vertex v of P determines a 1-spherical \mathcal{C} -colored χ -graph Γ_v^c and a vector

$$\mathbb{F}_{\mathcal{C}}(\Gamma_v^c) \in H(\Gamma_v^c)^* \simeq \otimes_{e_v} H_c(e_v)^*,$$

where e_v runs over all edges of P incident to v and oriented away from v . Let $V_c \in H_c^*$ be the image of the unordered tensor product $\otimes_v \mathbb{F}_{\mathcal{C}}(\Gamma_v^c)$ over all such v under the canonical \mathbb{k} -linear isomorphism

$$\bigotimes_v H(\Gamma_v^c)^* \simeq \bigotimes_v \bigotimes_{e_v} H_c(e_v)^* \simeq H_c^*.$$

Finally set

$$|c| = (V_c \otimes \text{id}_{H(\partial M, d)})(*_c) \in \mathbb{k} \otimes H(\partial M, d) \simeq H(\partial M, d).$$

We now define a vector in $H(\partial M, d)$ by

$$|M, \sigma, g, d|_{\mathcal{C}} = \dim(\mathcal{C}_1^1)^{-|M \setminus P|} \sum_c \dim(c) |c|,$$

where $|M \setminus P|$ is the number of connected components of $M \setminus P$ and c runs over all I -colorings of P extending d .

Theorem 9.3. *The vector $|M, \sigma, g, d|_{\mathcal{C}} \in H(\partial M, d)$ is a topological invariant of (M, σ, g, d) independent of the choice of P .*

Proof. The topological invariance here means that if $\psi: (M, \sigma, g) \rightarrow (M', \sigma', g')$ is a χ -homeomorphism between χ -manifolds, then $|M', \sigma', g', d'|_{\mathcal{C}} = H(\psi)(|M, \sigma, g, d|_{\mathcal{C}})$, where d' is the composition of d with the bijection $\{\text{edges of } \sigma'\} \rightarrow \{\text{edges of } \sigma\}$ induced by ψ and $H(\psi): H(\partial M, d) \rightarrow H(\partial M', d')$ is the isomorphism induced by ψ . This invariance follows from the independence of P , and the latter is verified exactly as in the proof of Theorem 7.1 using Lemma 9.2. \square

The invariant $|M, \sigma, g, d|_{\mathcal{C}}$ is multiplicative: for any χ -manifolds (M_1, σ_1, g_1) , (M_2, σ_2, g_2) and any I -colorings d_1 of ∂M_1 and d_2 of ∂M_2 , we have:

$$|(M_1, \sigma_1, g_1) \sqcup (M_2, \sigma_2, g_2), d_1 \sqcup d_2|_C = |M_1, \sigma_1, g_1, d_1|_C \otimes |M_2, \sigma_2, g_2, d_2|_C$$

up to the canonical isomorphism

$$H(\partial(M_1 \sqcup M_2), d_1 \sqcup d_2) \simeq H(\partial M_1, d_1) \otimes H(\partial M_2, d_2).$$

9.6. Functoriality

Let $\Sigma_0 = (\Sigma_0, \sigma_0, f_0)$ and $\Sigma_1 = (\Sigma_1, \sigma_1, f_1)$ be χ -surfaces and let $\xi: \Sigma_0 \rightarrow \Sigma_1$ be a morphism in Cob_3^χ represented by a χ -cobordism (M, h) where $M = (M, \sigma, g)$ is a χ -manifold M and $h: (-\Sigma_0) \sqcup \Sigma_1 \rightarrow \partial M$ is a χ -homeomorphism (see Section 8.4). Any I -colorings d_0 of Σ_0 and d_1 of Σ_1 yields an I -coloring $d_0 \cup d_1$ of $(-\Sigma_0) \sqcup \Sigma_1$ in the obvious way and so an I -coloring d of the χ -surface $\partial M = h((-\Sigma_0) \sqcup \Sigma_1)$. Theorem 9.3 gives a vector $|M, \sigma, g, d|_C \in H(\partial M, d)$. Now let Υ be the composition of the isomorphisms

$$\begin{aligned} H(\partial M, d) &\simeq H((-\Sigma_0) \sqcup \Sigma_1, d_0 \cup d_1) \simeq H(-\Sigma_0, d_0) \otimes H(\Sigma_1, d_1) \\ &\simeq H(\Sigma_0, d_0)^* \otimes H(\Sigma_1, d_1) \simeq \text{Hom}_{\mathbb{k}}(H(\Sigma_0, d_0), H(\Sigma_1, d_1)), \end{aligned}$$

