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On the graded center of graded categories

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ABSTRACT

We study the G-centers of G-graded monoidal categories where G is an arbitrary group. We prove that for any spherical G-fusion category $\mathcal{C}$ over an algebraically closed field such that the dimension of the neutral component of $\mathcal{C}$ is non-zero, the G-center of $\mathcal{C}$ is a G-modular category. This generalizes a theorem of M. Müger corresponding to $G = 1$. We also exhibit interesting objects of the G-center.

1. Introduction

The study of group-graded categories was initiated by the first author [11] with a view towards constructing 3-dimensional Homotopy Quantum Field Theory (HQFT) generalizing the 3-dimensional Topological Quantum Field Theory (TQFT) introduced by E. Witten and M. Atiyah. An HQFT applies to manifolds and cobordisms equipped with maps to a fixed target space. HQFTs with target the Eilenberg–MacLane space $K(G, 1)$, where G is a group, naturally arise from G-graded categories via two fundamental constructions based on state-sums on triangulations and on surgery; see [11,13] and references therein. The present paper is a part of the authors' work on the following claim: for any group G, the state sum HQFT associated with a spherical G-fusion category is isomorphic to the surgery HQFT associated with the G-center of that category. We provide here the algebraic background for this claim and specifically study the G-centers.

A G-graded category is a monoidal category whose objects are equipped with a multiplicative grading by elements of G. The objects of $\mathcal{C}$ graded by $\alpha \in G$ form a full subcategory, $\mathcal{C}_\alpha$, of $\mathcal{C}$ called the α-component of $\mathcal{C}$. The multiplicativity of the grading means that $X \otimes Y \in \mathcal{C}_{\alpha\beta}$ for any $X \in \mathcal{C}_\alpha$, $Y \in \mathcal{C}_\beta$ with $\alpha, \beta \in G$. The category $\mathcal{C}_1$ corresponding to $\alpha = 1$ is called the neutral component of $\mathcal{C}$.

A number of standard notions of the theory of monoidal categories (corresponding to $G = 1$) naturally generalize to this setting. This leads, in particular, to a notion of a G-fusion category. On the other hand, to define G-braidings in a G-graded category $\mathcal{C}$, one needs an additional ingredient: an action of G on $\mathcal{C}$ by strong monoidal auto-equivalences $\{\varphi_\alpha : \mathcal{C} \to \mathcal{C}\}_{\alpha \in G}$ such that $\varphi_\alpha(\mathcal{C}_\beta) \subset \mathcal{C}_{\alpha^{-1}\beta\alpha}$ for all $\alpha, \beta \in \mathcal{C}$. Using this action, called a crossing, we define G-braided and G-ribbon categories. Under the simplifying assumption that the crossing is strict, these notions were first introduced in [11].

The Drinfeld–Joyal–Street center construction applies to any monoidal category $\mathcal{C}$ and produces a braided monoidal category $\mathcal{Z}(\mathcal{C})$, the center of $\mathcal{C}$. An analogue of the center in the setting of G-graded categories was considered by Gelaki, Naidu, and Nikshych [4]: for a finite group G, they associate to any G-fusion category $\mathcal{C}$ a G-braided category $\mathcal{Z}_G(\mathcal{C})$. The construction of $\mathcal{Z}_G(\mathcal{C})$ as a monoidal category is rather straightforward and applies to an arbitrary group G and any G-graded category $\mathcal{C}$. The objects of $\mathcal{Z}_G(\mathcal{C})$ are pairs (A, σ) where A is an object of $\mathcal{C}$ and σ is a half-braiding in $\mathcal{C}$ relative to $\mathcal{C}_1$, that is a system of isomorphisms $\sigma_Y : A \otimes Y \to Y \otimes A$ in $\mathcal{C}$ permuting A with arbitrary objects Y of $\mathcal{C}_1$. The morphisms in $\mathcal{Z}_G(\mathcal{C})$ are the morphisms in $\mathcal{C}$ commuting with the half-braidings. The monoidal product in $\mathcal{Z}_G(\mathcal{C})$ is essentially the composition of

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half-braidings. The difficult part in the construction of $\mathcal{Z}_G(\mathcal{C})$, requiring additional assumptions on $\mathcal{C}$, concerns the crossing and the G-braiding. We define a crossing and a G-braiding in $\mathcal{Z}_G(\mathcal{C})$ for non-singular G-graded categories $\mathcal{C}$ generalizing the Gelaki–Naidu–Nikshych construction (see Theorem 4.1).

For topological applications, it is important to study the so-called G-modular categories. A G-modular category is a G-fusion G-ribbon G-graded category whose neutral component (which is a fusion ribbon category in the usual sense) has an invertible S-matrix. Our first main result is the following modularity theorem.

Theorem 1.1. *If $\mathcal{C}$ is a spherical G-fusion category over an algebraically closed field and the dimension of the neutral component of $\mathcal{C}$ is non-zero, then $\mathcal{Z}_G(\mathcal{C})$ is a G-modular category.*

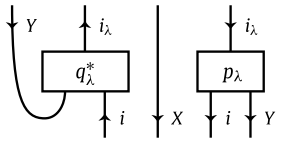
This theorem is highly non-trivial already in the case $G = 1$ where it was first proved by M. Müger; see [8, Theorem 1.2]. Our proof of the modularity theorem uses the technique of Hopf monads introduced in [1].

In general, it is not easy to exhibit objects of the G-center of a G-graded category $\mathcal{C}$. When $\mathcal{C}$ is non-singular and $\mathcal{C}_1$-centralizable (in the sense defined in this paper), the forgetful functor $\mathcal{Z}_G(\mathcal{C}) \to \mathcal{C}$, $(A, \sigma) \mapsto A$, has a left adjoint functor $\mathcal{F} : \mathcal{C} \to \mathcal{Z}_G(\mathcal{C})$. The objects of $\mathcal{Z}_G(\mathcal{C})$ isomorphic to objects in the image of $\mathcal{F}$ are said to be free. Our second group of results yields explicit computations of the free objects. In particular, if $\mathcal{C}$ is a G-fusion category over a field, then $\mathcal{C}$ is non-singular and $\mathcal{C}_1$-centralizable, and for any object X of $\mathcal{C}$,

$$\mathcal{F}(X) \simeq \Big(A = \bigoplus_{i \in I_1} i^* \otimes X \otimes i,\ \sigma = \{\sigma_Y : A \otimes Y \to Y \otimes A\}_{Y \in \mathcal{C}_1}\Big)$$

where I_1 is a representative set of isomorphism classes of simple objects of $\mathcal{C}_1$ and Y runs over objects of $\mathcal{C}_1$. The restriction of σ_Y to the direct summand $i^* \otimes X \otimes i \otimes Y$ of $A \otimes Y$ with $i \in I_1$ is computed by

$$\sigma_Y = \sum_{\lambda}$$



where $(p_\lambda : i \otimes Y^* \to i_\lambda, q_\lambda : i_\lambda \to i \otimes Y^*)_\lambda$ are the projections and the embeddings determined by a splitting of $i \otimes Y^*$ as a direct sum of simple objects $i_\lambda \in I_1$. Similar pictorial formulas compute the crossing and the G-braiding in $\mathcal{Z}_G(\mathcal{C})$ on the free objects. For example, for any $\alpha \in G$, the object $\varphi_\alpha(\mathcal{F}(X))$ is computed by the formula above with I_1 replaced everywhere by I_α, a representative set of isomorphism classes of simple objects of $\mathcal{C}_\alpha$. For precise statements, see Theorems 10.4 and 10.6.

Every G-modular category gives rise to a 3-dimensional HQFT with target $K(G, 1)$. The modularity theorem above allows us to derive such an HQFT from $\mathcal{Z}_G(\mathcal{C})$. Our computations with free objects lead to a computation of the vector spaces assigned by this HQFT to surfaces equipped with maps to $K(G, 1)$. These results are crucially used in the proof (given elsewhere) of the claim stated at the beginning of the introduction. In the present paper we focus on the algebraic side of the theory and do not study HQFTs.

The organization of the paper is as follows. In Section 2 we recall necessary notions from the theory of monoidal categories. In Section 3 we introduce the key notions of the theory of G-graded categories. In Section 4 we construct the G-center. In Section 5 we introduce spherical G-fusion categories, state the modularity theorem, and start its proof. In Sections 6–8 we discuss G-ribbonness, Hopf monads, and coends, respectively. In Section 9 we finish the proof of the modularity theorem. In Section 10 we compute the crossing and the G-braiding on the free objects and also discuss the G-fusion case. The appendix is devoted to the computation of certain objects of $\mathcal{Z}_G(\mathcal{C})$ which will be instrumental in the study of the associated HQFT.

Throughout the paper, we fix a (discrete) group G and a commutative ring $\Bbbk$.

2. Preliminaries on categories and functors

We recall here several standard notions and techniques of the theory of monoidal categories referring for details to [5].

2.1. Conventions

The unit object of a monoidal category $\mathcal{C}$ is denoted by $\mathbb{1} = \mathbb{1}_{\mathcal{C}}$. Notation $X \in \mathcal{C}$ means that X is an object of $\mathcal{C}$. To simplify the formulas, we will always pretend that the monoidal categories at hand are strict. Consequently, we omit brackets in the monoidal products and suppress the associativity constraints $(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$ and the unitality constraints $X \otimes \mathbb{1} \cong X \cong \mathbb{1} \otimes X$. By the monoidal product $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ of $n \geq 2$ objects $X_1, \ldots, X_n$ of a monoidal category we mean $(\ldots((X_1 \otimes X_2) \otimes X_3) \otimes \cdots \otimes X_{n-1}) \otimes X_n$.

2.2. Monoidal functors

Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. A *monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a triple (F, F_2, F_0), where $F\colon \mathcal{C} \to \mathcal{D}$ is a functor,

$$F_2 = \{F_2(X,Y)\colon F(X)\otimes F(Y) \to F(X\otimes Y)\}_{X,Y\in\mathcal{C}}$$

is a natural transformation from $\otimes(F\times F)$ to $F\otimes$, and $F_0\colon \mathbb{1}_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}})$ is a morphism in $\mathcal{D}$ such that the following diagrams commute for all $X, Y, Z \in \mathcal{C}$:

$$\begin{tikzcd} F(X)\otimes F(Y)\otimes F(Z) \arrow[r, "\mathrm{id}_{F(X)}\otimes F_2(Y,Z)"] \arrow[d, "F_2(X,Y)\otimes \mathrm{id}_{F(Z)}"'] & F(X)\otimes F(Y\otimes Z) \arrow[d, "F_2(X,Y\otimes Z)"] \\ F(X\otimes Y)\otimes F(Z) \arrow[r, "F_2(X\otimes Y,Z)"'] & F(X\otimes Y\otimes Z), \end{tikzcd} \tag{1}$$

$$\begin{tikzcd} F(X) \arrow[r, "\mathrm{id}_{F(X)}\otimes F_0"] \arrow[d, "F_0\otimes \mathrm{id}_{F(X)}"'] \arrow[dr, "\mathrm{id}_{F(X)}"] & F(X)\otimes F(\mathbb{1}_{\mathcal{C}}) \arrow[d, "F_2(X,\mathbb{1}_{\mathcal{C}})"] \\ F(\mathbb{1}_{\mathcal{C}})\otimes F(X) \arrow[r, "F_2(\mathbb{1}_{\mathcal{C}},X)"'] & F(X). \end{tikzcd} \tag{2}$$

A monoidal functor (F, F_2, F_0) is *strong* if F_2 and F_0 are isomorphisms and *strict* if F_2 and F_0 are identity morphisms.

A natural transformation $\varphi = \{\varphi_X\colon F(X) \to G(X)\}_{X\in\mathcal{C}}$ from a monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$ to a monoidal functor $G\colon \mathcal{C} \to \mathcal{D}$ is *monoidal* if $G_0 = \varphi_{\mathbb{1}} F_0$ and

$$\varphi_{X\otimes Y} F_2(X,Y) = G_2(X,Y)(\varphi_X \otimes \varphi_Y)$$

for all $X, Y \in \mathcal{C}$. A *monoidal natural isomorphism* between F and G is a monoidal natural transformation φ from F to G which is an isomorphism in the sense that each φ_X is an isomorphism. The inverse $\varphi^{-1} = \{\varphi_X^{-1}\colon G(X) \to F(X)\}_{X\in\mathcal{C}}$ is then a monoidal natural transformation from G to F.

If $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ are monoidal functors between monoidal categories, then their composition $GF\colon \mathcal{C} \to \mathcal{E}$ is a monoidal functor with

$$(GF)_0 = G(F_0)G_0 \quad \text{and} \quad (GF)_2 = \{G(F_2(X,Y))\, G_2(F(X),F(Y))\}_{X,Y\in\mathcal{C}}.$$

2.3. Rigid categories

Let $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1})$ be a monoidal category. A *left dual* of an object $X \in \mathcal{C}$ is an object ${}^\vee\! X \in \mathcal{C}$ together with morphisms $\mathrm{ev}_X\colon {}^\vee\! X \otimes X \to \mathbb{1}$ and $\mathrm{coev}_X\colon \mathbb{1} \to X \otimes {}^\vee\! X$ such that

$$(\mathrm{id}_X \otimes \mathrm{ev}_X)(\mathrm{coev}_X \otimes \mathrm{id}_X) = \mathrm{id}_X \quad \text{and} \quad (\mathrm{ev}_X \otimes \mathrm{id}_{{}^\vee\! X})(\mathrm{id}_{{}^\vee\! X} \otimes \mathrm{coev}_X) = \mathrm{id}_{{}^\vee\! X}.$$

One calls $\mathcal{C}$ *left rigid* if every object of $\mathcal{C}$ has a left dual. A choice of a left dual for each object of $\mathcal{C}$ defines a *left dual functor* ${}^\vee ?\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$, where $\mathcal{C}^{\mathrm{op}}$ is the category opposite to $\mathcal{C}$ with opposite monoidal structure. The functor ${}^\vee ?$ carries a morphism $f\colon X \to Y$ in $\mathcal{C}$ (i.e., a morphism $Y \to X$ in $\mathcal{C}^{\mathrm{op}}$) to

$${}^\vee\! f = (\mathrm{ev}_Y \otimes \mathrm{id}_{{}^\vee\! X})(\mathrm{id}_{{}^\vee Y} \otimes f \otimes \mathrm{id}_{{}^\vee\! X})(\mathrm{id}_{{}^\vee Y} \otimes \mathrm{coev}_X)\colon {}^\vee Y \to {}^\vee\! X.$$

The functor ${}^\vee ?$ is strong monoidal with $({}^\vee ?)_0 = \mathrm{coev}_{\mathbb{1}}\colon \mathbb{1} \to {}^\vee \mathbb{1}$ and with

$$({}^\vee ?)_2(X,Y)\colon {}^\vee\! X \otimes {}^\vee Y \to {}^\vee(X \otimes^{\mathrm{op}} Y) = {}^\vee(Y \otimes X)$$

defined to be equal to

$$(\mathrm{ev}_X \otimes \mathrm{id}_{{}^\vee(X\otimes Y)})(\mathrm{id}_{{}^\vee\! X} \otimes \mathrm{coev}_Y \otimes \mathrm{id}_{X\otimes {}^\vee(X\otimes Y)})(\mathrm{id}_{{}^\vee\! X\otimes {}^\vee Y} \otimes \mathrm{coev}_{X\otimes Y}).$$

The isomorphisms $({}^\vee ?)_0$ and $({}^\vee ?)_2(X,Y)$ are called *left monoidal constraints*.

Similarly, a *right dual* of $X \in \mathcal{C}$ is an object $X^\vee$ of $\mathcal{C}$ equipped with morphisms $\widetilde{\mathrm{ev}}_X\colon X \otimes X^\vee \to \mathbb{1}$ and $\widetilde{\mathrm{coev}}_X\colon \mathbb{1} \to X^\vee \otimes X$ such that

$$(\widetilde{\mathrm{ev}}_X \otimes \mathrm{id}_X)(\mathrm{id}_X \otimes \widetilde{\mathrm{coev}}_X) = \mathrm{id}_X \quad \text{and} \quad (\mathrm{id}_{X^\vee} \otimes \widetilde{\mathrm{ev}}_X)(\widetilde{\mathrm{coev}}_X \otimes \mathrm{id}_{X^\vee}) = \mathrm{id}_{X^\vee}.$$

One calls $\mathcal{C}$ *right rigid* if every object of $\mathcal{C}$ has a right dual. Similarly to the above, for a right rigid category $\mathcal{C}$, one defines a strong monoidal *right dual functor* $?^\vee\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ and *right monoidal constraints*.

A monoidal category is *rigid* if it is both left rigid and right rigid. Note that the left and right duals of an object are unique up to an isomorphism preserving the (co)evaluation morphisms. Different choices of left/right dual objects lead to monoidally isomorphic left/right dual functors.

2.4. Pivotal categories

By a *pivotal* category we mean a rigid monoidal category $\mathcal{C}$ such that the left and right dual functors are equal as monoidal functors. Then for each object $X \in \mathcal{C}$, we have a *dual object* $X^* = {}^\vee X = X^\vee$ and morphisms

$$\begin{aligned} &\mathrm{ev}_X : X^* \otimes X \to \mathbb{1}, \qquad \mathrm{coev}_X : \mathbb{1} \to X \otimes X^*, \\ &\widetilde{\mathrm{ev}}_X : X \otimes X^* \to \mathbb{1}, \qquad \widetilde{\mathrm{coev}}_X : \mathbb{1} \to X^* \otimes X, \end{aligned}$$

such that $(X^*, \mathrm{ev}_X, \mathrm{coev}_X)$ is a left dual for X and $(X^*, \widetilde{\mathrm{ev}}_X, \widetilde{\mathrm{coev}}_X)$ is a right dual for X. The *dual* $f^* : Y^* \to X^*$ of any morphism $f : X \to Y$ in $\mathcal{C}$ is computed by

$$\begin{aligned} f^* &= (\mathrm{ev}_Y \otimes \mathrm{id}_{X^*})(\mathrm{id}_{Y^*} \otimes f \otimes \mathrm{id}_{X^*})(\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X) \\ &= (\mathrm{id}_{X^*} \otimes \widetilde{\mathrm{ev}}_Y)(\mathrm{id}_{X^*} \otimes f \otimes \mathrm{id}_{Y^*})(\widetilde{\mathrm{coev}}_X \otimes \mathrm{id}_{Y^*}). \end{aligned}$$

Working with a pivotal category $\mathcal{C}$, we will suppress the duality constraints $\mathbb{1}^* \cong \mathbb{1}$ and $X^* \otimes Y^* \cong (Y \otimes X)^*$. For example, we write $(f \otimes g)^* = g^* \otimes f^*$ for morphisms f, g in $\mathcal{C}$.

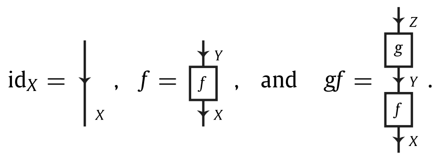
For an endomorphism g of an object X of a pivotal category $\mathcal{C}$, one defines the *left* and *right traces*

$$\mathrm{tr}_l(g) = \mathrm{ev}_X(\mathrm{id}_{X^*} \otimes g)\widetilde{\mathrm{coev}}_X \quad \text{and} \quad \mathrm{tr}_r(g) = \widetilde{\mathrm{ev}}_X(g \otimes \mathrm{id}_{X^*})\mathrm{coev}_X.$$

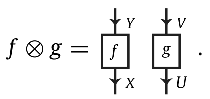
Both traces take values in $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$ and are symmetric: $\mathrm{tr}_{l/r}(fh) = \mathrm{tr}_{l/r}(hf)$ for any morphisms $f : X \to Y$, $h : Y \to X$ in $\mathcal{C}$. Also $\mathrm{tr}_{l/r}(g) = \mathrm{tr}_{r/l}(g^*)$ for any endomorphism g of an object. The *left* and *right dimensions* of an object $X \in \mathcal{C}$ are defined by $\dim_{l/r}(X) = \mathrm{tr}_{l/r}(\mathrm{id}_X)$. Clearly, $\dim_{l/r}(X) = \dim_{r/l}(X^*)$ for all X.