where the first isomorphism is induced by h^{-1} , the second and fourth are obvious, and the third is induced by the canonical isomorphism $H(-\Sigma_0, d_0) \simeq H(\Sigma_0, d_0)^*$ discussed in Section 9.1. Set

$$|\xi, d_0, d_1| = \frac{\dim(\mathcal{C}_1^1)^{v(\Sigma_1)}}{\dim(d_1)} \Upsilon(|M, \sigma, g, d|_C): H(\Sigma_0, d_0) \rightarrow H(\Sigma_1, d_1), \tag{38}$$

where $v(\Sigma_1)$ is the number of vertices of σ_1 and $\dim(d_1)$ is the product over all edges of σ_1 of the dimensions of their colors. Theorem 9.3 implies that the homomorphism $|f, d_0, d_1|$ does not depend on the choice of the pair (M, h) representing f . The normalization factor in the definition of $|f, d_0, d_1|$ is justified by the next lemma.

Lemma 9.4. *Let $\xi_0: \Sigma_0 \rightarrow \Sigma$ and $\xi_1: \Sigma \rightarrow \Sigma_1$ be morphisms in Cob_3^χ . For any I -colorings d_0 of Σ_0 and d_1 of Σ_1 ,*

$$|\xi_1 \xi_0, d_0, d_1| = \sum_d |\xi_1, d, d_0| \circ |\xi_0, d_0, d|: H(\Sigma_0, d_0) \rightarrow H(\Sigma_1, d_1),$$

where d runs over all I -colorings of Σ .

Proof. Represent ξ_0 and ξ_1 , respectively, by pairs (M_0, h_0) and (M_1, h_1) as above. The morphism $\xi_1 \xi_0: \Sigma_0 \rightarrow \Sigma_1$ is represented by the pair (M, h) , where M is obtained by gluing M_0 to M_1 along $h_1 h_0^{-1}: h_0(\Sigma) \rightarrow h_1(\Sigma)$ and $h = h_0|_{\Sigma_0} \sqcup h_1|_{\Sigma_2}$. To simplify notation, we identify $h_0(\Sigma) = h_1(\Sigma) \subset \text{Int}(M)$ with Σ via h_0 . Pick χ -skeletons P_0 of M_0 and P_1 of M_1 such that $\partial P_0 \cap \Sigma = \partial P_1 \cap \Sigma$. Then $P = P_0 \cup P_1 \subset M$ is a χ -skeleton of M .

Since $v(\Sigma)$ is equal to the number of connected components of $\Sigma \setminus A$ where $A = \partial P_0 \cap \Sigma$ (see Section 8.2), we have:

$$|M \setminus P| = |M_0 \setminus P_0| + |M_1 \setminus P_1| - v(\Sigma).$$

The term $-v(\Sigma)$ explains the need for the factor $\dim(\mathcal{C}_1^1)^{v(\Sigma)}$ in the definition of $|\xi_1, d, d_0|$. Similarly, given a region r_0 of P_0 and a region r_1 of P_1 adjacent to the same edge e of A , the union $r = r_0 \cup r_1 \cup e$ is a region of P with Euler characteristic $\text{Eul}(r) = \text{Eul}(r_0) + \text{Eul}(r_1) - 1$. The term -1 explains the need for the factor $\dim(d)^{-1}$ in the definition of $|\xi_1, d, d_0|$. \square

9.7. *The HQFT $|\cdot|_{\mathcal{C}}$*

For a χ -surface Σ , denote by $\text{Col}(\Sigma)$ the set of all I -colorings of Σ and consider the \mathbb{k} -module

$$|\Sigma|^\circ = \bigoplus_{d \in \text{Col}(\Sigma)} H(\Sigma, d).$$

For a morphism $\xi: \Sigma_0 \rightarrow \Sigma_1$ in Cob_3^χ , consider the \mathbb{k} -linear homomorphism

$$|\xi|^\circ = \sum_{\substack{d_0 \in \text{Col}(\Sigma_0) \\ d_1 \in \text{Col}(\Sigma_1)}} |\xi, d_0, d_1|: |\Sigma_0|^\circ \rightarrow |\Sigma_1|^\circ.$$