2.5. Penrose graphical calculus

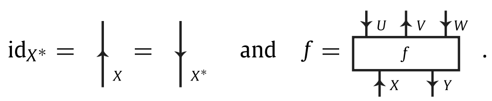
We will represent morphisms in a category $\mathcal{C}$ by plane diagrams to be read from the bottom to the top. The diagrams are made of oriented arcs colored by objects of $\mathcal{C}$ and of boxes colored by morphisms of $\mathcal{C}$. The arcs connect the boxes and have no mutual intersections or self-intersections. The identity id_X of $X \in \mathcal{C}$, a morphism $f : X \to Y$, and the composition of two morphisms $f : X \to Y$ and $g : Y \to Z$ are represented as follows:



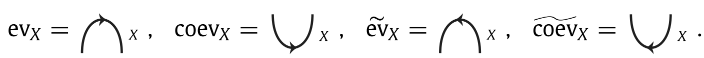
If $\mathcal{C}$ is monoidal, then the monoidal product of two morphisms $f : X \to Y$ and $g : U \to V$ is represented by juxtaposition:



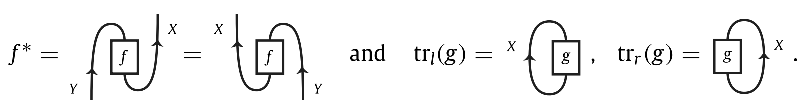
Suppose that $\mathcal{C}$ is pivotal. By convention, if an arc colored by $X \in \mathcal{C}$ is oriented upwards, then the corresponding object in the source/target of morphisms is X^*. For example, id_{X^*} and a morphism $f : X^* \otimes Y \to U \otimes V^* \otimes W$ may be depicted as



The duality morphisms are depicted as follows:



The dual of a morphism $f : X \to Y$ and the traces of a morphism $g : X \to X$ can be depicted as follows:



It is easy to see that the morphisms represented by such diagrams are invariant under isotopies of the diagrams in $\mathbb{R}^2$ keeping fixed the bottom and top endpoints.

2.6. Pivotal functors

Given a strong monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$ between pivotal categories, we define for each $X \in \mathcal{C}$ a morphism

$$F^l(X) = (F_0^{-1}F(\mathrm{ev}_X)F_2(X^*, X) \otimes \mathrm{id}_{F(X)^*})(\mathrm{id}_{F(X^*)} \otimes \mathrm{coev}_{F(X)}) : F(X^*) \to F(X)^*.$$

It is well-known that $F^l = \{F^l(X)\colon F(X^*) \to F(X)^*\}_{X\in\mathcal{C}}$ is a monoidal natural isomorphism. Likewise, the morphisms $\{F^r(X)\colon F(X^*) \to F(X)^*\}_{X\in\mathcal{C}}$, defined by

$$F^r(X) = (\mathrm{id}_{F(X)^*} \otimes F_0^{-1}F(\widetilde{\mathrm{ev}}_X)F_2(X, X^*))(\widetilde{\mathrm{coev}}_{F(X)} \otimes \mathrm{id}_{F(X^*)}),$$

form a monoidal natural isomorphism F^r. For all $X \in \mathcal{C}$, we have

$$F^l(X^*)F(\phi_X) = F^r(X)^*\phi_{F(X)}$$

where $\{\phi_X : X \to X^{**}\}_X$ is the *pivotal structure* in $\mathcal{C}$ defined by

$$\phi_X = (\widetilde{\mathrm{ev}}_X \otimes \mathrm{id}_{X^{**}})(\mathrm{id}_X \otimes \mathrm{coev}_{X^*})\colon X \to X^{**}. \tag{3}$$

The monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$ is *pivotal* if $F^l(X) = F^r(X)$ for any $X \in \mathcal{C}$. In this case, $F^l = F^r$ is denoted by F^1.

2.7. Additive categories

A category $\mathcal{C}$ is $\Bbbk$-*additive* if the Hom-sets of $\mathcal{C}$ are modules over the ring $\Bbbk$, the composition of morphisms is $\Bbbk$-bilinear, and any finite family of objects has a direct sum. In particular, such a $\mathcal{C}$ has a *zero object*, that is, an object $\mathbf{0} \in \mathcal{C}$ such that $\mathrm{End}_{\mathcal{C}}(\mathbf{0}) = 0$. A monoidal category is $\Bbbk$-*additive* if it is $\Bbbk$-additive as a category and the monoidal product is $\Bbbk$-bilinear.

A functor $F\colon \mathcal{C} \to \mathcal{D}$ between $\Bbbk$-additive categories is $\Bbbk$-*linear* if the map from $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ to $\mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ induced by F is $\Bbbk$-linear for all $X, Y \in \mathcal{C}$. Such a functor necessarily preserves direct sums.

3. *G*-structures on monoidal categories

In this section we define the classes of G-graded, G-crossed, G-braided, and G-ribbon categories.

3.1. G-graded categories

By a *G-graded category (over* $\Bbbk$*)*, we mean a $\Bbbk$-additive monoidal category $\mathcal{C}$ endowed with a system of pairwise disjoint full $\Bbbk$-additive subcategories $\{\mathcal{C}_\alpha\}_{\alpha\in G}$ such that

(a) each object $X \in \mathcal{C}$ splits as a direct sum $\oplus_\alpha X_\alpha$ where $X_\alpha \in \mathcal{C}_\alpha$ and α runs over a finite subset of G;
(b) if $X \in \mathcal{C}_\alpha$ and $Y \in \mathcal{C}_\beta$, then $X \otimes Y \in \mathcal{C}_{\alpha\beta}$;
(c) if $X \in \mathcal{C}_\alpha$ and $Y \in \mathcal{C}_\beta$ with $\alpha \neq \beta$, then $\mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$;
(d) $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}_1$.

The category $\mathcal{C}_1$ corresponding to the neutral element $1 \in G$ is called the *neutral component* of $\mathcal{C}$. Clearly, $\mathcal{C}_1$ is a $\Bbbk$-additive monoidal category.

3.2. G-crossed categories

Given a monoidal $\Bbbk$-additive category $\mathcal{C}$, denote by $\mathrm{Aut}(\mathcal{C})$ the category whose objects are $\Bbbk$-linear strong monoidal functors $\mathcal{C} \to \mathcal{C}$ that are equivalences of categories. The morphisms in $\mathrm{Aut}(\mathcal{C})$ are monoidal natural isomorphisms. The category $\mathrm{Aut}(\mathcal{C})$ has a canonical structure of a monoidal category, in which the monoidal product is the composition of monoidal functors and the monoidal unit is the identity endofunctor $1_{\mathcal{C}}$ of $\mathcal{C}$.

Denote by $\overline{G}$ the category whose objects are elements of G and morphisms are identities. We turn $\overline{G}$ into a monoidal category with the monoidal product given by the opposite group multiplication in G, i.e., $\alpha \otimes \beta = \beta\alpha$ for all $\alpha, \beta \in G$.

A *G-crossed category* is a G-graded category $\mathcal{C}$ (over $\Bbbk$) endowed with a *crossing*, that is, a strong monoidal functor $\varphi\colon \overline{G} \to \mathrm{Aut}(\mathcal{C})$ such that $\varphi_\alpha(\mathcal{C}_\beta) \subset \mathcal{C}_{\alpha^{-1}\beta\alpha}$ for all $\alpha, \beta \in G$. Thus, for each $\alpha \in G$, the crossing φ provides a strong monoidal equivalence $\varphi_\alpha : \mathcal{C} \to \mathcal{C}$ equipped with an isomorphism $(\varphi_\alpha)_0\colon \mathbb{1} \xrightarrow{\sim} \varphi_\alpha(\mathbb{1})$ in $\mathcal{C}$ and with natural isomorphisms

$$(\varphi_\alpha)_2 = \{(\varphi_\alpha)_2(X, Y)\colon \varphi_\alpha(X) \otimes \varphi_\alpha(Y) \xrightarrow{\sim} \varphi_\alpha(X \otimes Y)\}_{X,Y\in\mathcal{C}},$$

$$\varphi_2 = \left\{\varphi_2(\alpha, \beta) = \{\varphi_2(\alpha, \beta)_X\colon \varphi_\alpha\varphi_\beta(X) \xrightarrow{\sim} \varphi_{\beta\alpha}(X)\}_{X\in\mathcal{C}}\right\}_{\alpha,\beta\in G},$$

$$\varphi_0 = \{(\varphi_0)_X\colon X \xrightarrow{\sim} \varphi_1(X)\}_{X\in\mathcal{C}}$$

such that $(\varphi_0)_{\mathbb{1}} = (\varphi_1)_0$ and for all $\alpha, \beta, \gamma \in G$ and all $X, Y, Z \in \mathcal{C}$, the following diagrams commute:

$$\begin{tikzcd}
\varphi_\alpha(X)\otimes\varphi_\alpha(Y)\otimes\varphi_\alpha(Z) \arrow[r, "\mathrm{id}_{\varphi_\alpha(X)}\otimes(\varphi_\alpha)_2(Y,Z)"] \arrow[d, "(\varphi_\alpha)_2(X,Y)\otimes\mathrm{id}_{\varphi_\alpha(Z)}"] & \varphi_\alpha(X)\otimes\varphi_\alpha(Y\otimes Z) \arrow[d, "(\varphi_\alpha)_2(X,Y\otimes Z)"] \\
\varphi_\alpha(X\otimes Y)\otimes\varphi_\alpha(Z) \arrow[r, "(\varphi_\alpha)_2(X\otimes Y,Z)"'] & \varphi_\alpha(X\otimes Y\otimes Y),
\end{tikzcd} \tag{4}$$

$$\begin{tikzcd}
\varphi_\alpha(X) \arrow[r, "\mathrm{id}_{\varphi_\alpha(X)}\otimes(\varphi_\alpha)_0"] \arrow[d, "(\varphi_\alpha)_0\otimes\mathrm{id}_{\varphi_\alpha(X)}"'] \arrow[dr, "\mathrm{id}_{\varphi_\alpha(X)}"] & \varphi_\alpha(X)\otimes\varphi_\alpha(\mathbb{1}) \arrow[d, "(\varphi_\alpha)_2(X,\mathbb{1})"] \\
\varphi_\alpha(\mathbb{1})\otimes\varphi_\alpha(X) \arrow[r, "(\varphi_\alpha)_2(\mathbb{1},X)"'] & \varphi_\alpha(X),
\end{tikzcd} \tag{5}$$

$$\begin{tikzcd}
 & & \varphi_{\beta\alpha}(X)\otimes\varphi_{\beta\alpha}(Y) \arrow[drr, "(\varphi_{\beta\alpha})_2(X,Y)"] & & \\
\varphi_\alpha\varphi_\beta(X)\otimes\varphi_\alpha\varphi_\beta(Y) \arrow[urr, "\varphi_2(\alpha,\beta)_X\otimes\varphi_2(\alpha,\beta)_Y"] \arrow[dr, "(\varphi_\alpha)_2(\varphi_\beta(X),\varphi_\beta(Y))"'] & & & & \varphi_{\beta\alpha}(X\otimes Y) \\
 & \varphi_\alpha(\varphi_\beta(X)\otimes\varphi_\beta(Y)) \arrow[rr, "\varphi_\alpha((\varphi_\beta)_2(X,Y))"'] & & \varphi_\alpha\varphi_\beta(X\otimes Y), \arrow[ur, "\varphi_2(\alpha,\beta)_{X\otimes Y}"'] &
\end{tikzcd} \tag{6}$$

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "(\varphi_{\beta\alpha})_0"] \arrow[d, "(\varphi_\alpha)_0"'] & \varphi_{\beta\alpha}(\mathbb{1}) \\
\varphi_\alpha(\mathbb{1}) \arrow[r, "\varphi_\alpha\big((\varphi_\beta)_0\big)"'] & \varphi_\alpha\varphi_\beta(\mathbb{1}), \arrow[u, "\varphi_2(\alpha,\beta)_{\mathbb{1}}"']
\end{tikzcd} \tag{7}$$

$$\begin{tikzcd}
X\otimes Y \arrow[rr, "(\varphi_0)_{X\otimes Y}"] \arrow[dr, "(\varphi_0)_X\otimes(\varphi_0)_Y"'] & & \varphi_1(X\otimes Y) \\
 & \varphi_1(X)\otimes\varphi_1(Y), \arrow[ur, "(\varphi_1)_2(X,Y)"'] &
\end{tikzcd} \tag{8}$$

$$\begin{tikzcd}
\varphi_\alpha\varphi_\beta\varphi_\gamma(X) \arrow[r, "\varphi_\alpha(\varphi_2(\beta,\gamma)_X)"] \arrow[d, "\varphi_2(\alpha,\beta)_{\varphi_\gamma(X)}"'] & \varphi_\alpha\varphi_{\gamma\beta}(X) \arrow[d, "\varphi_2(\alpha,\gamma\beta)_X"] \\
\varphi_{\beta\alpha}\varphi_\gamma(X) \arrow[r, "\varphi_2(\beta\alpha,\gamma)_X"'] & \varphi_{\gamma\beta\alpha}(X),
\end{tikzcd} \tag{9}$$

$$\begin{tikzcd}
\varphi_\alpha(X) \arrow[r, "\varphi_\alpha((\varphi_0)_X)"] \arrow[d, "(\varphi_0)_{\varphi_\alpha(X)}"'] \arrow[dr, "\mathrm{id}_{\varphi_\alpha(X)}"] & \varphi_\alpha\varphi_1(X) \arrow[d, "\varphi_2(\alpha,1)_X"] \\
\varphi_1\varphi_\alpha(X) \arrow[r, "\varphi_2(1,\alpha)_X"'] & \varphi_\alpha(X).
\end{tikzcd} \tag{10}$$

The commutativity of the diagrams (4) and (5) means that $(\varphi_\alpha, (\varphi_\alpha)_2, (\varphi_\alpha)_0)$ is a monoidal endofunctor of $\mathcal{C}$. The diagrams (6) and (7) indicate that the natural transformation $\varphi_2(\alpha, \beta)$ is monoidal. The commutativity of (8) and the equality $(\varphi_0)_{\mathbb{1}} = (\varphi_1)_0$ mean that the natural transformation φ_0 is monoidal. Finally, the diagrams (9) and (10) indicate that $(\varphi, \varphi_2, \varphi_0)$ is a monoidal functor.

3.3. G-braidings

An object X of a G-graded category $\mathcal{C}$ is *homogeneous* if $X \in \mathcal{C}_\alpha$ for some $\alpha \in G$. Such an α is then uniquely determined by X and denoted $|X|$. If two homogeneous objects $X, Y \in \mathcal{C}$ are isomorphic, then either they are zero objects or $|X| = |Y|$. Let $\mathcal{C}_{\mathrm{hom}} = \amalg_{\alpha \in G} \mathcal{C}_\alpha$ denote the full subcategory of homogeneous objects of $\mathcal{C}$.

A *G-braided category* is a G-crossed category $(\mathcal{C}, \varphi)$ endowed with a *G-braiding*, i.e., a family of isomorphisms

$$\tau = \{\tau_{X,Y} : X \otimes Y \to Y \otimes \varphi_{|Y|}(X)\}_{X \in \mathcal{C}, Y \in \mathcal{C}_{\mathrm{hom}}}$$

which is natural in X, Y and satisfies the following three conditions:

(a) for all $X \in \mathcal{C}$ and $Y, Z \in \mathcal{C}_{\mathrm{hom}}$, the following diagram commutes:

$$\begin{tikzcd}
X \otimes Y \otimes Z \arrow[r, "\tau_{X,Y\otimes Z}"] \arrow[d, "\tau_{X,Y} \otimes \mathrm{id}_Z"'] & Y \otimes Z \otimes \varphi_{|Y \otimes Z|}(X) \\
Y \otimes \varphi_{|Y|}(X) \otimes Z \arrow[r, "\mathrm{id}_Y \otimes \tau_{\varphi_{|Y|}(X), Z}"'] & Y \otimes Z \otimes \varphi_{|Z|}\varphi_{|Y|}(X), \arrow[u, "\mathrm{id}_{Y\otimes Z} \otimes \varphi_2(|Z|,|Y|)_X"']
\end{tikzcd} \tag{11}$$

(b) for all $X, Y \in \mathcal{C}$ and $Z \in \mathcal{C}_{\mathrm{hom}}$, the following diagram commutes:

$$\begin{tikzcd}
X \otimes Y \otimes Z \arrow[r, "\tau_{X\otimes Y, Z}"] \arrow[d, "\mathrm{id}_X \otimes \tau_{Y,Z}"'] & Z \otimes \varphi_{|Z|}(X \otimes Y) \\
X \otimes Z \otimes \varphi_{|Z|}(Y) \arrow[r, "\tau_{X,Z} \otimes \mathrm{id}_{\varphi_{|Z|}(Y)}"'] & Z \otimes \varphi_{|Z|}(X) \otimes \varphi_{|Z|}(Y), \arrow[u, "\mathrm{id}_Z \otimes (\varphi_{|Z|})_2(X,Y)"']
\end{tikzcd} \tag{12}$$

(c) for all $\alpha \in G$, $X \in \mathcal{C}$, and $Y \in \mathcal{C}_{\mathrm{hom}}$, the following diagram commutes:

$$\begin{tikzcd}
\varphi_\alpha(X) \otimes \varphi_\alpha(Y) \arrow[r, "(\varphi_\alpha)_2(X,Y)"] \arrow[d, "\tau_{\varphi_\alpha(X), \varphi_\alpha(Y)}"'] & \varphi_\alpha(X \otimes Y) \arrow[d, "\varphi_\alpha(\tau_{X,Y})"] \\
\varphi_\alpha(Y) \otimes \varphi_{\alpha^{-1}|Y|\alpha}\varphi_\alpha(X) \arrow[d, "\mathrm{id}_{\varphi_\alpha(Y)} \otimes \varphi_2(\alpha^{-1}|Y|\alpha, \alpha)_X"] & \varphi_\alpha(Y \otimes \varphi_{|Y|}(X)) \\
\varphi_\alpha(Y) \otimes \varphi_{|Y|\alpha}(X) \arrow[r, "\mathrm{id}_{\varphi_\alpha(Y)} \otimes \varphi_2(\alpha, |Y|)_X^{-1}"'] & \varphi_\alpha(Y) \otimes \varphi_\alpha\varphi_{|Y|}(X). \arrow[u, "(\varphi_\alpha)_2(Y, \varphi_{|Y|}(X))"']
\end{tikzcd} \tag{13}$$

For a G-braided category $(\mathcal{C}, \varphi, \tau)$, the category $\mathcal{C}_1$ is a braided category (in the usual sense of the word) with braiding

$$\{c_{X,Y} = (\mathrm{id}_Y \otimes (\varphi_0)_X^{-1})\tau_{X,Y} : X \otimes Y \to Y \otimes X\}_{X,Y \in \mathcal{C}_1}. \tag{14}$$

3.4. Pivotality and ribbonness

A G-graded category $\mathcal{C}$ is *rigid* (resp., *pivotal*) if its underlying monoidal category is rigid (resp., pivotal). If $\mathcal{C}$ is pivotal, then for all $X \in \mathcal{C}_\alpha$ with $\alpha \in G$, we always choose X^* to be in $\mathcal{C}_{\alpha^{-1}}$. If $\mathcal{C}$ is pivotal, then so is $\mathcal{C}_1$.