Lemma 9.4 implies that for any ξ_0, ξ_1 as in this lemma,

$$|\xi_1 \xi_0|^\circ = |\xi_1|^\circ \circ |\xi_0|^\circ. \tag{39}$$

In particular, this implies that $p_\Sigma = |\text{id}_\Sigma|^\circ: |\Sigma|^\circ \rightarrow |\Sigma|^\circ$ is a projector onto a direct summand $|\Sigma| = \text{Im}(p_\Sigma)$ of $|\Sigma|^\circ$. We next associate with each morphism $\xi: \Sigma_0 \rightarrow \Sigma_1$ in Cob_3^χ a homomorphism $|\xi|: |\Sigma_0| \rightarrow |\Sigma_1|$. Formulas (39) and $\text{id}_{\Sigma_1} \circ \xi = \xi$ imply that $|\xi|^\circ = p_{\Sigma_1} |\xi|^\circ$, and so the image of $|\xi|^\circ$ is contained in $|\Sigma_1|$. Denote by $|\xi|: |\Sigma_0| \rightarrow |\Sigma_1|$ the restriction of $|\xi|^\circ: |\Sigma_0|^\circ \rightarrow |\Sigma_1|^\circ$ to $|\Sigma_0|$ and $|\Sigma_1|$. It is clear that the rule $\Sigma \mapsto |\Sigma|$, $\xi \mapsto |\xi|$ defines a functor $|\cdot|: \text{Cob}_3^\chi \rightarrow \text{Mod}_{\mathbb{k}}$. We endow this functor with monoidal constraints as follows. By definition, $|\emptyset| = |\emptyset|^\circ = \mathbb{k}$ for the empty χ -surface \emptyset . Given two χ -surfaces Σ and Σ' , the canonical isomorphisms $H(\Sigma \sqcup \Sigma', d \sqcup d') \simeq H(\Sigma, d) \otimes H(\Sigma', d')$, where d runs over I -colorings of Σ and d' runs over I -colorings of Σ' , yield an isomorphism $|\Sigma \sqcup \Sigma'|^\circ \simeq |\Sigma|^\circ \otimes |\Sigma'|^\circ$. The latter factorizes into an isomorphism $|\Sigma \sqcup \Sigma'| \simeq |\Sigma| \otimes |\Sigma'|$ which, together with the canonical isomorphism $|\Sigma| \otimes_{\mathbb{k}} |\Sigma'| \simeq |\Sigma| \otimes |\Sigma'|$, defines the monoidal constraint $|\Sigma| \otimes_{\mathbb{k}} |\Sigma'| \simeq |\Sigma \sqcup \Sigma'|$. These monoidal constraints turn $|\cdot|$ into a symmetric strong monoidal functor. We denote this functor by $|\cdot|_{\mathcal{C}, I}$ or, shorter, by $|\cdot|_{\mathcal{C}}$. Combining the considerations above, we obtain the following theorem.

Theorem 9.5. *The functor $|\cdot|_{\mathcal{C}}$ is a 3-dimensional HQFT with target $B\chi$.*

The HQFT $|\cdot|_{\mathcal{C}}$ is called the *state sum HQFT* derived from \mathcal{C} . Considered up to isomorphism, the HQFT $|\cdot|_{\mathcal{C}}$ does not depend on the choice of the χ -representative set I for \mathcal{C} . The scalar invariant of closed χ -manifolds produced by this HQFT is precisely the invariant of Section 7.

9.8. The case of push-forwards

Let $\phi = (\psi: E \rightarrow E', \varphi: H \rightarrow H')$ be a morphism from a crossed module $\chi: E \rightarrow H$ to a crossed module $\chi': E' \rightarrow H'$ such that ψ and φ are surjective, $\text{Ker}(\psi) \cap \text{Ker}(\chi) = 1$, and both $\text{Ker}(\varphi)$ and $\text{Ker}(\chi')$ are finite with invertible cardinal in \mathbb{k} . Consider a spherical χ -fusion category \mathcal{C} (over \mathbb{k}) such that $\dim(\mathcal{C}_1^1)$ is invertible. Then the push-forward $\phi_*(\mathcal{C})$ of \mathcal{C} is a χ' -fusion category such that $\dim(\phi_*(\mathcal{C})_1^1)$ is invertible (see Section 7.9). By Theorem 9.5, the category $\phi_*(\mathcal{C})$ defines the 3-dimensional HQFT $|\cdot|_{\phi_*(\mathcal{C})}$ with target $B\chi'$ and the category \mathcal{C} defines the 3-dimensional HQFT $|\cdot|_{\mathcal{C}}$ with target $B\chi$. Recall that Theorem 7.2 computes the values of the former from the latter for connected closed χ' -manifolds. There is a similar computation for χ' -surfaces. More precisely, let $B\phi: (B\chi)_* \rightarrow (B\chi')_*$ be the filtered map induced by ϕ . Then for any χ' -surface (Σ, σ, f') ,