A crossing $\varphi : \overline{G} \to \mathrm{Aut}(\mathcal{C})$ in a pivotal G-graded category $\mathcal{C}$ is *pivotal* if the monoidal functor φ_α is pivotal for all $\alpha \in G$. A pivotal crossing φ gives for each $\alpha \in G$ a natural isomorphism

$$\varphi_\alpha^1 = \{\varphi_\alpha^1(X) : \varphi_\alpha(X^*) \xrightarrow{\sim} \varphi_\alpha(X)^*\}_{X \in \mathcal{C}}$$

which preserves both left and right dualities (as in Section 2.6) and is monoidal: $\big((\varphi_\alpha)_0^{-1}\big)^* = \varphi_\alpha^1(\mathbb{1})(\varphi_\alpha)_0 : \mathbb{1}^* = \mathbb{1} \to \varphi_\alpha(\mathbb{1})^*$ and for all $X, Y \in \mathcal{C}$,

$$\varphi_\alpha^1(X \otimes Y)(\varphi_\alpha)_2(Y^*, X^*) = \big((\varphi_\alpha)_2(X, Y)\big)^*(\varphi_\alpha^1(Y) \otimes \varphi_\alpha^1(X)).$$

A pivotal crossing φ preserves the trace in the following sense: for any $\alpha \in G$ and for any endomorphism g of an object of $\mathcal{C}$,

$$\mathrm{tr}(\varphi_\alpha(g)) = (\varphi_\alpha)_0^{-1}\varphi_\alpha(\mathrm{tr}(g))(\varphi_\alpha)_0.$$

If $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \Bbbk\, \mathrm{id}_{\mathbb{1}}$, then this formula implies that $\mathrm{tr}(\varphi_\alpha(g)) = \mathrm{tr}(g)$ for any g.

A pivotal G-braided category $(\mathcal{C}, \varphi, \tau)$ has a *twist* which is the family of morphisms $\theta = \{\theta_X\}_{X \in \mathcal{C}_{\mathrm{hom}}}$ where

$$\theta_X = (\mathrm{ev}_X \otimes \mathrm{id}_{\varphi_{|X|}(X)})(\mathrm{id}_{X^*} \otimes \tau_{X,X})(\widetilde{\mathrm{coev}}_X \otimes \mathrm{id}_X) : X \to \varphi_{|X|}(X). \tag{15}$$

The naturality of τ implies that θ_X is natural in X.

A *G-ribbon category* is a pivotal G-braided category $(\mathcal{C}, \varphi, \tau)$ whose crossing φ is pivotal and whose twist is *self-dual* in the sense that for all $X \in \mathcal{C}_{\mathrm{hom}}$,

$$(\theta_X)^* = (\varphi_0)_X^*(\varphi_2(|X|^{-1}, |X|)_X^{-1})^*\varphi^1_{|X|^{-1}}(\varphi_{|X|}(X))\theta_{\varphi_{|X|}(X)^*}. \tag{16}$$

For a G-ribbon category $(\mathcal{C}, \varphi, \tau)$, the category $\mathcal{C}_1$ is a ribbon category (in the usual sense of the word) with braiding (14) and twist $\{(\varphi_0)_X^{-1}\theta_X : X \to X\}_{X \in \mathcal{C}_1}$.

3.5. Example

The following example of a G-ribbon category is adapted from [6]. Let $\pi : H \to G$ be a group epimorphism with kernel K. Let $\mathcal{D}$ be the category of H-graded finitely generated projective $\Bbbk$-modules $M = \oplus_{h \in H} M_h$ endowed with a right action of K such that $M_h \cdot k \subset M_{k^{-1}hk}$ for all $h \in H$ and $k \in K$. Since M is finitely generated, $M_h = 0$ for all but a finite number of $h \in H$. Morphisms in $\mathcal{D}$ are H-graded K-linear morphisms. The category $\mathcal{D}$ is monoidal: the monoidal product of $M, N \in \mathcal{D}$ is the $\Bbbk$-module $M \otimes N = M \otimes_\Bbbk N$ with diagonal K-action and H-grading $(M \otimes N)_h = \oplus_{h_1 h_2 = h} M_{h_1} \otimes_\Bbbk N_{h_2}$ for $h \in H$. The monoidal unit of $\mathcal{D}$ is $\Bbbk$ in degree $1 \in H$ with trivial action of K. The category $\mathcal{D}$ is $\Bbbk$-additive in the obvious way and pivotal: the dual of $M \in \mathcal{D}$ is the $\Bbbk$-module $M^* = \mathrm{Hom}_\Bbbk(M, \Bbbk)$ with H-grading $(M^*)_h = \mathrm{Hom}(M_{h^{-1}}, \Bbbk)$ for $h \in H$ and action of K defined by $(f \cdot k)(m) = f(m \cdot k^{-1})$ for $k \in K, f \in M^*, m \in M$. The left and right (co)evaluation morphisms are the usual ones (i.e., are inherited from the pivotal category of finitely generated projective $\Bbbk$-modules). The category $\mathcal{D}$ is G-graded as follows: for $\alpha \in G$, $\mathcal{D}_\alpha$ is the full subcategory of all $M \in \mathcal{D}$ such that $M_h = 0$ whenever $\pi(h) \neq \alpha$.

Any set-theoretic section s of π, i.e., a map $s : G \to H$ such that $\pi s = \mathrm{id}_G$ defines a structure of a G-ribbon category on $\mathcal{D}$ as follows. For $\alpha \in G$ and $M \in \mathcal{D}$, set $\varphi_\alpha(M) = M$ as a $\Bbbk$-module with H-grading $\varphi_\alpha(M)_h = M_{s(\alpha)^{-1}hs(\alpha)}$ for $h \in H$ and right K-action $mk = m \cdot s(\alpha)ks(\alpha)^{-1}$ for $m \in \varphi_\alpha(M)$ and $k \in K$. For a morphism f in $\mathcal{D}$, set $\varphi_\alpha(f) = f$. This defines a strict monoidal endofunctor φ_α of $\mathcal{D}$. For $\alpha, \beta \in G$ and $M \in \mathcal{D}$, the formulas $m \mapsto m \cdot s(\beta)s(\alpha)s(\beta\alpha)^{-1}$ and $m \mapsto m \cdot s(1)^{-1}$ define isomorphisms, respectively,

$$\varphi_2(\alpha, \beta)_M : \varphi_\alpha\varphi_\beta(M) \to \varphi_{\beta\alpha}(M) \quad \text{and} \quad (\varphi_0)_M : M \to \varphi_1(M).$$

This defines a pivotal crossing in $\mathcal{D}$. Given $M \in \mathcal{D}$ and $N \in \mathcal{D}_\alpha$ with $\alpha \in G$, the G-braiding $\tau_{M,N} : M \otimes N \to N \otimes \varphi_\alpha(M)$ carries $m \otimes n$ to $n \otimes (m \cdot hs(\alpha)^{-1})$ for $m \in M$, $h \in \pi^{-1}(\alpha)$, and $n \in N_h$. For $M \in \mathcal{D}_\alpha$ with $\alpha \in G$, the (self-dual) twist $\theta_M : M \to \varphi_\alpha(M)$ carries $m \in M_h$ with $h \in \pi^{-1}(\alpha)$ to $m \cdot hs(\alpha)^{-1}$. In this way, $\mathcal{D}$ becomes a G-ribbon category. Though the structure of a G-ribbon category on $\mathcal{D}$ depends on the choice of a section $s : G \to H$, an appropriately defined equivalence class of this structure is independent of s; cf. Section 5.6.

3.6. Remarks

1. For $G = 1$, the definition of a G-crossed category $\mathcal{C}$ means that $\mathcal{C}$ is a $\Bbbk$-additive monoidal category such that every object $X \in \mathcal{C}$ gives rise to an object $X' \in \mathcal{C}$ and an isomorphism $X \approx X'$. Indeed, a strong monoidal functor $\varphi : \overline{1} \to \mathrm{Aut}(\mathcal{C})$ yields an object $X' = \varphi_1(X)$ and an isomorphism $(\varphi_0)_X : X \to X'$ for each $X \in \mathcal{C}$. It is easy to see that any system of objects and isomorphisms $\{(X' \in \mathcal{C}, X \approx X')\}_{X \in \mathcal{C}}$ arises in this way from a unique strong monoidal functor $\overline{1} \to \mathrm{Aut}(\mathcal{C})$. For $G = 1$, the notions of G-braided/G-ribbon categories are equivalent to the standard notions of braided/ribbon categories.

2. The notions of G-braided/G-ribbon categories were first introduced in [11] in a special case. Denoting by G^{op} the group G with opposite multiplication, a G-braided (resp. G-ribbon) category in [11] is a G^{op}-braided (resp. G^{op}-ribbon)

category in the sense above whose crossing φ is *strict*, meaning that φ is strict monoidal (i.e., $\varphi_2(\alpha, \beta)$ and φ_0 are identity morphisms) and each φ_α is strict monoidal (i.e., $(\varphi_\alpha)_2$ and $(\varphi_\alpha)_0$ are identity morphisms). For instance, the crossing in Example 3.5 is strict if and only if s is a group homomorphism. Such an s does not exist unless H is a semidirect product of K and G.

4. Centers of G-graded categories

We define and study G-centers of G-graded categories. We begin by recalling several notions of the theory of categories.

4.1. Preliminaries

An *idempotent* in a category $\mathcal{C}$ is an endomorphism e of an object $X \in \mathcal{C}$ such that $e^2 = e$. An idempotent $e\colon X \to X$ in $\mathcal{C}$ *splits* if there is an object $E \in \mathcal{C}$ and morphisms $p\colon X \to E$ and $q\colon E \to X$ such that $qp = e$ and $pq = \mathrm{id}_E$. Such a *splitting triple* (E, p, q) of e is unique up to isomorphism: if (E', p', q') is another splitting triple of e, then $\phi = p'q\colon E \to E'$ is the (unique) isomorphism between E and E' such that $p' = \phi p$ and $q' = q\phi^{-1}$. A *category with split idempotents* is a category in which all idempotents split.

A monoidal category $\mathcal{C}$ is *pure* if $f \otimes \mathrm{id}_X = \mathrm{id}_X \otimes f$ for all $f \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$ and $X \in \mathcal{C}$. For example, this condition is satisfied if $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \Bbbk\, \mathrm{id}_{\mathbb{1}}$.

If a monoidal category $\mathcal{C}$ is pure and pivotal, then the left and right traces in $\mathcal{C}$ are $\otimes$-multiplicative: $\mathrm{tr}_{l/r}(f \otimes g) = \mathrm{tr}_{l/r}(f)\,\mathrm{tr}_{l/r}(g)$ for any endomorphisms f, g of objects of $\mathcal{C}$. In particular, $\dim_{l/r}(X \otimes Y) = \dim_{l/r}(X) \dim_{l/r}(Y)$ for any $X, Y \in \mathcal{C}$.

We call a pivotal G-graded category $\mathcal{C}$ *non-singular* if it is pure, has split idempotents, and for all $\alpha \in G$, the subcategory $\mathcal{C}_\alpha$ of $\mathcal{C}$ has at least one object whose left dimension is invertible in $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$. Examples of such categories will be given in Section 5.6.

4.2. Relative centers

Recall the notion of a relative center of a monoidal category due to P. Schauenburg [9]. Let $\mathcal{C}$ be a monoidal category and $\mathcal{D}$ be a monoidal subcategory of $\mathcal{C}$. A (left) *half braiding of $\mathcal{C}$ relative to $\mathcal{D}$* is a pair (A, σ), where $A \in \mathcal{C}$ and $\sigma = \{\sigma_X\colon A \otimes X \to X \otimes A\}_{X \in \mathcal{D}}$ is a family of isomorphisms in $\mathcal{C}$ which is natural with respect to X and satisfies for all $X, Y \in \mathcal{D}$,

$$\sigma_{X \otimes Y} = (\mathrm{id}_X \otimes \sigma_Y)(\sigma_X \otimes \mathrm{id}_Y). \tag{17}$$

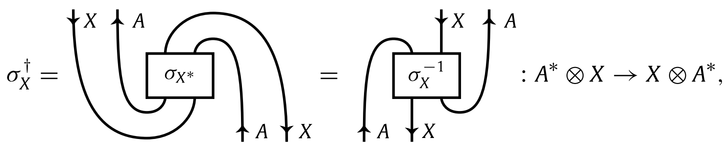
This implies that $\sigma_{\mathbb{1}} = \mathrm{id}_A$.

The (left) *center of $\mathcal{C}$ relative to $\mathcal{D}$* is the monoidal category $\mathcal{Z}(\mathcal{C}; \mathcal{D})$ whose objects are half braidings of $\mathcal{C}$ relative to $\mathcal{D}$. A morphism $(A, \sigma) \to (A', \sigma')$ in $\mathcal{Z}(\mathcal{C}; \mathcal{D})$ is a morphism $f\colon A \to A'$ in $\mathcal{C}$ such that $(\mathrm{id}_X \otimes f)\sigma_X = \sigma'_X(f \otimes \mathrm{id}_X)$ for all $X \in \mathcal{D}$. The monoidal product in $\mathcal{Z}(\mathcal{C}; \mathcal{D})$ is defined by

$$(A, \sigma) \otimes (B, \rho) = \big(A \otimes B, (\sigma \otimes \mathrm{id}_B)(\mathrm{id}_A \otimes \rho)\big),$$

and the unit object of $\mathcal{Z}(\mathcal{C}; \mathcal{D})$ is $\mathbb{1}_{\mathcal{Z}(\mathcal{C};\mathcal{D})} = (\mathbb{1}, \{\mathrm{id}_X\}_{X \in \mathcal{D}})$. The *forgetful functor* $\mathcal{Z}(\mathcal{C}; \mathcal{D}) \to \mathcal{C}$ carries (A, σ) to $A \in \mathcal{C}$ and acts in the obvious way on the morphisms. This functor is strict monoidal and reflects isomorphisms, meaning that a morphism in $\mathcal{Z}(\mathcal{C}; \mathcal{D})$ carried to an isomorphism in $\mathcal{C}$ is itself an isomorphism.

The category $\mathcal{Z}(\mathcal{C}; \mathcal{D})$ inherits most of the standard properties of $\mathcal{C}$. If $\mathcal{C}$ is a category with split idempotents, then so is $\mathcal{Z}(\mathcal{C}; \mathcal{D})$. If $\mathcal{C}$ is pure, then $\mathcal{Z}(\mathcal{C}; \mathcal{D})$ is pure and $\mathrm{End}_{\mathcal{Z}(\mathcal{C};\mathcal{D})}(\mathbb{1}_{\mathcal{Z}(\mathcal{C};\mathcal{D})}) = \mathrm{End}_{\mathcal{C}}(\mathbb{1})$. If $\mathcal{C}$ is rigid and $\mathcal{D}$ is a rigid subcategory of $\mathcal{C}$ (that is, a monoidal subcategory stable under left and right dualities), then $\mathcal{Z}(\mathcal{C}; \mathcal{D})$ is rigid. If $\mathcal{C}$ is pivotal and $\mathcal{D}$ is a pivotal subcategory of $\mathcal{C}$ (that is, a monoidal subcategory stable under duality), then $\mathcal{Z}(\mathcal{C}; \mathcal{D})$ is pivotal with $(A, \sigma)^* = (A^*, \sigma^\dagger)$ for $(A, \sigma) \in \mathcal{Z}(\mathcal{C}; \mathcal{D})$, where



(18)

and $\mathrm{ev}_{(A,\sigma)} = \mathrm{ev}_A$, $\mathrm{coev}_{(A,\sigma)} = \mathrm{coev}_A$, $\widetilde{\mathrm{ev}}_{(A,\sigma)} = \widetilde{\mathrm{ev}}_A$, $\widetilde{\mathrm{coev}}_{(A,\sigma)} = \widetilde{\mathrm{coev}}_A$. The traces of morphisms and dimensions of objects in $\mathcal{Z}(\mathcal{C}; \mathcal{D})$ are the same as in $\mathcal{C}$.

If $\mathcal{C}$ is $\Bbbk$-additive, then so is $\mathcal{Z}(\mathcal{C};\,\mathcal{D})$, and the forgetful functor $\mathcal{Z}(\mathcal{C};\,\mathcal{D}) \to \mathcal{C}$ is $\Bbbk$-linear. If $\mathcal{C}$ is an abelian category, then so is $\mathcal{Z}(\mathcal{C};\,\mathcal{D})$.

The center $\mathcal{Z}(\mathcal{C};\,\mathcal{C})$ of $\mathcal{C}$ relative to itself is the center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ in the sense of A. Joyal, R. Street, and V. Drinfeld. The center $\mathcal{Z}(\mathcal{C};\,*)$ of $\mathcal{C}$ relative to its trivial subcategory $*$ formed by the single object $\mathbb{1}$ and the single morphism $\mathrm{id}_{\mathbb{1}}$ is canonically isomorphic to $\mathcal{C}$. When $\mathcal{C}$ is pure, $\mathcal{Z}(\mathcal{C};\,\{\mathbb{1}\})$ is also canonically isomorphic to $\mathcal{C}$, where $\{\mathbb{1}\}$ is the full subcategory of $\mathcal{C}$ having a single object $\mathbb{1}$. The canonical isomorphism $\mathcal{C} \to \mathcal{Z}(\mathcal{C};\,\{\mathbb{1}\})$ carries any object A of $\mathcal{C}$ to the half-braiding $(A, \mathrm{id}_A \colon A \otimes \mathbb{1} \to \mathbb{1} \otimes A)$; the naturality of id_A with respect to endomorphisms of $\mathbb{1}$ is verified using the purity of $\mathcal{C}$.

4.3. The G-center

For a G-graded category $\mathcal{C}$ (over $\Bbbk$), we set $\mathcal{Z}_G(\mathcal{C}) = \mathcal{Z}(\mathcal{C};\,\mathcal{C}_1)$ and call $\mathcal{Z}_G(\mathcal{C})$ the *G-center of* $\mathcal{C}$. By Section 4.2, the category $\mathcal{Z}_G(\mathcal{C})$ is $\Bbbk$-additive. For $\alpha \in G$, let $\mathcal{Z}_\alpha(\mathcal{C})$ be the full subcategory of $\mathcal{Z}_G(\mathcal{C})$ formed by the half braidings (A, σ) relative to $\mathcal{C}_1$ with $A \in \mathcal{C}_\alpha$. This system of subcategories turns $\mathcal{Z}_G(\mathcal{C})$ into a G-graded category (over $\Bbbk$). By definition, $|(A, \sigma)| = |A| = \alpha$ for any $(A, \sigma) \in \mathcal{Z}_\alpha(\mathcal{C})$. If $\mathcal{C}$ is pivotal (or pure, or with split idempotents), then so is $\mathcal{Z}_G(\mathcal{C})$. The main result of this section is the following theorem.

Theorem 4.1. *Let $\mathcal{C}$ be a non-singular pivotal G-graded category. Then $\mathcal{Z}_G(\mathcal{C})$ has a canonical structure of a pivotal G-braided category with pivotal crossing.*

When G is finite, $\Bbbk$ is a field of characteristic zero, and $\mathcal{C}$ is a G-fusion category (see Section 5.3 for the definition), Theorem 4.1 was first obtained by Gelaki, Naidu, and Nikshych [4]. In difference to [4], we give an explicit construction of the crossing and the G-braiding in $\mathcal{Z}_G(\mathcal{C})$.

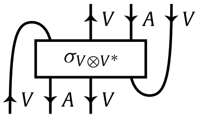
Theorem 4.1 is proved in Sections 4.4 and 4.5. Several lemmas are stated in these sections without proof which is postponed to Section 4.6. Throughout the proof of Theorem 4.1 we keep the assumptions of this theorem and use the following notation. For $V \in \mathcal{C}$, we set $d_V = \dim_l(V) \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$. For $\alpha \in G$, we denote by $\mathcal{E}_\alpha$ the class of objects of $\mathcal{C}_\alpha$ with invertible left dimension.

4.4. The crossing

The crossing in $\mathcal{Z}_G(\mathcal{C})$ is constructed in three steps. In Step 1, we construct a family of monoidal endofunctors of $\mathcal{Z}_G(\mathcal{C})$ numerated by objects of $\mathcal{C}$ belonging to $\amalg_{\alpha \in G}\, \mathcal{E}_\alpha$. In Step 2, we construct a system of isomorphisms between these endofunctors. In Step 3, we define the crossing as the limit of the resulting projective system of endofunctors and isomorphisms.

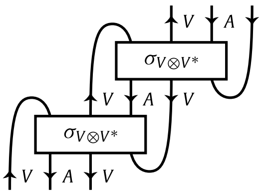
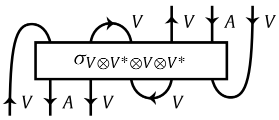
Step 1. For any $V \in \mathcal{E}_\alpha$ with $\alpha \in G$, we define a monoidal endofunctor φ_V of $\mathcal{Z}_G(\mathcal{C})$. We begin with a lemma.

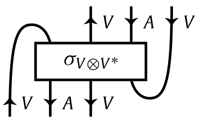
Lemma 4.2. *For any $(A, \sigma) \in \mathcal{Z}_G(\mathcal{C})$, the morphism*

$\pi^V_{(A,\sigma)} = d_V^{-1}$  $\in \mathrm{End}_{\mathcal{C}}(V^* \otimes A \otimes V)$

is an idempotent.

Proof. Using (17) and the naturality of σ, we obtain

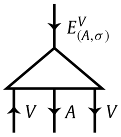
$\left(\pi^V_{(A,\sigma)}\right)^2 = d_V^{-2}$  $= d_V^{-2}$ 

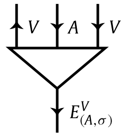
$= d_V^{-2}$   $= \pi^V_{(A,\sigma)}.$ □

Since all idempotents in $\mathcal{C}$ split, there exist an object $E^V_{(A,\sigma)} \in \mathcal{C}$ and morphisms $p^V_{(A,\sigma)} : V^* \otimes A \otimes V \to E^V_{(A,\sigma)}$ and $q^V_{(A,\sigma)} : E^V_{(A,\sigma)} \to V^* \otimes A \otimes V$ such that

$$\pi^V_{(A,\sigma)} = q^V_{(A,\sigma)} p^V_{(A,\sigma)} \quad \text{and} \quad p^V_{(A,\sigma)} q^V_{(A,\sigma)} = \mathrm{id}_{E^V_{(A,\sigma)}}. \tag{19}$$

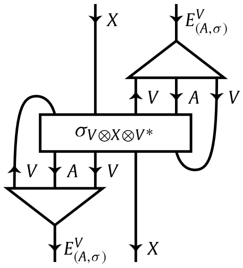
We will depict the morphisms $p^V_{(A,\sigma)}$ and $q^V_{(A,\sigma)}$ as

$p^V_{(A,\sigma)} =$ 

and $q^V_{(A,\sigma)} =$ 

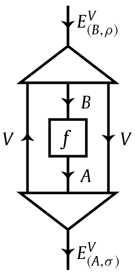
.

We can now define an endofunctor φ_V of $\mathcal{Z}_G(\mathcal{C})$ as follows. For (A, σ), set $\varphi_V(A, \sigma) = (E^V_{(A,\sigma)}, \gamma^V_{(A,\sigma)}) \in \mathcal{Z}_G(\mathcal{C})$ where, for each $X \in \mathcal{C}_1$,

$\gamma^V_{(A,\sigma),X} = d_V^{-1}$ 

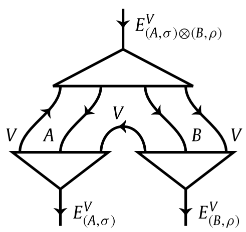
$: E^V_{(A,\sigma)} \otimes X \to X \otimes E^V_{(A,\sigma)}$.

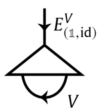
We show in Section 4.6 that $\gamma^V_{(A,\sigma)}$ is a half-braiding of $\mathcal{C}$ relative to $\mathcal{C}_1$ so that $\varphi_V(A, \sigma) \in \mathcal{Z}_G(\mathcal{C})$. If $A \in \mathcal{C}_\beta$ with $\beta \in G$, then we always choose $E^V_{(A,\sigma)}$ in $\mathcal{C}_{\alpha^{-1}\beta\alpha}$ so that $\varphi_V(A, \sigma) \in \mathcal{Z}_{\alpha^{-1}\beta\alpha}(\mathcal{C})$. For a morphism $f : (A, \sigma) \to (B, \rho)$ in $\mathcal{Z}_G(\mathcal{C})$, set

$\varphi_V(f) =$ 

$: \varphi_V(A, \sigma) \to \varphi_V(B, \rho)$.

This defines φ_V as a functor. To turn φ_V into a monoidal functor, set for any $(A, \sigma), (B, \rho) \in \mathcal{Z}_\beta(\mathcal{C})$,

$(\varphi_V)_2\big((A, \sigma), (B, \rho)\big) =$ 

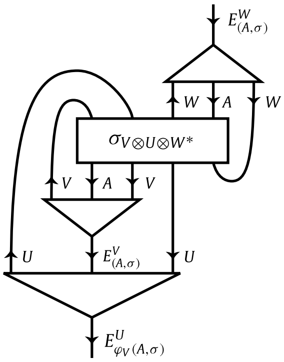
and $(\varphi_V)_0 =$ 

.

Lemma 4.3. *$(\varphi_V, (\varphi_V)_2, (\varphi_V)_0)$ is a well-defined pivotal strong monoidal $\Bbbk$-linear endofunctor of $\mathcal{Z}_G(\mathcal{C})$ such that $\varphi_V\big(\mathcal{Z}_\beta(\mathcal{C})\big) \subset \mathcal{Z}_{\alpha^{-1}\beta\alpha}(\mathcal{C})$ for all $\beta \in G$.*

We now study the endofunctors $\{\varphi_V\}_V$. Pick any $U \in \mathcal{E}_\alpha$, $V \in \mathcal{E}_\beta$, $W \in \mathcal{E}_{\beta\alpha}$ with $\alpha, \beta \in G$. For each $(A,\sigma) \in \mathcal{Z}_G(\mathcal{C})$, consider the morphism

$$\zeta^{U,V,W}_{(A,\sigma)} = d_U^{-1} d_V^{-1}$$



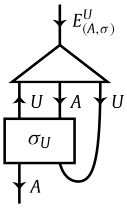
$$: \varphi_U\varphi_V(A,\sigma) \to \varphi_W(A,\sigma).$$

Lemma 4.4. (a) *The family $\zeta^{U,V,W} = \{\zeta^{U,V,W}_{(A,\sigma)}\}_{(A,\sigma)\in\mathcal{Z}_G(\mathcal{C})}$ is a monoidal natural isomorphism from $\varphi_U\varphi_V$ to φ_W.*
(b) *For all $U \in \mathcal{E}_\alpha$, $V \in \mathcal{E}_\beta$, $W \in \mathcal{E}_\gamma$ with $\alpha, \beta, \gamma \in G$ and for all $R \in \mathcal{E}_{\beta\alpha}$, $S \in \mathcal{E}_{\gamma\beta}$, and $T \in \mathcal{E}_{\gamma\beta\alpha}$, the following diagram commutes:*

$$\begin{tikzcd}
\varphi_U\varphi_V\varphi_W \arrow[r, "\varphi_U\left(\zeta^{V,W,S}\right)"] \arrow[d, "\zeta^{U,V,R}_{\varphi_W}"'] & \varphi_U\varphi_S \arrow[d, "\zeta^{U,S,T}"] \\
\varphi_R\varphi_W \arrow[r, "\zeta^{R,W,T}"'] & \varphi_T .
\end{tikzcd}$$

For $U \in \mathcal{E}_1$ and each $(A,\sigma) \in \mathcal{Z}_G(\mathcal{C})$, consider the morphism

$$\eta^U_{(A,\sigma)} =$$



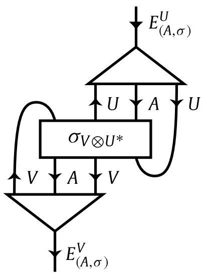
$$: (A,\sigma) \to \varphi_U(A,\sigma).$$

Lemma 4.5. (a) *The family $\eta^U = \{\eta^U_{(A,\sigma)}\}_{(A,\sigma)\in\mathcal{Z}_G(\mathcal{C})}$ is a monoidal natural isomorphism from $1_{\mathcal{Z}_G(\mathcal{C})}$ to φ_U.*
(b) *For all $U \in \mathcal{E}_1$ and $V \in \mathcal{E}_\alpha$ with $\alpha \in G$, the following diagram commutes:*

$$\begin{tikzcd}
\varphi_V \arrow[r, "\varphi_V\left(\eta^U\right)"] \arrow[d, "\eta^U_{\varphi_V}"'] \arrow[dr, "\mathrm{id}_{\varphi_V}"] & \varphi_V\varphi_U \arrow[d, "\zeta^{V,U,V}"] \\
\varphi_U\varphi_V \arrow[r, "\zeta^{U,V,V}"'] & \varphi_V .
\end{tikzcd}$$

Step 2. For each $\alpha \in G$, we construct isomorphisms between the endofunctors $\{\varphi_V \mid V \in \mathcal{E}_\alpha\}$. For $U, V \in \mathcal{E}_\alpha$, define $\delta^{U,V} = \{\delta^{U,V}_{(A,\sigma)}\}_{(A,\sigma)\in\mathcal{Z}_G(\mathcal{C})}$ by

$$\delta^{U,V}_{(A,\sigma)} = d_V^{-1}$$



$$: \varphi_V(A,\sigma) \to \varphi_U(A,\sigma).$$

Lemma 4.6. (a) *$\delta^{U,V}$ is a monoidal natural isomorphism from φ_U to φ_V.*
(b) *$\delta^{U,U} = \mathrm{id}_{\varphi_U}$ and $\delta^{U,V}\delta^{V,W} = \delta^{U,W}$ for any $U, V, W \in \mathcal{E}_\alpha$.*

(c) *For all $U, U' \in \mathcal{E}_\alpha$, $V, V' \in \mathcal{E}_\beta$, and $W, W' \in \mathcal{E}_{\beta\alpha}$, the following diagram of monoidal natural isomorphisms commutes:*

$$\begin{tikzcd} \varphi_U\varphi_V \arrow[r, "\zeta^{U,V,W}"] \arrow[d, "\varphi_{U'}\left(\delta^{V',V}\right)\delta^{U',U}_{\varphi_V}"'] & \varphi_W \arrow[d, "\delta^{W',W}"] \\ \varphi_{U'}\varphi_{V'} \arrow[r, "\zeta^{U',V',W'}"'] & \varphi_{W'}. \end{tikzcd}$$

(d) $\delta^{U',U}\eta^U = \eta^{U'}$ *for all $U, U' \in \mathcal{E}_1$.*

Note as a consequence of (a) and (b) that $(\delta^{U,V}_{(A,\sigma)})^{-1} = \delta^{V,U}_{(A,\sigma)}$.

Step 3. By Lemmas 4.3 and 4.6(a), (b), the family $(\varphi_V, \delta^{U,V})_{U,V\in\mathcal{E}_\alpha}$ is a projective system in the category of pivotal strong monoidal $\Bbbk$-linear endofunctors of $\mathcal{Z}_G(\mathcal{C})$. Since all $\delta^{U,V}$'s are isomorphisms, this system has a well-defined projective limit

$$\varphi_\alpha = \varprojlim(\varphi_V, \delta^{U,V})_{U,V\in\mathcal{E}_\alpha}$$

which is a pivotal strong monoidal $\Bbbk$-linear endofunctor of $\mathcal{Z}_G(\mathcal{C})$. By Lemma 4.3, we can assume that $\varphi_\alpha\big(\mathcal{Z}_\beta(\mathcal{C})\big) \subset \mathcal{Z}_{\alpha^{-1}\beta\alpha}(\mathcal{C})$ for all $\beta \in G$.

Denote by $\iota^\alpha = \{\iota^\alpha_V\}_{V\in\mathcal{E}_\alpha}$ the universal cone associated with the projective limit above: for $V \in \mathcal{E}_\alpha$,

$$\iota^\alpha_V = \{(\iota^\alpha_V)_{(A,\sigma)} \colon \varphi_\alpha(A,\sigma) \to \varphi_V(A,\sigma)\}_{(A,\sigma)\in\mathcal{Z}_G(\mathcal{C})} \tag{20}$$

is a monoidal natural isomorphism from φ_α to φ_V.

By Lemma 4.6(c), (d), the transformations ζ and η induce monoidal natural isomorphisms $\varphi_2(\alpha,\beta) \colon \varphi_\alpha\varphi_\beta \to \varphi_{\beta\alpha}$ and $\varphi_0 \colon 1_{\mathcal{Z}_G(\mathcal{C})} \to \varphi_1$, respectively. These isomorphisms are related to the universal cone as follows: for $U \in \mathcal{E}_\alpha$, $V \in \mathcal{E}_\beta$, $W \in \mathcal{E}_{\beta\alpha}$, and $R \in \mathcal{E}_1$, the following diagrams commute:

$$\begin{tikzcd} \varphi_\alpha\varphi_\beta \arrow[r, "(\varphi_2)(\alpha,\beta)"] \arrow[d, "\varphi_U(\iota^\beta_V)(\iota^\alpha_U)_{\varphi_\beta}"'] & \varphi_{\beta\alpha} \arrow[d, "\iota^{\beta\alpha}_W"] \\ \varphi_U\varphi_V \arrow[r, "\zeta^{U,V,W}"'] & \varphi_W \end{tikzcd} \qquad \begin{tikzcd} & 1_{\mathcal{Z}_G(\mathcal{C})} \arrow[dl, "\varphi_0"'] \arrow[dr, "\eta^R"] & \\ \varphi_1 \arrow[rr, "\iota^1_R"'] & & \varphi_R. \end{tikzcd}$$

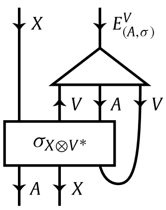
By Lemmas 4.4(b) and 4.5(b), φ_2 and φ_0 satisfy (9) and (10). Note that φ_2 and φ_0 induce natural isomorphisms $\varphi_\alpha\varphi_{\alpha^{-1}} \simeq \varphi_1 \simeq 1_{\mathcal{Z}_G(\mathcal{C})}$ and $\varphi_{\alpha^{-1}}\varphi_\alpha \simeq \varphi_1 \simeq 1_{\mathcal{Z}_G(\mathcal{C})}$ for $\alpha \in G$. Hence, the endofunctor φ_α of $\mathcal{Z}_G(\mathcal{C})$ is an equivalence. Therefore

$$\varphi = (\varphi, \varphi_2, \varphi_0) \colon \overline{G} \to \mathrm{Aut}\big(\mathcal{Z}_G(\mathcal{C})\big), \quad \alpha \mapsto \varphi_\alpha$$

is a strong monoidal functor such that $\varphi_\alpha\big(\mathcal{Z}_\beta(\mathcal{C})\big) \subset \mathcal{Z}_{\alpha^{-1}\beta\alpha}(\mathcal{C})$ for all $\alpha, \beta \in G$. Thus, φ is a crossing in $\mathcal{Z}_G(\mathcal{C})$. It is pivotal because all φ_α's are pivotal.

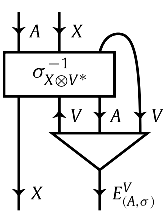
4.5. The G-braiding

We construct a G-braiding in $\mathcal{Z}_G(\mathcal{C})$ following the scheme of Section 4.4. For $(A,\sigma) \in \mathcal{Z}_G(\mathcal{C})$, $V \in \mathcal{E}_\alpha$ with $\alpha \in G$, and $X \in \mathcal{C}_{\mathrm{hom}}$, set

$\Gamma^V_{(A,\sigma),X} =$  $: A \otimes X \to X \otimes E^V_{(A,\sigma)}.$

The next lemma shows that these morphisms are isomorphisms compatible with the transformations introduced in Section 4.4.

Lemma 4.7. (a) *$\Gamma^V_{(A,\sigma),X}$ is an isomorphism natural in (A,σ) and in X, and*

$\big(\Gamma^V_{(A,\sigma),X}\big)^{-1} = d_V^{-1}$  .

(b) *For any $U, V \in \mathcal{E}_\alpha$, the following diagram commutes:*

$$\begin{tikzcd}
 & A \otimes X \arrow[dl, "\Gamma^V_{(A,\sigma),X}"'] \arrow[dr, "\Gamma^U_{(A,\sigma),X}"] & \\
X \otimes E^V_{(A,\sigma)} \arrow[rr, "\mathrm{id}_X \otimes \delta^{U,V}"'] & & X \otimes E^U_{(A,\sigma)}.
\end{tikzcd}$$

(c) *For all $(A,\sigma) \in \mathcal{Z}_G(\mathcal{C})$, $U \in \mathcal{E}_\alpha$, $V \in \mathcal{E}_\beta$, $W \in \mathcal{E}_{\alpha\beta}$, $X \in \mathcal{C}_\alpha$, and $Y \in \mathcal{C}_\beta$, the following diagram commutes:*

$$\begin{tikzcd}
A \otimes X \otimes Y \arrow[r, "\Gamma^W_{(A,\sigma),X\otimes Y}"] \arrow[d, "\Gamma^U_{(A,\sigma),X} \otimes \mathrm{id}_Y"'] & X \otimes Y \otimes E^W_{(A,\sigma)} \\
X \otimes E^U_{(A,\sigma)} \otimes Y \arrow[r, "\mathrm{id}_X \otimes \Gamma^V_{\varphi_U(A,\sigma),Y}"'] & X \otimes Y \otimes E^V_{\varphi_U(A,\sigma)}. \arrow[u, "\mathrm{id}_{X\otimes Y} \otimes \zeta^{V,U,W}_{(A,\sigma)}"']
\end{tikzcd}$$

(d) *For all $(A,\sigma), (B,\rho) \in \mathcal{Z}_G(\mathcal{C})$, $V \in \mathcal{E}_\alpha$, and $X \in \mathcal{C}_\alpha$, the following diagram commutes:*

$$\begin{tikzcd}
A \otimes B \otimes X \arrow[r, "\Gamma^V_{(A,\sigma)\otimes(B,\rho),X}"] \arrow[d, "\mathrm{id}_A \otimes \Gamma^V_{(B,\rho),X}"'] & X \otimes E^V_{(A,\sigma)\otimes(B,\rho)} \\
A \otimes X \otimes E^V_{(B,\rho)} \arrow[r, "\Gamma^V_{(A,\sigma),X} \otimes \mathrm{id}_{E^V_{(B,\rho)}}"'] & X \otimes E^V_{(A,\sigma)} \otimes E^V_{(B,\rho)}. \arrow[u, "\mathrm{id}_X \otimes (\varphi_V)_2((A,\sigma),(B,\rho))"']
\end{tikzcd}$$

(e) $\Gamma^V_{(A,\sigma),\mathbb{1}} = \eta^V_{(A,\sigma)}$ *for any $V \in \mathcal{E}_1$.*

(f) $\Gamma^V_{(\mathbb{1},\mathrm{id}),X} = \mathrm{id}_X \otimes (\varphi_V)_0$ *for all $V \in \mathcal{E}_\alpha$ and $X \in \mathcal{C}_\alpha$.*