$$|\Sigma, \sigma, f'|_{\phi_*(\mathcal{C})} = \bigoplus_{\substack{f \in [\Sigma, B\chi]_{\sigma} \\ B\phi \circ f = f'}} |\Sigma, \sigma, f|_{\mathcal{C}}.$$

The proof of this equality follows that of Theorem 7.2.

Acknowledgments

This work was supported in part by the FNS-ANR grant OCHoTop (ANR-18-CE93-0002) and the Labex CEMPI (ANR-11-LABX-0007-01).

References

- [1] J. Barrett, B. Westbury, Invariants of piecewise-linear 3-manifolds, *Trans. Am. Math. Soc.* 348 (1996) 3997–4022.
- [2] M. Brightwell, P. Turner, Representations of the homotopy surface category of a simply connected space, *J. Knot Theory Ramif.* 9 (2000) 855–864.
- [3] R. Brown, P.J. Higgins, The classifying space of a crossed complex, *Math. Proc. Camb. Philos. Soc.* 110 (1991) 95–120.
- [4] P. Etingof, D. Nikshych, V. Ostrik, On fusion categories, *Ann. Math. (2)* 162 (2005) 581–642.
- [5] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, *Tensor Categories*, Mathematical Surveys and Monographs, vol. 205, American Mathematical Society, Providence, RI, 2015.
- [6] J. Faria Martins, T. Porter, On Yetter’s invariant and an extension of the Dijkgraaf-Witten invariant to categorical groups, *Theory Appl. Categ.* 18 (2007) 118–150.
- [7] A. Joyal, R. Street, The geometry of tensor calculus I, *Adv. Math.* 88 (1991) 55–112.
- [8] S. MacLane, *Categories for the Working Mathematician*, second edition, Springer-Verlag, New York, 1998.

- [9] S. Mac Lane, J.H.C. Whitehead, On the 3-type of a complex, *Proc. Natl. Acad. Sci.* (1950) 41–48.
- [10] S.V. Matveev, *Algorithmic Topology and Classification of 3-Manifolds*, second edition, *Algorithms and Computation in Math.*, vol. 9, Springer, Berlin, 2007.
- [11] T. Porter, Formal homotopy quantum field theories, II: simplicial formal maps, *Contemp. Math.* 431 (2007) 375–404.
- [12] T. Porter, V. Turaev, Formal homotopy quantum field theories, I: formal maps and crossed \mathcal{C} -algebras, *J. Homotopy Relat. Struct.* 3 (2008) 113–159.
- [13] K. Sözer, Two-dimensional extended homotopy field theories, arXiv:1909.04187, *Algebraic Geom. Topol.* (2023), in press.
- [14] K. Sözer, A. Virelizier, Hopf crossed module (co)algebras, preprint, arXiv:2305.15485, 2023.
- [15] M. Staic, V. Turaev, Remarks on 2-dimensional HQFTs, *Algebraic Geom. Topol.* 10 (2010) 1367–1393.
- [16] V. Turaev, *Homotopy Quantum Field Theory*, EMS Tracts in Math., vol. 10, European Math. Soc. Publ. House, Zürich, 2010.
- [17] V. Turaev, A. Virelizier, On 3-dimensional homotopy quantum field theory, I, *Int. J. Math.* 23 (9) (2012) 1250094.
- [18] V. Turaev, A. Virelizier, On 3-dimensional homotopy quantum field theory II: the surgery approach, *Int. J. Math.* 25 (4) (2014) 1450027.
- [19] V. Turaev, A. Virelizier, On 3-dimensional homotopy quantum field theory III: comparison of two approaches, *Int. J. Math.* 31 (10) (2020) 2050076.
- [20] V. Turaev, A. Virelizier, *Monoidal Categories and Topological Field Theory*, *Progress in Mathematics*, vol. 322, Birkhäuser, Basel, 2017, xii+523 pp.
- [21] D.N. Yetter, TQFTs from homotopy 2-types, *J. Knot Theory Ramif.* 2 (1993) 113–123.