(g) *For all $\alpha, \beta \in G$, $(A,\sigma) \in \mathcal{Z}_G(\mathcal{C})$, $(B,\rho) \in \mathcal{Z}_\beta(\mathcal{C})$, $U \in \mathcal{E}_\alpha$, $V \in \mathcal{E}_\beta$, $W \in \mathcal{E}_{\beta\alpha}$, and $S \in \mathcal{E}_{\alpha^{-1}\beta\alpha}$, the following diagram commutes:*

$$\begin{tikzcd}
E^U_{(A,\sigma)} \otimes E^U_{(B,\rho)} \arrow[r, "(\varphi_U)_2((A,\sigma),(B,\rho))"] \arrow[d, "\Gamma^S_{\varphi_U(A,\sigma),E^U_{(B,\rho)}}"] & E^U_{(A,\sigma)\otimes(B,\rho)} \arrow[d, "\varphi_U(\Gamma^V_{(A,\sigma),B})"] \\
E^U_{(B,\rho)} \otimes E^S_{\varphi_U(A,\sigma)} \arrow[d, "\mathrm{id}_{E^U_{(B,\rho)}} \otimes \zeta^{S,U,W}_{(A,\sigma)}"] & E^U_{(B,\rho)\otimes\varphi_V(A,\sigma)} \\
E^U_{(B,\rho)} \otimes E^W_{(A,\sigma)} \arrow[r, "\mathrm{id}_{E^U_{(B,\rho)}} \otimes (\zeta^{U,V,W}_{(A,\sigma)})^{-1}"'] & E^U_{(B,\rho)} \otimes E^U_{\varphi_V(A,\sigma)}. \arrow[u, "(\varphi_U)_2((B,\rho),\varphi_V(A,\sigma))"']
\end{tikzcd}$$

(h) *For all $(A,\sigma) \in \mathcal{Z}_G(\mathcal{C})$, $\alpha \in G$, $(B,\rho) \in \mathcal{Z}_\alpha(\mathcal{C})$, $V \in \mathcal{E}_\alpha$, and $X \in \mathcal{C}_1$, the following diagram commutes:*

$$\begin{tikzcd}
A \otimes B \otimes X \arrow[r, "\Gamma^V_{(A,\sigma),B} \otimes \mathrm{id}_X"] \arrow[d, "\mathrm{id}_A \otimes \rho_X"] & B \otimes E^V_{(A,\sigma)} \otimes X \arrow[d, "\mathrm{id}_B \otimes \gamma^V_{(A,\sigma),X}"] \\
A \otimes X \otimes B \arrow[d, "\sigma_X \otimes \mathrm{id}_B"] & B \otimes X \otimes E^V_{(A,\sigma)} \arrow[d, "\rho_X \otimes \mathrm{id}_{E^V_{(A,\sigma)}}"] \\
X \otimes A \otimes B \arrow[r, "\mathrm{id}_X \otimes \Gamma^V_{(A,\sigma),B}"'] & X \otimes B \otimes E^V_{(A,\sigma)}.
\end{tikzcd}$$

By Lemma 4.7(a), (b), the transformation Γ induces a family of isomorphisms

$$\{\tau_{(A,\sigma),X} : A \otimes X \to X \otimes \mathcal{U}\big(\varphi_{|X|}(A,\sigma)\big)\}_{(A,\sigma)\in\mathcal{Z}_G(\mathcal{C}),X\in\mathcal{C}_{\mathrm{hom}}}, \tag{21}$$

where $\mathcal{U}\colon \mathcal{Z}_G(\mathcal{C}) \to \mathcal{C}$ is the forgetful functor. This family is natural in (A,σ) and in X and is related to the universal cones $\{\iota^\alpha\}_{\alpha\in G}$ as follows: for any $(A,\sigma)\in\mathcal{Z}_G(\mathcal{C})$, $X\in\mathcal{C}_{\mathrm{hom}}$, and $V\in\mathcal{E}_{|X|}$,

$$\big(\mathrm{id}_X\otimes(\iota_V^{|X|})_{(A,\sigma)}\big)\tau_{(A,\sigma),X}=\Gamma^V_{(A,\sigma),X}.$$

We call the family (21) the *enhanced G-braiding* in $\mathcal{Z}_G(\mathcal{C})$.

Lemma 4.8. *For all $(A,\sigma)\in\mathcal{Z}_G(\mathcal{C})$ and $X, Y\in\mathcal{C}_{\mathrm{hom}}$,*

(a) $\tau_{(A,\sigma),X\otimes Y}=(\mathrm{id}_{X\otimes Y}\otimes\varphi_2(|Y|,|X|)_{(A,\sigma)})(\mathrm{id}_X\otimes\tau_{\varphi_{|X|}(A,\sigma),Y})(\tau_{(A,\sigma),X}\otimes\mathrm{id}_Y)$;
(b) *Given $(B,\rho)\in\mathcal{Z}_G(\mathcal{C})$,*

$$\tau_{(A,\sigma)\otimes(B,\rho),X}=\big(\mathrm{id}_X\otimes(\varphi_{|X|})_2((A,\sigma),(B,\rho))\big)\big(\tau_{(A,\sigma),X}\otimes\mathrm{id}_{\varphi_{|X|}(B,\rho)}\big)\circ\big(\mathrm{id}_A\otimes\tau_{(B,\rho),X}\big);$$

(c) $\tau_{(A,\sigma),\mathbb{1}}=(\varphi_0)_{(A,\sigma)}$;
(d) $\tau_{(\mathbb{1},\mathrm{id}),X}=\mathrm{id}_X\otimes(\varphi_{|X|})_0$;
(e) *The inverse of $\tau_{(A,\sigma),X}$ is computed by*

$$\begin{aligned}\tau^{-1}_{(A,\sigma),X}=&(\widetilde{\mathrm{ev}}_X\otimes(\varphi_0)^{-1}_{(A,\sigma)}\varphi_2(|X|^{-1},|X|)_{(A,\sigma)}\otimes\mathrm{id}_X)\circ(\mathrm{id}_X\otimes\tau_{\varphi_{|X|}(A,\sigma),X^*}\otimes\mathrm{id}_X)(\mathrm{id}_{X\otimes A}\otimes\widetilde{\mathrm{coev}}_X)\\ =&(\mathrm{id}_{A\otimes X}\otimes\mathrm{ev}_{\varphi_{|X|}(A,\sigma)}(\varphi^l_{|X|}(A,\sigma)\otimes\mathrm{id}_{\varphi_{|X|}(A,\sigma)}))\circ(\mathrm{id}_A\otimes\tau_{(A,\sigma)^*,X}\otimes\mathrm{id}_{\varphi_{|X|}(A,\sigma)})(\mathrm{coev}_A\otimes\mathrm{id}_{X\otimes\varphi_{|X|}(A,\sigma)}).\end{aligned}$$

Proof. Claims (a)–(d) follow respectively from Lemma 4.7(c)–(f). Claim (e) follows from the expression of the inverse of $\Gamma^V_{(A,\sigma),X}$ given in Lemma 4.7(a) and from the computation of σ_Y^{-1} in terms of σ_{Y^*} and $\sigma_Y^\dagger$ provided by (18) for any $Y\in\mathcal{C}_1$. □

Lemma 4.9. *The family $\tau=\{\tau_{(A,\sigma),(B,\rho)}\}_{(A,\sigma)\in\mathcal{Z}_G(\mathcal{C}),(B,\rho)\in\mathcal{Z}_G(\mathcal{C})_{\mathrm{hom}}}$ defined by*

$$\tau_{(A,\sigma),(B,\rho)}=\tau_{(A,\sigma),B}\colon (A,\sigma)\otimes(B,\rho)\to(B,\rho)\otimes\varphi_{|(B,\rho)|}(A,\sigma).$$

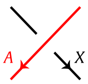
is a G-braiding in $\mathcal{Z}_G(\mathcal{C})$.

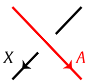
Proof. Lemma 4.7(h) implies that $\tau_{(A,\sigma),(B,\rho)}$ is a morphism in $\mathcal{Z}_G(\mathcal{C})$. It is an isomorphism since $\mathcal{U}(\tau_{(A,\sigma),(B,\rho)})=\tau_{(A,\sigma),B}$ is an isomorphism in $\mathcal{C}$ and the forgetful functor $\mathcal{U}$ reflects isomorphisms. The naturality of the enhanced braiding implies the naturality of τ. The formulas (11), (12), (13) follow respectively from Lemmas 4.8(a), 4.8(b), and 4.7(e). □

Theorem 4.1 is a direct consequence of Lemma 4.9.

4.6. *Proof of Lemmas 4.3–4.7*

Let $(A,\sigma)\in\mathcal{Z}_G(\mathcal{C})$. For $X\in\mathcal{C}_1$, we depict the morphism $\sigma_X: A\otimes X\to X\otimes A$ and its inverse $\sigma_X^{-1}: X\otimes A\to A\otimes X$ by

$\sigma_X=$ 

and $\sigma_X^{-1}=$ 

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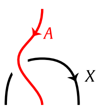
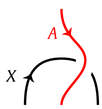
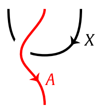
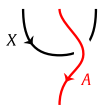
Formula (18) implies that



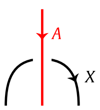
$=$ 

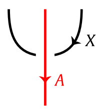
and 

$=$ 

.

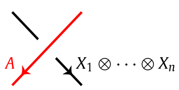
These two morphisms are pictorially represented respectively as

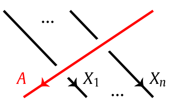


and 

.

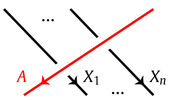
Axiom (17) implies that for any $X_1,\ldots,X_n\in\mathcal{C}_1$,



$=$ 

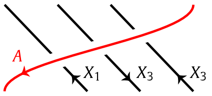
.

In generalization of this notation, if $X\in\mathcal{C}_1$ decomposes as $X=X_1\otimes\cdots\otimes X_n$ where $X_1,\ldots,X_n$ are any homogeneous objects of $\mathcal{C}$, then we will depict σ_X as

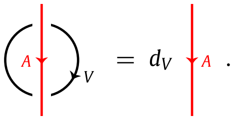
$\sigma_X=$ 

.

As usual, if an arc colored by X_i is oriented upwards, then the corresponding object in the source/target of morphisms is X_i^*. For example, if $X \in \mathcal{C}_1$ decomposes as $X = X_1^* \otimes X_2 \otimes X_3^*$ where $X_1, X_2, X_3 \in \mathcal{C}$, then we depict σ_X as

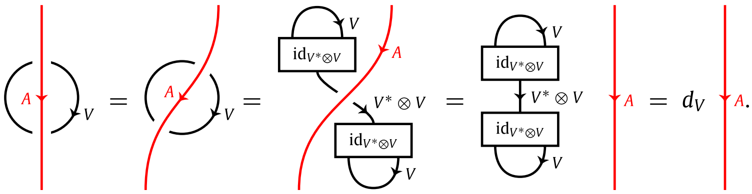
$\sigma_X =$  .

In this pictorial formalism, for $V \in \mathcal{C}$,

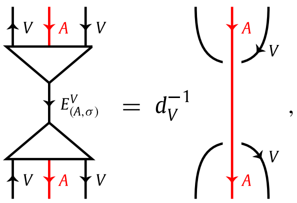


(22)

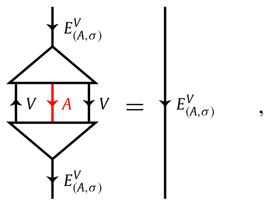
Indeed, by the naturality of σ,



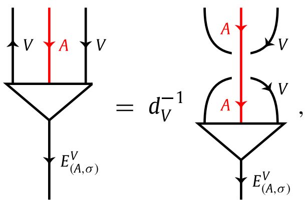
Lemma 4.10. *Let $(A, \sigma) \in \mathcal{Z}_G(\mathcal{C})$ and $V \in \mathcal{E}_\alpha$ with $\alpha \in G$. Then*



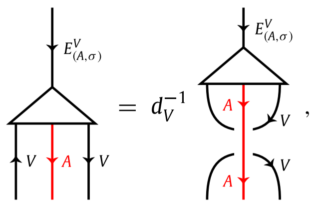
(23)



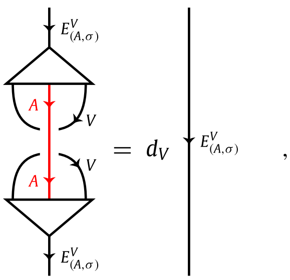
(24)



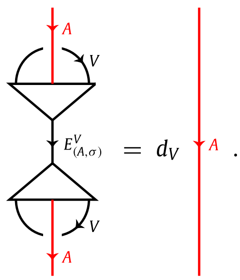
(25)



(26)



(27)

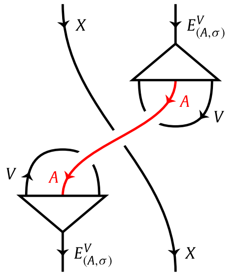


(28)

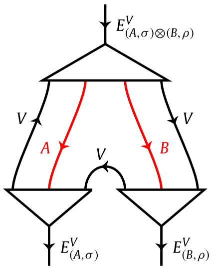
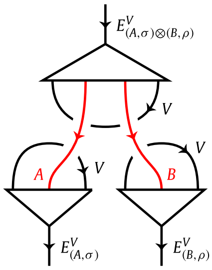
Note that the left hand side of (28) does not necessarily depict a morphism in itself because V may not belong to $\mathcal{C}_1$. The equality (28) means that in any diagram, a piece as in the left-hand side of (28) may be replaced with the piece as in right-hand side of (28) and vice versa.

Proof. Equalities (23) and (24) follow directly from (19) and the definition of $\pi^V_{(A,\sigma)}$. Composing on the right (23) with $q^V_{(A,\sigma)}$ and then using (24) gives (25). Similarly (26) is obtained by composing on the left (23) with $p^V_{(A,\sigma)}$. Composing (26) with (25) and then using (24) and (22) gives (27). Finally (28) is a direct consequence of (23) and (22). $\square$

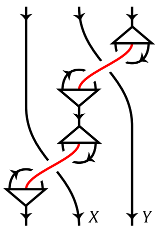
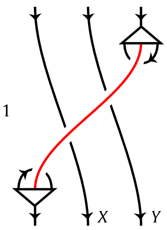
We compute now the functor φ_V in this pictorial formalism for $V \in \mathcal{E}_\alpha$ with $\alpha \in G$. For $(A, \sigma) \in \mathcal{Z}_G(\mathcal{C})$, we have $\varphi_V(A, \sigma) = (E^V_{(A,\sigma)}, \gamma^V_{(A,\sigma)})$ where, for $X \in \mathcal{C}_1$,

$\gamma^V_{(A,\sigma),X} = d_V^{-1}$  .

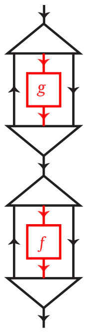
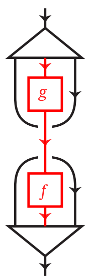
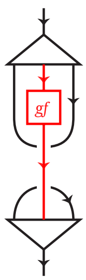
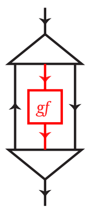
Using (25) twice, we obtain that

$(\varphi_V)_2\big((A, \sigma), (B, \rho)\big) =$  $= d_V^{-2}$  .

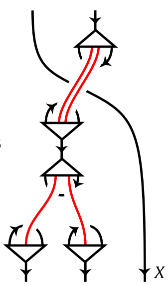
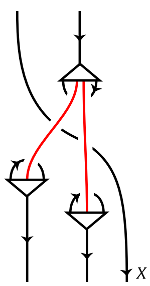
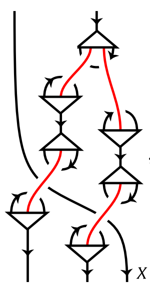
Let us prove Lemma 4.3. First, $\gamma = \gamma^V_{(A,\sigma)}$ is a half-braiding. Indeed, the naturality of σ implies that of γ. Also, using (27) we obtain $\gamma_{\mathbb{1}} = \mathrm{id}_A$ and

$(\gamma_X \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes \gamma_Y) = d_V^{-2}$  $= d_V^{-1}$  $= \gamma_{X\otimes Y}$.

The category $\mathcal{C}$ being rigid, these two equalities imply that γ is invertible. Hence γ is a half-braiding. Second, φ_V is a functor since $\varphi_V(\mathrm{id}_{(A,\sigma)}) = \mathrm{id}_{\varphi_V(A,\sigma)}$ by (24) and, for two composable morphisms g, f in $\mathcal{Z}_G(\mathcal{C})$, (23) and (25) give

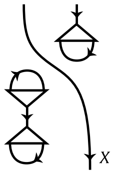
$\varphi_V(g)\varphi_V(f) =$  $= d_V^{-1}$  $= d_V^{-1}$  $=$  $= \varphi_V(gf)$.

Let us prove that φ_V is strong monoidal. Let $(A, \sigma), (B, \rho), (C, \varrho) \in \mathcal{Z}_G(\mathcal{C})$. Applying (28), we obtain

d_V^{-3}  $= d_V^{-2}$  $= d_V^{-4}$  , that is,

$$\gamma^V_{(A,\sigma)\otimes(B,\rho),X}\left((\varphi_V)_2\big((A,\sigma),(B,\rho)\big)\otimes \mathrm{id}_X\right)$$
$$=\left(\mathrm{id}_X\otimes(\varphi_V)_2\big((A,\sigma),(B,\rho)\big)\right)\left(\gamma^V_{(A,\sigma),X}\otimes \mathrm{id}_{\varphi_V(B,\rho)}\right)\left(\mathrm{id}_{\varphi_V(A,\sigma)}\otimes\gamma^V_{(B,\rho),X}\right).$$

Thus $(\varphi_V)_2\big((A,\sigma),(B,\rho)\big)$ is a morphism in $\mathcal{Z}_G(\mathcal{C})$. Similarly, $(\varphi_V)_0$ is a morphism in $\mathcal{Z}_G(\mathcal{C})$ because, by (23),

$$\gamma^V_{(\mathbb{1},\mathrm{id}),X}\big((\varphi_V)_0\otimes\mathrm{id}_X\big)=d_V^{-1}$$


$$=\mathrm{id}_X\otimes(\varphi_V)_0.$$

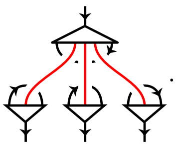
Now $(\varphi_V)_2$ satisfies (1) since, by using (23), we obtain that both

$$(\varphi_V)_2\big((A,\sigma)\otimes(B,\rho),(C,\varrho)\big)\big((\varphi_V)_2\big((A,\sigma),(B,\rho)\big)\otimes\mathrm{id}_{\varphi_V(C,\varrho)}\big)$$

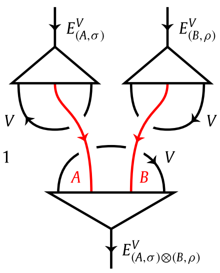
and

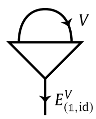
$$(\varphi_V)_2\big((A,\sigma),(B,\rho)\otimes(C,\varrho)\big)\big(\mathrm{id}_{\varphi_V(A,\sigma)}\otimes(\varphi_V)_2\big((B,\rho),(C,\varrho)\big)\big)$$

are equal to

$$d_V^{-3}$$

.

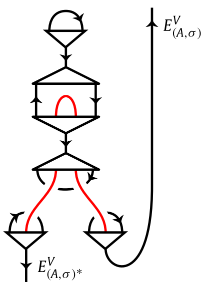
Axiom (2) is a direct consequence of (23) applied to $(\mathbb{1},\mathrm{id})$. Hence φ_V is a monoidal functor. It remains to prove that both $(\varphi_V)_2\big((A,\sigma),(B,\rho)\big)$ and $(\varphi_V)_0$ are isomorphisms in $\mathcal{Z}_G(\mathcal{C})$. Since the forgetful functor $\mathcal{Z}_G(\mathcal{C})\to\mathcal{C}$ reflects isomorphisms, we only need to verify that these morphisms are isomorphisms in $\mathcal{C}$. This can be done by verifying (using Lemma 4.10) that

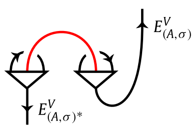
$$(\varphi_V)_2\big((A,\sigma),(B,\rho)\big)^{-1}=d_V^{-1}$$


and $$(\varphi_V)_0^{-1}=d_V^{-1}$$


.

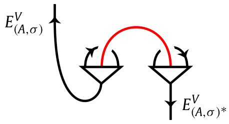
Finally, let us prove that φ_V is pivotal. For $(A,\sigma)\in\mathcal{Z}_G(\mathcal{C})$, we obtain that

$$\varphi_V^l(A,\sigma)=d_V^{-3}$$


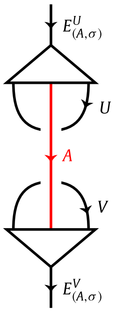
$$=d_V^{-3}$$


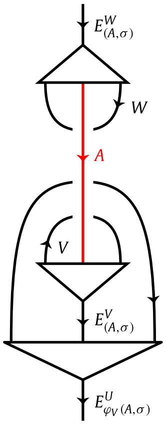
.

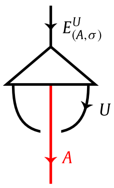
A similar computation gives that

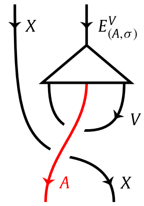
$$\varphi^r_V(A,\sigma) = d_V^{-3}$$  .

Thus $\varphi^l_V(A,\sigma) = \varphi^r_V(A,\sigma)$ by the pivotality of $\mathcal{C}$. This concludes the proof of Lemma 4.3. The proof of Lemmas 4.4–4.9 follows the same lines depicting the morphisms involved as

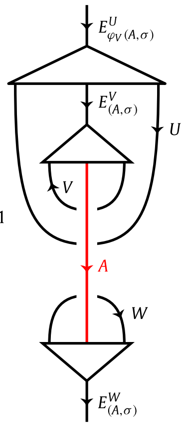
$$\delta^{U,V}_{(A,\sigma)} = d_V^{-1}$$  ,

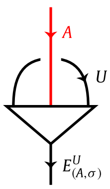
$$\zeta^{U,V,W}_{(A,\sigma)} = d_U^{-1} d_V^{-1}$$  ,

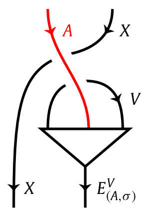
$$\eta^U_{(A,\sigma)} =$$  ,

$$\Gamma^V_{(A,\sigma),X} =$$ 

and depicting the inverse isomorphisms by

$$\big(\zeta^{U,V,W}_{(A,\sigma)}\big)^{-1} = d_W^{-1}$$  ,

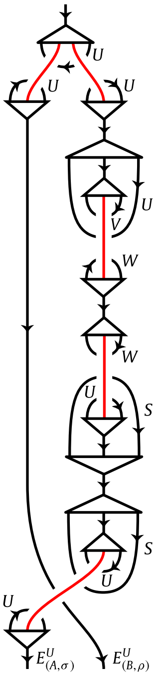
$$\big(\eta^U_{(A,\sigma)}\big)^{-1} = d_U^{-1}$$  ,

$$\big(\Gamma^V_{(A,\sigma),X}\big)^{-1} = d_V^{-1}$$  .

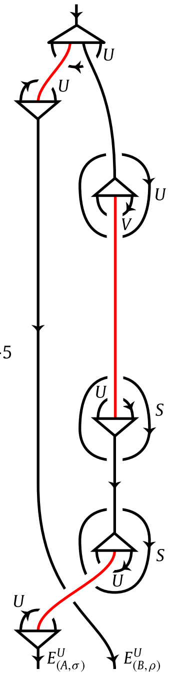
For example, let us check Lemma 4.7(e). Let $(A,\sigma)\in\mathcal{Z}_G(\mathcal{C})$, $(B,\rho)\in\mathcal{Z}_\beta(\mathcal{C})$, $U\in\mathcal{E}_\alpha$, $V\in\mathcal{E}_\beta$, $W\in\mathcal{E}_{\beta\alpha}$, and $S\in\mathcal{E}_{\alpha^{-1}\beta\alpha}$, with $\alpha,\beta\in G$. Then

$$(\varphi_U)_2\big((B,\rho),\varphi_V(A,\sigma)\big)\left(\mathrm{id}_{E^U_{(B,\rho)}}\otimes(\xi^{U,V,W}_{(A,\sigma)})^{-1}\xi^{S,U,W}_{(A,\sigma)}\right)\Gamma^S_{\varphi_U(A,\sigma),E^U_{(B,\rho)}}$$

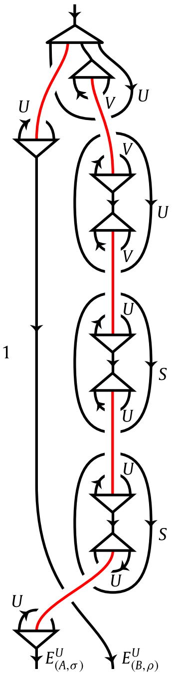
$= d_S^{-1}d_U^{-4}d_W^{-1}$



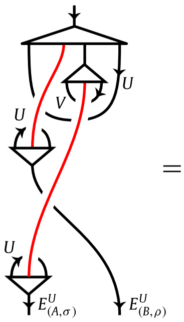
$\overset{(i)}{=} d_S^{-2}d_U^{-5}$



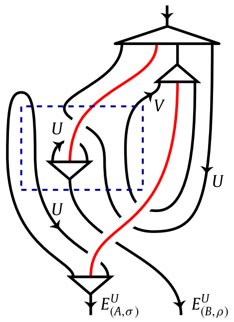
$\overset{(ii)}{=} d_S^{-2}d_U^{-5}d_V^{-1}$



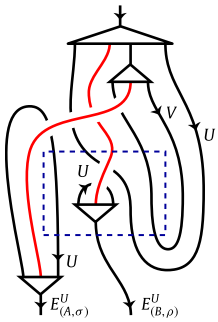
$\overset{(iii)}{=} d_U^{-2}$



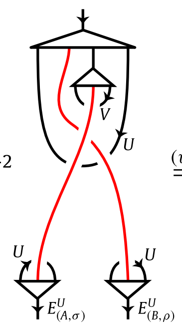
$= d_U^{-2}$



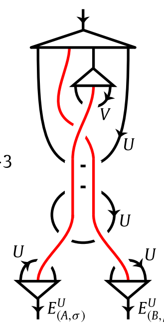
$\overset{(iv)}{=} d_U^{-2}$



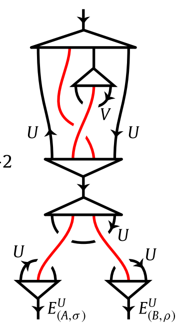
$= d_U^{-2}$



$\overset{(v)}{=} d_U^{-3}$



$\overset{(vi)}{=} d_U^{-2}$



$$= \varphi_U\big(\Gamma^V_{(A,\sigma),B}\big)(\varphi_U)_2\big((A,\sigma),(B,\rho)\big).$$

Here, the equality *(i)* is obtained by applying (23) twice and (28), *(ii)* follows from the definition of the half-braidings of $\varphi_U(A,\sigma)$ and $\varphi_V(A,\sigma)$, *(iii)* is obtained by applying (27) and then (22), *(iv)* follows from the naturality of σ applied to the morphism delimited by the dotted blue box (which is indeed a morphism in $\mathcal{C}_1$), *(v)* is obtained by applying (22), and *(vi)* is obtained by applying (23).

4.7. Remarks

1. The G-center $\mathcal{Z}_G(\mathcal{C})$ is the *left G-center* of $\mathcal{C}$ while the center studied in [4] is the *right G-center* $\mathcal{Z}^r_G(\mathcal{C})$. The objects of $\mathcal{Z}^r_G(\mathcal{C})$ are right half braidings of $\mathcal{C}$ relative to $\mathcal{C}_1$, i.e., pairs $(A \in \mathcal{C}, \sigma)$ where $\sigma = \{\sigma_X : X \otimes A \to A \otimes X\}_{X \in \mathcal{C}_1}$ is a natural isomorphism such that $\sigma_{X\otimes Y} = (\sigma_X \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes \sigma_Y)$ for all $X, Y \in \mathcal{C}_1$. The left and right G-centers are related as follows. Given a G-graded category $\mathcal{D}$, denote by $\mathcal{D}^{\otimes\mathrm{op}}$ the G^{op}-graded category obtained from $\mathcal{D}$ by replacing $\otimes$ with the opposite product $\otimes^{\mathrm{op}}$ defined by $X \otimes^{\mathrm{op}} Y = Y \otimes X$ and by setting $(\mathcal{D}^{\otimes\mathrm{op}})_\alpha = \mathcal{D}_\alpha$ for $\alpha \in G$. Then $\mathcal{Z}^r_G(\mathcal{C}) = (\mathcal{Z}_{G^{\mathrm{op}}}(\mathcal{C}^{\otimes\mathrm{op}}))^{\otimes\mathrm{op}}$.

2. Each G-graded pure pivotal category $\mathcal{C}$ with split idempotents contains a maximal non-singular graded subcategory. Namely, let H be the set of all $\alpha \in G$ such that $\mathcal{C}_\alpha$ contains an object with invertible left dimension and an object with invertible right dimension. Then H is a subgroup of G and the H-category $\oplus_{\alpha\in H} \mathcal{C}_\alpha \subset \mathcal{C}$ is non-singular.

5. The modularity theorem

We state in this section our main result concerning the modularity of the G-center. We first discuss several important conditions on categories.

5.1. Sphericity

A *spherical* category is a pivotal category $\mathcal{C}$ such that the left and right traces of endomorphisms in $\mathcal{C}$ coincide. For any endomorphism g of an object of such a $\mathcal{C}$ set $\mathrm{tr}(g) = \mathrm{tr}_l(g) = \mathrm{tr}_r(g)$ and for any object $X \in \mathcal{C}$ set $\dim(X) = \dim_l(X) = \dim_r(X) = \mathrm{tr}(\mathrm{id}_X)$. For instance, all ribbon categories are spherical; see [10].

For a spherical category, the graphical calculus of Section 2.5 has the following additional feature: the morphisms represented by diagrams in $\mathbb{R}^2$ are invariant under isotopies of the diagrams in the 2-sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$. In other words, these morphisms are preserved under isotopies of diagrams in $\mathbb{R}^2$ and under isotopies pushing arcs of diagrams across ∞. For example, the diagrams in Section 2.5 representing $\mathrm{tr}_l(g)$ and $\mathrm{tr}_r(g)$ are related by such an isotopy. The sphericity condition $\mathrm{tr}_l(g) = \mathrm{tr}_r(g)$ for all g ensures the isotopy invariance.

A G-graded category is *spherical* if it is spherical as a monoidal category.

5.2. Split semisimplicity

An object i of a $\Bbbk$-additive category $\mathcal{C}$ is *simple* if $\mathrm{End}_{\mathcal{C}}(i)$ is a free $\Bbbk$-module of rank 1. Then the map $\Bbbk \to \mathrm{End}_{\mathcal{C}}(i)$, $k \mapsto k\,\mathrm{id}_i$ is a $\Bbbk$-algebra isomorphism which we use to identify $\mathrm{End}_{\mathcal{C}}(i) = \Bbbk$. All objects isomorphic to a simple object are simple. If $\mathcal{C}$ is rigid, then the left/right duals of a simple object of $\mathcal{C}$ are simple.

A $\Bbbk$-additive category $\mathcal{C}$ is *split semisimple* if each object of $\mathcal{C}$ is a finite direct sum of simple objects of $\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{C}}(i, j) = 0$ for any non-isomorphic simple objects i, j of $\mathcal{C}$. A set I of simple objects of a split semisimple category $\mathcal{C}$ is *representative* if every simple object of $\mathcal{C}$ is isomorphic to a unique element of I. Then any $X \in \mathcal{C}$ splits as a (finite) direct sum of objects of I. In other words, there exists a finite family of morphisms $(p_\alpha : X \to i_\alpha \in I, q_\alpha : i_\alpha \to X)_{\alpha\in\Lambda}$ in $\mathcal{C}$ such that

$$\mathrm{id}_X = \sum_{\alpha\in\Lambda} q_\alpha p_\alpha \quad \text{and} \quad p_\alpha q_\beta = \delta_{\alpha,\beta}\, \mathrm{id}_{i_\alpha} \quad \text{for all} \quad \alpha, \beta \in \Lambda. \tag{29}$$

Such a family $(p_\alpha, q_\alpha)_{\alpha\in\Lambda}$ is called an *I-partition* of X.

A split semisimple category $\mathcal{C}$ is *finite* if the set of isomorphism classes of simple objects of $\mathcal{C}$ is finite. If a finite split semisimple category $\mathcal{C}$ is pivotal, then the *dimension* of $\mathcal{C}$ is defined by

$$\dim(\mathcal{C}) = \sum_{i\in I} \dim_l(i) \dim_r(i) \in \mathrm{End}_{\mathcal{C}}(\mathbb{1}),$$

where I is a representative set of simple objects of $\mathcal{C}$. The sum here is well defined because I is finite and does not depend on the choice of I.

5.3. *G-fusion*

A *G-pre-fusion category* is a G-graded category $\mathcal{C}$ (over $\Bbbk$) such that the unit object $\mathbb{1}$ is simple and

(a) $\mathcal{C}$ is pivotal and split semisimple as a $\Bbbk$-additive category;
(b) for all $\alpha \in G$, the category $\mathcal{C}_\alpha$ has at least one simple object.

A set I of simple objects of a G-pre-fusion category $\mathcal{C}$ is *G-representative* if all elements of I are homogeneous and every simple object of $\mathcal{C}$ is isomorphic to a unique element of I. Any such set I splits as a disjoint union $I = \amalg_{\alpha \in G} I_\alpha$ where I_α is the (non-empty) set of all elements of I belonging to $\mathcal{C}_\alpha$. The existence of a G-representative set I follows from the fact that any simple object of $\mathcal{C}$ is isomorphic to a simple object of $\mathcal{C}_\alpha$ for a unique $\alpha \in G$. Note also that $\mathcal{C}_\alpha$ is split semisimple for all $\alpha \in G$.

Any G-pre-fusion category $\mathcal{C}$ is pure and both the left and right dimensions of simple objects of $\mathcal{C}$ are invertible (see Lemma 4.1 of [12]). If $\Bbbk$ is a field (or, more generally, a local ring), then $\mathcal{C}$ has split idempotents. Therefore any G-pre-fusion category $\mathcal{C}$ over a field is non-singular. Such a $\mathcal{C}$ satisfies the hypothesis of Theorem 4.1, and so, $\mathcal{Z}_G(\mathcal{C})$ is a pivotal G-braided category with pivotal crossing.

A *G-fusion category* is a G-pre-fusion category $\mathcal{C}$ (over $\Bbbk$) such that the set of isomorphism classes of simple objects of $\mathcal{C}_\alpha$ is finite for all $\alpha \in G$.

5.4. *Modularity*

A *G-modular category* is a G-ribbon G-fusion category $\mathcal{D}$ whose neutral component $\mathcal{D}_1$ is modular in the sense of [10], that is, the *S-matrix* $(\mathrm{tr}\,(c_{j,i}c_{i,j}))_{i,j}$ is invertible over $\Bbbk$. Here i, j run over a representative set of simple objects of $\mathcal{D}_1$ and $c_{i,j} : i \otimes j \to i \otimes j$ is the braiding (14) in $\mathcal{D}_1$.

Theorem 5.1. *Let $\mathcal{C}$ be a spherical G-fusion category over an algebraically closed field such that* $\dim(\mathcal{C}_1) \neq 0$. *Then $\mathcal{Z}_G(\mathcal{C})$ is a G-modular category.*

The proof of Theorem 5.1 given below is based on the following two key lemmas.

Lemma 5.2. *Let $\mathcal{C}$ be a spherical G-pre-fusion category with split idempotents. Then $\mathcal{Z}_G(\mathcal{C})$ is a G-ribbon category.*

Lemma 5.3. *Let $\mathcal{C}$ be a G-fusion category over an algebraically closed field such that* $\dim(\mathcal{C}_1) \neq 0$. *Then $\mathcal{Z}_G(\mathcal{C})$ is a G-fusion category.*

The proof of Lemma 5.3 uses the following claim of independent interest.

Lemma 5.4. *Let $\mathcal{C}$ be a split semisimple pivotal category over an algebraically closed field such that the unit object $\mathbb{1}_\mathcal{C}$ is simple. Let $\mathcal{D}$ be a finite split semisimple pivotal subcategory of $\mathcal{C}$ (not necessarily full) such that* $\dim(\mathcal{D}) \neq 0$. *Then the relative center $\mathcal{Z}(\mathcal{C}; \mathcal{D})$ is split semisimple.*

Lemma 5.2 is proved in Section 6 and Lemmas 5.3 and 5.4 are proved in Section 9. The arguments in Section 9 use the results of Sections 7 and 8 concerned with monads and coends, respectively.

5.5. *Proof of Theorem 5.1*

By Lemmas 5.2 and 5.3, $\mathcal{Z}_G(\mathcal{C})$ is a G-ribbon G-fusion category. The neutral component $\mathcal{Z}_1(\mathcal{C})$ of $\mathcal{Z}_G(\mathcal{C})$ is isomorphic to the center $\mathcal{Z}(\mathcal{C}_1) = \mathcal{Z}(\mathcal{C}_1; \mathcal{C}_1)$ of $\mathcal{C}_1$. Since $\mathcal{C}$ is spherical, so is $\mathcal{C}_1$. By [8, Theorem 1.2], $\mathcal{Z}(\mathcal{C}_1)$ is modular. Therefore $\mathcal{Z}_G(\mathcal{C})$ is G-modular.

5.6. *Example*

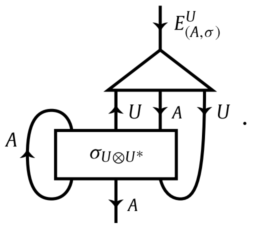
The G-ribbon category $\mathcal{D} = \mathcal{D}(\pi)$ derived from a group epimorphism $\pi : H \to G$ in Section 3.5 can be realized as the G-center of a non-singular pivotal G-graded category. To see this, observe that any pivotal H-graded category $\mathcal{C}$ gives rise to a pivotal G-graded category $\pi_*(\mathcal{C})$ which is equal to $\mathcal{C}$ as a pivotal category and has the grading $\pi_*(\mathcal{C}) = \oplus_{\alpha \in G} \pi_*(\mathcal{C})_\alpha$

where $\pi_*(\mathcal{C})_\alpha = \oplus_{h\in\pi^{-1}(\alpha)}\mathcal{C}_h$. We apply this observation to the H-ribbon category $\mathcal{C} = \mathcal{D}(\mathrm{id}_H : H \to H)$ of H-graded finitely generated projective $\Bbbk$-modules. It is easy to see that $\mathcal{C}$ is spherical and has split idempotents. For $h \in H$, let $\Bbbk[h] \in \mathcal{C}$ be the $\Bbbk$-module which is $\Bbbk$ in degree h and zero in all other degrees. The category $\mathcal{C}$ is non-singular since its monoidal unit $\Bbbk[1]$ is simple and for $h \in H$, the module $\Bbbk[h]$ has categorical dimension 1. This implies that the category $\mathcal{C}^\pi = \pi_*(\mathcal{C})$ is non-singular. By Theorem 4.1, the G-center of $\mathcal{C}^\pi$ is a G-braided category. We claim that the G-braided categories $\mathcal{D}$ and $\mathcal{Z}_G(\mathcal{C}^\pi)$ are equivalent. To see this, let $s\colon G \to H$ be the map used in the definition of $\mathcal{D}$ in Section 3.5. Using the objects $(\Bbbk[s(\alpha)])_{\alpha\in G}$ of $\mathcal{C}$, one can explicitly describe $\mathcal{Z}_G(\mathcal{C}^\pi)$; cf. Sections 4.4–6.1. In particular, a relative half braiding $(M, \sigma) \in \mathcal{Z}_\alpha(\mathcal{C}^\pi)$ defines a right action of $K = \mathrm{Ker}\,\pi$ on M by $m \cdot k = \sigma_{\Bbbk[s(\alpha)]}(m \otimes 1_\Bbbk)$ for $m \in M$ and $k \in K$. This determines the required equivalence $\mathcal{Z}_G(\mathcal{C}^\pi) \approx \mathcal{D}$. Furthermore, if K is finite and $\Bbbk$ is an algebraically closed field whose characteristic does not divide the order $\#K$ of K, then it is easy to see that $\mathcal{C}^\pi$ is a spherical G-fusion category and $\dim(\mathcal{C}^\pi_1) = \#K$. Theorem 5.1 implies that $\mathcal{Z}_G(\mathcal{C}^\pi)$ is a G-modular category. We deduce that in this case the category $\mathcal{D}$ is G-modular.

6. G-ribbonness re-examined

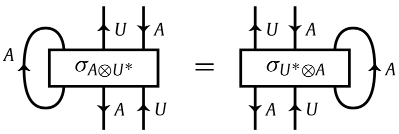
6.1. A ribbonness criterion

Consider a non-singular G-graded pivotal category $\mathcal{C}$. Let, for $\alpha \in G$, the symbol $\mathcal{E}_\alpha$ denote the class of all $V \in \mathcal{C}_\alpha$ with invertible left dimension $d_V = \dim_l(V) \in \Bbbk$. By Theorem 4.1, the G-center $\mathcal{Z}_G(\mathcal{C})$ is a pivotal G-braided category with pivotal crossing. The corresponding twist θ in $\mathcal{Z}_G(\mathcal{C})$ is computed as follows: if $(A, \sigma) \in \mathcal{Z}_\alpha(\mathcal{C})$ with $\alpha \in G$, then for any $U \in \mathcal{E}_\alpha$, we have $\theta_{(A,\sigma)} = (\iota^\alpha_U)^{-1}_{(A,\sigma)}\vartheta^U_{(A,\sigma)}$ where ι^α is the universal cone (20) and

$\vartheta^U_{(A,\sigma)} =$  .

By definition, $\mathcal{Z}_G(\mathcal{C})$ is G-ribbon if θ is self-dual; see Section 3.4. The following lemma gives a necessary and sufficient condition for $\mathcal{Z}_G(\mathcal{C})$ to be G-ribbon.

Lemma 6.1. *$\mathcal{Z}_G(\mathcal{C})$ is G-ribbon if and only if*



for all $\alpha \in G$, $(A, \sigma) \in \mathcal{Z}_\alpha(\mathcal{C})$, and $U \in \mathcal{E}_\alpha$.

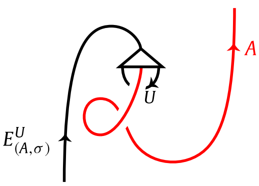
Proof. Recall that θ is self-dual if for all $\alpha \in G$ and $(A, \sigma) \in \mathcal{Z}_\alpha(\mathcal{C})$,

$$(\theta_{(A,\sigma)})^* = \big((\varphi_0)_{(A,\sigma)}\big)^*\big(\varphi_2(\alpha^{-1}, \alpha)^{-1}_{(A,\sigma)}\big)^*\varphi^1_{\alpha^{-1}}\big(\varphi_\alpha(A, \sigma)\big)\theta_{(\varphi_\alpha(A,\sigma))^*}. \tag{30}$$

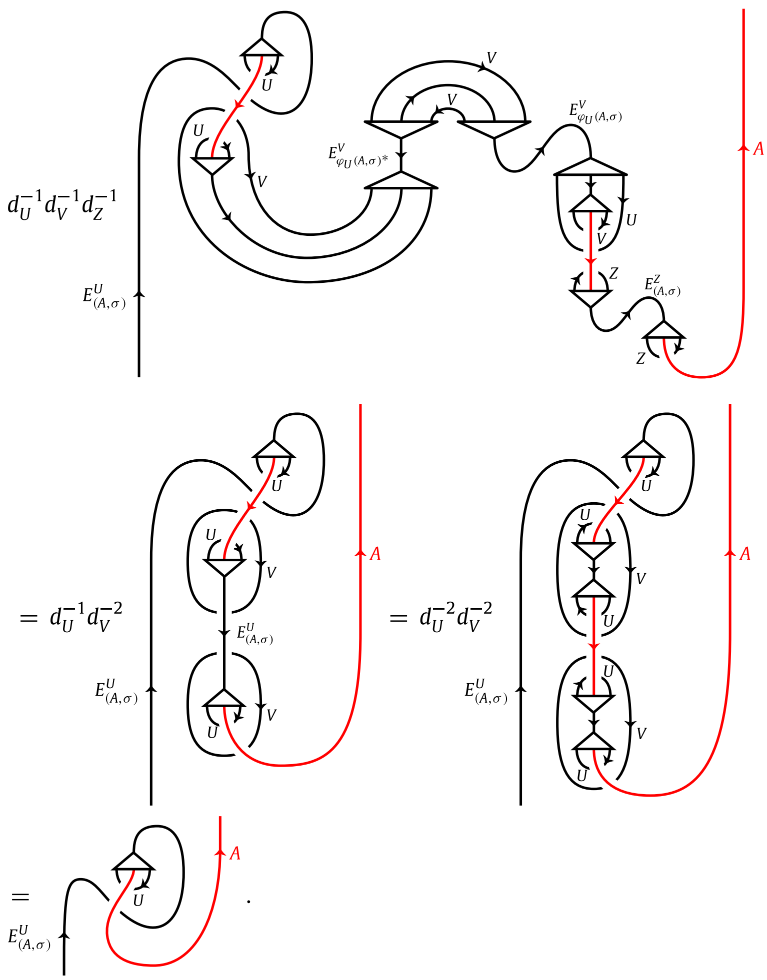
Pick any $U \in \mathcal{E}_\alpha$, $V \in \mathcal{E}_{\alpha^{-1}}$, and $R \in \mathcal{E}_1$. Composing (30) on the right with $(\iota^\alpha_U)^*$, we rewrite (30) in the equivalent form

$$(\vartheta^U_{(A,\sigma)})^* = \big(\eta^R_{(A,\sigma)}\big)^*\big((\xi^{V,U,R}_{(A,\sigma)})^{-1}\big)^*\varphi^1_V\big(\varphi_U(A, \sigma)\big)\vartheta^V_{(\varphi_U(A,\sigma))^*}. \tag{31}$$

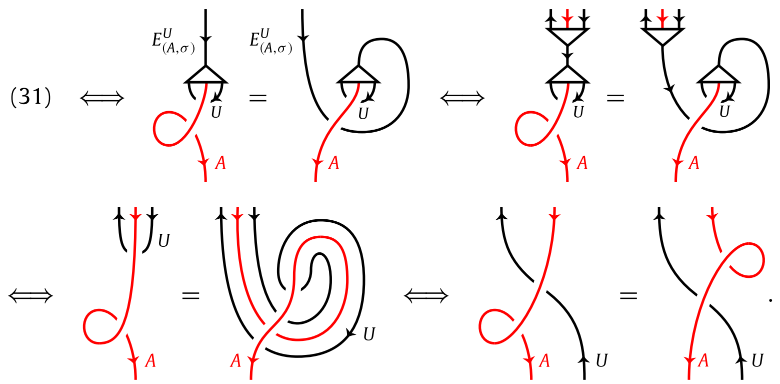
Now, using the pictorial formalism of Section 4.6, Lemma 4.10, and the definition of $\vartheta, \eta, \xi, \varphi^1_V$, we obtain that the left-hand side of (31) is equal to



while the right-hand side of (31) is equal to



Therefore



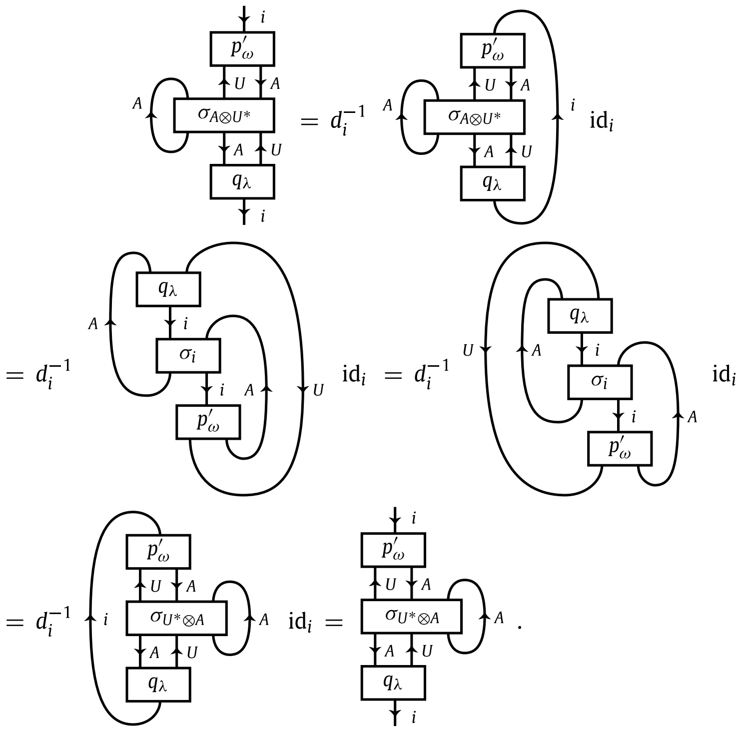
The last equality is exactly the condition of the lemma. □

6.2. Proof of Lemma 5.2

We check the criterion of Lemma 6.1 for any (A, σ) and U. Pick a representative set I of simple objects of $\mathcal{C}$. Let

$$(p_\lambda\colon A \otimes U^* \to i_\lambda, q_\lambda\colon i_\lambda \to A \otimes U^*)_{\lambda\in\Lambda},\ (p'_\omega\colon U^* \otimes A \to i_\omega, q'_\omega\colon i_\omega \to U^* \otimes A)_{\omega\in\Omega}$$

be I-partitions of $A \otimes U^*$ and $U^* \otimes A$, respectively. For any $\lambda \in \Lambda$ and $\omega \in \Omega$ such that $i_\lambda = i_\omega = i \in I$, we have



Here the first and last equalities follow from the simplicity of i and the formulas $d_i = \dim_l(i) = \dim_r(i)$. The other equalities follow from the isotopy invariance in S^2 and the naturality of σ. We conclude using that any morphism $f\colon A\otimes U^* \to U^*\otimes A$ expands as $f = \sum_{\omega\in\Omega,\lambda\in\Lambda} q'_\omega(p'_\omega f q_\lambda)p_\lambda$ where $p'_\omega f q_\lambda = 0$ for $i_\lambda \neq i_\omega$.

7. Monads and Hopf monads

Monads, bimonads, and Hopf monads generalize respectively algebras, bialgebras, and Hopf algebras to the categorical setting. The concept of a monad originated in Godement's work on sheaf cohomology in the 1950s. Bimonads were introduced by Moerdijk [7] in 2002. Hopf monads were introduced by A. Bruguières and the second author [1] in 2006; see also [2,3]. We recall here the basics of the theory of monads, bimonads, and Hopf monads needed in the sequel.

7.1. Monads and modules

Given a category $\mathcal{C}$, we denote by $\mathrm{End}(\mathcal{C})$ the category whose objects are endofunctors of $\mathcal{C}$ (that is, functors $\mathcal{C} \to \mathcal{C}$) and morphisms are natural transformations between the endofunctors. The category $\mathrm{End}(\mathcal{C})$ is a strict monoidal category with monoidal product being composition of endofunctors and unit object being the identity functor $1_\mathcal{C}\colon \mathcal{C} \to \mathcal{C}$. A *monad* on $\mathcal{C}$ is a monoid in the category $\mathrm{End}(\mathcal{C})$, that is, a triple $(T \in \mathrm{End}(\mathcal{C}), \mu, \eta)$ consisting of a functor $T\colon \mathcal{C} \to \mathcal{C}$ and two natural transformations

$$\mu = \{\mu_X\colon T^2(X) \to T(X)\}_{X\in\mathcal{C}} \quad\text{and}\quad \eta = \{\eta_X\colon X \to T(X)\}_{X\in\mathcal{C}}$$

called the *product* and the *unit* of T, such that for all $X \in \mathcal{C}$,

$$\mu_X T(\mu_X) = \mu_X\mu_{T(X)} \quad\text{and}\quad \mu_X\eta_{T(X)} = \mathrm{id}_{T(X)} = \mu_X T(\eta_X).$$

For example, the identity functor $1_\mathcal{C}\colon \mathcal{C} \to \mathcal{C}$ is a monad on $\mathcal{C}$ with identity as product and unit. This is the *trivial monad*.

Given a monad $T = (T, \mu, \eta)$ on $\mathcal{C}$, a *T-module* is a pair $(M \in \mathcal{C}, r)$ where $r\colon T(M) \to M$ is a morphism in $\mathcal{C}$ such that $rT(r) = r\mu_M$ and $r\eta_M = \mathrm{id}_M$. We call such a morphism r *an action of T on M*. A morphism from a T-module (M, r) to a T-module (N, s) is a morphism $f\colon M \to N$ in $\mathcal{C}$ such that $fr = sT(f)$. This defines the *category of T-modules*, $\mathcal{C}^T$, with composition induced by that in $\mathcal{C}$. We define a *forgetful functor* $U_T : \mathcal{C}^T \to \mathcal{C}$ by $U_T(M, r) = M$ and $U_T(f) = f$. We also define the *free module functor* $F_T : \mathcal{C} \to \mathcal{C}^T$ by $F_T(X) = (T(X), \mu_X)$ for $X \in \mathcal{C}$ and $F_T(f) = T(f)$ for any morphism f in $\mathcal{C}$. The functors F_T and U_T are adjoint: there is a system of bijections $\mathrm{Hom}_{\mathcal{C}^T}(F(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, U_T(Y))$ natural in $X \in \mathcal{C}$, $Y \in \mathcal{C}^T$. For an endofunctor Q of $\mathcal{C}^T$, the functor $Q \rtimes T = U_T Q F_T : \mathcal{C} \to \mathcal{C}$ is called the *cross product* of Q with T. For example, $1_{\mathcal{C}^T} \rtimes T = T$. If T is the trivial monad, then $\mathcal{C}^T = \mathcal{C}$, $F_T = U_T = 1_{\mathcal{C}}$, and $Q \rtimes T = Q$ for all $Q \in \mathrm{End}(\mathcal{C})$.

If $\mathcal{C}$ is $\Bbbk$-additive and T is $\Bbbk$-linear, then the category $\mathcal{C}^T$ is $\Bbbk$-additive and the functors U_T, F_T are $\Bbbk$-linear.

7.2. Comonoidal functors

To introduce bimonads and Hopf monads we ought to replace endofunctors in the definitions above by comonoidal endofunctors. We recall here the relevant definitions.

Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. A *comonoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a triple (F, F_2, F_0), where $F : \mathcal{C} \to \mathcal{D}$ is a functor,

$$F_2 = \{F_2(X, Y) : F(X \otimes Y) \to F(X) \otimes F(Y)\}_{X,Y \in \mathcal{C}}$$

is a natural transformation from $F\otimes$ to $F \otimes F$, and $F_0 : F(\mathbb{1}) \to \mathbb{1}$ is a morphism in $\mathcal{D}$, such that

$$\big(\mathrm{id}_{F(X)} \otimes F_2(Y, Z)\big) F_2(X, Y \otimes Z) = \big(F_2(X, Y) \otimes \mathrm{id}_{F(Z)}\big) F_2(X \otimes Y, Z);$$
$$(\mathrm{id}_{F(X)} \otimes F_0) F_2(X, \mathbb{1}) = \mathrm{id}_{F(X)} = (F_0 \otimes \mathrm{id}_{F(X)}) F_2(\mathbb{1}, X);$$

for all objects X, Y, Z of $\mathcal{C}$. A comonoidal functor (F, F_2, F_0) is *strong* (resp. *strict*) if F_0 and all the morphisms $F_2(X, Y)$ are isomorphisms (resp. identities). The formula $(F, F_2, F_0) \mapsto (F, F_2^{-1}, F_0^{-1})$ establishes a bijective correspondence between strong (resp. strict) comonoidal functors and strong (resp. strict) monoidal functors.

A natural transformation $\varphi = \{\varphi_X : F(X) \to G(X)\}_{X \in \mathcal{C}}$ from a comonoidal functor $F : \mathcal{C} \to \mathcal{D}$ to a comonoidal functor $G : \mathcal{C} \to \mathcal{D}$ is *comonoidal* if $G_0 \varphi_{\mathbb{1}} = F_0$ and $G_2(X, Y)\varphi_{X \otimes Y} = (\varphi_X \otimes \varphi_Y) F_2(X, Y)$ for all $X, Y \in \mathcal{C}$.

If $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are comonoidal functors between monoidal categories, then their composition $GF : \mathcal{C} \to \mathcal{E}$ is a comonoidal functor with

$$(GF)_0 = G_0 G(F_0) \quad \text{and} \quad (GF)_2 = \{G_2(F(X), F(Y)) G(F_2(X, Y))\}_{X,Y \in \mathcal{C}}.$$

7.3. Bimonads

For a monoidal category $\mathcal{C}$, denote by $\mathrm{End}_{\mathrm{co}\otimes}(\mathcal{C})$ the category whose objects are comonoidal endofunctors of $\mathcal{C}$ and morphisms are comonoidal natural transformations. The category $\mathrm{End}_{\mathrm{co}\otimes}(\mathcal{C})$ is strict monoidal with composition of comonoidal endofunctors as monoidal product and the identity functor $1_{\mathcal{C}}$ as monoidal unit. A *bimonad* on $\mathcal{C}$ is a monoid in the category $\mathrm{End}_{\mathrm{co}\otimes}(\mathcal{C})$. In other words, a bimonad on $\mathcal{C}$ is a monad (T, μ, η) on $\mathcal{C}$ such that the functor $T : \mathcal{C} \to \mathcal{C}$ and the natural transformations μ and η are comonoidal. For example, the trivial monad on $\mathcal{C}$ with identity morphisms for comonoidal structure and for μ and η is a bimonad called the *trivial bimonad*.

Let $T = ((T, T_2, T_0), \mu, \eta)$ be a bimonad on a monoidal category $\mathcal{C}$. The category of T-modules $\mathcal{C}^T$ has a monoidal structure with unit object $(\mathbb{1}, T_0)$ and with monoidal product

$$(M, r) \otimes (N, s) = \big(M \otimes N, (r \otimes s)\, T_2(M, N)\big).$$

By [2, Section 3.3], the forgetful functor $U_T : \mathcal{C}^T \to \mathcal{C}$ is strict monoidal while the free module functor $F_T : \mathcal{C} \to \mathcal{C}^T$ is comonoidal with $(F_T)_0 = T_0$ and $(F_T)_2(X, Y) = T_2(X, Y)$ for any $X, Y \in \mathcal{C}$. By [2, Section 3.7], for any $Q \in \mathrm{End}_{\mathrm{co}\otimes}(\mathcal{C}^T)$, the cross product $Q \rtimes T = U_T Q F_T \in \mathrm{End}(\mathcal{C})$ is comonoidal with $(Q \rtimes T)_0 = Q_0 Q((F_T)_0)$ and $(Q \rtimes T)_2 = Q_2 Q((F_T)_2)$. The formula $Q \mapsto Q \rtimes T$ defines a monoidal functor

$$? \rtimes T : \mathrm{End}_{\mathrm{co}\otimes}(\mathcal{C}^T) \to \mathrm{End}_{\mathrm{co}\otimes}(\mathcal{C}) \tag{32}$$

with monoidal structure

$$((? \rtimes T)_0)_X = \eta_X : X \to T(X) \quad \text{and} \quad ((? \rtimes T)_2(Q, R))_X = U_T Q(\varepsilon_{RF_T(X)})$$

for any $X \in \mathcal{C}$ and $Q, R \in \mathrm{End}_{\mathrm{co}\otimes}(\mathcal{C}^T)$, where ε is the counit of the adjunction (F_T, U_T), that is, the natural transformation $F_T U_T \to 1_{\mathcal{C}^T}$ carrying $(M, r) \in \mathcal{C}^T$ to r.

7.4. Hopf monads

Given a bimonad (T, μ, η) on a monoidal category $\mathcal{C}$ and objects $X, Y \in \mathcal{C}$, one defines the *left fusion morphism*

$$H^l_{X,Y} = (T(X) \otimes \mu_Y)\, T_2(X, T(Y)) \colon T(X \otimes T(Y)) \to T(X) \otimes T(Y)$$

and the *right fusion morphism*

$$H^r_{X,Y} = (\mu_X \otimes T(Y))\, T_2(T(X), Y) \colon T(T(X) \otimes Y) \to T(X) \otimes T(Y);$$

see [3]. A *Hopf monad* on $\mathcal{C}$ is a bimonad on $\mathcal{C}$ whose left and right fusions are isomorphisms for all $X, Y \in \mathcal{C}$. For example, the trivial bimonad on $\mathcal{C}$ is a Hopf monad called the *trivial Hopf monad*.

When $\mathcal{C}$ is a rigid category (see Section 2.3) and T is a Hopf monad on $\mathcal{C}$, the monoidal category $\mathcal{C}^T$ has a canonical structure of a rigid category. This structure can be computed from the natural transformations

$$s^l = \{s^l_X \colon T({}^\vee T(X)) \to {}^\vee X\}_{X \in \mathcal{C}} \quad \text{and} \quad s^r = \{s^r_X \colon T(T(X)^\vee) \to X^\vee\}_{X \in \mathcal{C}}$$

called the *left and right antipodes* and determined by the fusion morphisms:

$$s^l_X = \big(T_0 T(\mathrm{ev}_{T(X)})(H^l_{{}^\vee T(X),X})^{-1} \otimes {}^\vee\eta_X\big)\big(\mathrm{id}_{T({}^\vee T(X))} \otimes \mathrm{coev}_{T(X)}\big)$$

and

$$s^r_X = \big(\eta^\vee_X \otimes T_0 T(\widetilde{\mathrm{ev}}_{T(X)})(H^r_{X,T(X)^\vee})^{-1}\big)\big(\widetilde{\mathrm{coev}}_{T(X)} \otimes \mathrm{id}_{T(T(X)^\vee)}\big).$$

Then the left and right duals of any T- module $(M, r) \in \mathcal{C}^T$ are defined by

$${}^\vee(M, r) = ({}^\vee M, s^l_M T({}^\vee r) \colon T({}^\vee M) \to {}^\vee M),$$
$$(M, r)^\vee = (M^\vee, s^r_M T(r^\vee) \colon T(M^\vee) \to M^\vee).$$

Though we shall not need it, note that, conversely, a bimonad T on $\mathcal{C}$ such that $\mathcal{C}^T$ is rigid is a Hopf monad.

8. Coends, centralizers, and free objects

We outline the theory of coends [5] and discuss connections with Hopf monads and relative centers.

8.1. Coends

Let $\mathcal{C}, \mathcal{D}$ be categories and $F \colon \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{C}$ be a functor. A *dinatural transformation* from F to an object A of $\mathcal{C}$ is a family $d = \{d_Y \colon F(Y, Y) \to A\}_{Y \in \mathcal{D}}$ of morphisms in $\mathcal{C}$ such that for every morphism $f \colon X \to Y$ in $\mathcal{D}$ (viewed also as a morphism $Y \to X$ in $\mathcal{D}^{\mathrm{op}}$), the following diagram commutes:

$$\begin{tikzcd} F(Y,X) \arrow[r, "F(\mathrm{id}_Y, f)"] \arrow[d, "F(f, \mathrm{id}_X)"'] & F(Y,Y) \arrow[d, "d_Y"] \\ F(X,X) \arrow[r, "d_X"'] & A. \end{tikzcd}$$

The *composition* of such a d with a morphism $\phi \colon A \to B$ in $\mathcal{C}$ is the dinatural transformation $\phi \circ d = \{\phi \circ d_X \colon F(Y, Y) \to B\}_{Y \in \mathcal{D}}$ from F to B. A *coend* of F is a pair $(C \in \mathcal{C}, \rho)$ where ρ is a dinatural transformation from F to C satisfying the following universality condition: every dinatural transformation d from F to an object of $\mathcal{C}$ is the composition of ρ with a morphism in $\mathcal{C}$ uniquely determined by d. If F has a coend (C, ρ), then it is unique up to (unique) isomorphism. One writes $C = \int^{Y \in \mathcal{D}} F(Y, Y)$. In particular, if the category $\mathcal{C}$ is monoidal, then for any functors $F_1 \colon \mathcal{D}^{\mathrm{op}} \to \mathcal{C}$, $F_2 \colon \mathcal{D} \to \mathcal{C}$, we write $\int^{Y \in \mathcal{D}} F_1(Y) \otimes F_2(Y)$ for the coend (if it exists) of the functor $\mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{C}$ defined on objects and morphisms by $(X, Y) \mapsto F_1(X) \otimes F_2(Y)$.

The following lemma gives a sufficient condition for the existence of coends.

Lemma 8.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be $\Bbbk$-additive categories. If $\mathcal{D}$ is finite split semisimple, then any $\Bbbk$-linear functor $F \colon \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{C}$ has a coend.*

Proof. Pick a (finite) representative set I of simple objects of $\mathcal{D}$ and set $C = \oplus_{i \in I} F(i, i) \in \mathcal{C}$. For each object $Y \in \mathcal{D}$, set $\rho_Y = \sum_\alpha F(q_\alpha, p_\alpha) \colon F(Y, Y) \to C$ where $(p_\alpha : Y \to i_\alpha, q_\alpha : i_\alpha \to Y)_\alpha$ is an arbitrary I-partition of Y. It is easy to check that ρ_Y does not depend on the choice of the I-partition and $(C, \rho = \{\rho_Y\}_Y)$ is a coend of F. Indeed, each dinatural transformation d from F to any $A \in \mathcal{C}$ is the composition of ρ with $\oplus_{i \in I} d_i \colon C \to A$. $\square$

The next lemma is a partial inverse to Lemma 8.1. It shows that a finiteness condition is necessary for the existence of coends.

Lemma 8.2. *Let $\mathcal{C}$ be a $\Bbbk$-additive pivotal category whose Hom-spaces are projective $\Bbbk$-modules of finite rank, and let $\mathcal{D}$ be a split semisimple full subcategory of $\mathcal{C}$. If the coend $\int^{Y\in\mathcal{D}} Y^*\otimes Y$ exists in $\mathcal{C}$, then $\mathcal{D}$ is finite.*

Proof. Let $C = \int^{Y\in\mathcal{D}} Y^*\otimes Y$ be the coend of the functor $F(X,Y) = X^*\otimes Y$ with universal dinatural transformation $\rho = \{\rho_Y : Y^*\otimes Y \to C\}_{Y\in\mathcal{D}}$. Let I be a representative set of isomorphism classes of simple objects of $\mathcal{D}$ and let $J\subset I$ be an arbitrary finite subset of I. Set $A = \oplus_{i\in J}\, i^*\otimes i \in \mathcal{C}$. For any $Y\in\mathcal{D}$, set

$$d_Y = \sum_{\alpha, i_\alpha\in J} q_\alpha^*\otimes p_\alpha : Y^*\otimes Y \to A$$

where $(p_\alpha : Y\to i_\alpha, q_\alpha : i_\alpha \to Y)_\alpha$ is an I-partition of Y. It is easy to check that d_Y does not depend on the choice of the I-partition and that the family $\{d_Y\}_Y$ is a dinatural transformation from F to A. Therefore there is a morphism $p\colon C\to A$ such that $d_Y = p\rho_Y$ for all $Y\in\mathcal{D}$. Set $q = \sum_{i\in J}\rho_i\colon A\to C$. Then $pq = \sum_{i\in J} p\rho_i = \sum_{i\in J} d_i = \mathrm{id}_A$. Thus, the composition with q induces a split injection $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)\to \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, C)$. Hence

$$\begin{aligned}\mathrm{card}(J) &= \sum_{i\in J}\mathrm{rank}_\Bbbk\big(\mathrm{Hom}_{\mathcal{C}}(i,i)\big) = \sum_{i\in J}\mathrm{rank}_\Bbbk\big(\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, i^*\otimes i)\big)\\ &= \mathrm{rank}_\Bbbk\big(\mathrm{Hom}_{\mathcal{C}}(\mathbb{1},A)\big) \le \mathrm{rank}_\Bbbk\big(\mathrm{Hom}_{\mathcal{C}}(\mathbb{1},C)\big).\end{aligned}$$

This bound implies the claim of the lemma. □

8.2. Lift of coends

Let $T = ((T, T_2, T_0), \mu, \eta)$ be a Hopf monad on a rigid category $\mathcal{C}$ and let $\mathcal{D}$ be a subcategory of $\mathcal{C}$ such that $T(\mathcal{D})\subset\mathcal{D}$. The functor T restricts to a monad on $\mathcal{D}$, also denoted T, and the corresponding category of modules $\mathcal{D}^T$ is a subcategory of $\mathcal{C}^T$. The following lemma allows us to lift from $\mathcal{C}$ to $\mathcal{C}^T$ the coends of certain functors $\mathcal{D}^{\mathrm{op}}\times\mathcal{D}\to\mathcal{C}$ associated with endofunctors of $\mathcal{C}^T$.

Lemma 8.3. *Let Q be an endofunctor of $\mathcal{C}^T$ such that there exists a coend*

$$C = \int^{Y\in\mathcal{D}} {}^\vee(Q\rtimes T)(Y)\otimes Y \in \mathcal{C}.$$

Then there exists a coend $\int^{M\in\mathcal{D}^T} {}^\vee Q(M)\otimes M\in\mathcal{C}^T$ carried by the forgetful functor $\mathcal{C}^T\to\mathcal{C}$ to C. More precisely, if

$$\rho = \{\rho_Y\colon {}^\vee(Q\rtimes T)(Y)\otimes Y\to C\}_{Y\in\mathcal{D}}$$

is the universal dinatural transformation of C, then there is a unique morphism $r\colon T(C)\to C$ in $\mathcal{C}$ such that for all $Y\in\mathcal{D}$,

$$rT(\rho_Y) = \rho_{T(Y)}\big({}^\vee Q(\mu_Y)s^l_{Q\rtimes T(Y)}T({}^\vee a_Y)\otimes\mathrm{id}_{T(Y)}\big)\,T_2\big({}^\vee(Q\rtimes T)(Y), Y\big),$$

where a_Y is the T-action of the T-module $QF_T(Y)$. Then r is an action of T on C and $(C,r) = \int^{M\in\mathcal{D}^T} {}^\vee Q(M)\otimes M$ with universal dinatural transformation

$$\varrho = \{\varrho_{(N,s)} = \rho_N({}^\vee Q(s)\otimes\mathrm{id}_N)\colon {}^\vee Q(N,s)\otimes(N,s)\to(C,r)\}_{(N,s)\in\mathcal{D}^T}.$$

Proof. This is a direct corollary of Lemma 3.9 and Proposition 3.10 of [2]. □

8.3. Centralizers of endofunctors

Let $\mathcal{C}$ be a rigid category and $\mathcal{D}$ be a subcategory of $\mathcal{C}$. An endofunctor E of $\mathcal{C}$ is *$\mathcal{D}$-centralizable* if for each $X\in\mathcal{C}$, the functor $\mathcal{D}^{\mathrm{op}}\times\mathcal{D}\to\mathcal{C}$ defined by $(Y,Y')\mapsto {}^\vee E(Y)\otimes X\otimes Y'$ has a coend

$$Z_E^{\mathcal{D}}(X) = \int^{Y\in\mathcal{D}} {}^\vee E(Y)\otimes X\otimes Y\in\mathcal{C}.$$

The correspondence $X\mapsto Z_E^{\mathcal{D}}(X)$ extends to a functor $Z_E^{\mathcal{D}}\colon\mathcal{C}\to\mathcal{C}$, called the *$\mathcal{D}$-centralizer* of E, so that the associated universal dinatural transformation

$$\rho_{X,Y}\colon {}^\vee E(Y)\otimes X\otimes Y\to Z_E^{\mathcal{D}}(X) \tag{33}$$

is natural in $X\in\mathcal{C}$ and dinatural in $Y\in\mathcal{D}$. For $\mathcal{D}=\mathcal{C}$, the notion of a centralizer of an endofunctor was introduced in [2].

When the identity endofunctor $1_{\mathcal{C}}$ of $\mathcal{C}$ is $\mathcal{D}$-centralizable, we say that $\mathcal{C}$ is $\mathcal{D}$-*centralizable*. For example, the endofunctor $1_{\mathcal{C}}$ is $\mathcal{D}$-centralizable if the category $\mathcal{D}$ is finite split semisimple; see Lemma 8.1. Moreover, any (finite) representative set I of simple objects of $\mathcal{D}$ determines a $\mathcal{D}$-centralizer $Z : \mathcal{C} \to \mathcal{C}$ of $1_{\mathcal{C}}$. The functor Z carries any $X \in \mathcal{C}$ to $Z(X) = \oplus_{i\in I}\, i^* \otimes X \otimes i$ and carries any morphism f in $\mathcal{C}$ to $Z(f) = \oplus_{i\in I}\, \mathrm{id}_{i^*} \otimes f \otimes \mathrm{id}_i$.

8.4. Relative centers and free objects

Let $\mathcal{C}$ be a rigid category and $\mathcal{D}$ be a rigid subcategory of $\mathcal{C}$, i.e., $\mathcal{D}$ is a monoidal subcategory of $\mathcal{C}$ stable under both left and right dualities. Suppose that $\mathcal{C}$ is $\mathcal{D}$-centralizable. We construct a Hopf monad $Z = Z_{\mathcal{D}}$ on $\mathcal{C}$ such that the relative center $\mathcal{Z}(\mathcal{C};\mathcal{D})$ is monoidally isomorphic to the category $\mathcal{C}^Z$.

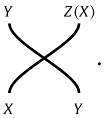
Let $Z : \mathcal{C} \to \mathcal{C}$ be a $\mathcal{D}$-centralizer of $1_{\mathcal{C}}$ with universal dinatural transformation

$$\rho = \{\rho_{X,Y} : {}^\vee Y \otimes X \otimes Y \to Z(X)\}_{X\in\mathcal{C}, Y\in\mathcal{D}}.$$

For $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, set

$$\partial_{X,Y} = (\mathrm{id}_Y \otimes \rho_{X,Y})(\mathrm{coev}_Y \otimes \mathrm{id}_{X\otimes Y}) : X \otimes Y \to Y \otimes Z(X).$$

We depict the morphism $\partial_{X,Y}$ as follows:

$\partial_{X,Y} =$  .

For any $X, X_1, X_2 \in \mathcal{C}$, the parameter theorem and the Fubini theorem for coends (see [5]) imply the existence of (unique) morphisms

$$\mu_X : Z(Z(X)) \to Z(X),\ Z_2(X_1, X_2) : Z(X_1 \otimes X_2) \to Z(X_1) \otimes Z(X_2),$$
$$Z_0 : Z(\mathbb{1}) \to \mathbb{1}, \qquad s^l_X : Z({}^\vee Z(X)) \to {}^\vee X, \qquad s^r_X : Z(Z(X)^\vee) \to X^\vee,$$

such that the equalities of morphisms shown in Fig. 1 hold for all $Y, Y_1, Y_2 \in \mathcal{D}$, where the trivalent vertex in the third picture stands for $\partial_{\mathbb{1},Y} : Y \to Y \otimes Z(\mathbb{1})$.

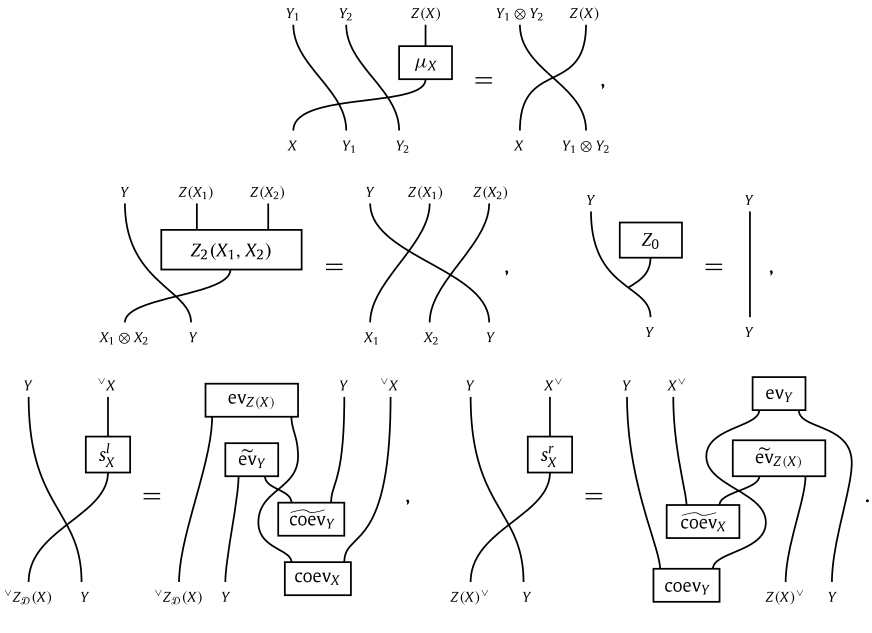


Fig. 1. Structural morphisms of $Z = Z_{\mathcal{D}}$.

Lemma 8.4. (a) *Let $\eta = \{\eta_X\}_{X\in\mathcal{C}}$ where $\eta_X = \partial_{X,\mathbb{1}} : X \to Z(X)$ for all $X \in \mathcal{C}$. Then $Z = \big((Z, Z_2, Z_0), \mu, \eta\big)$ is a Hopf monad on $\mathcal{C}$ with left antipode $s^l = \{s^l_X\}_{X\in\mathcal{C}}$ and right antipode $s^r = \{s^r_X\}_{X\in\mathcal{C}}$.*

(b) *Let $\Psi : \mathcal{C}^Z \to \mathcal{Z}(\mathcal{C};\mathcal{D})$ be the functor carrying any object $(M,r) \in \mathcal{C}^Z$ to (M,σ^r) with $\sigma^r = \{\sigma^r_Y = (\mathrm{id}_Y \otimes r)\partial_{M,Y} : M \otimes Y \to Y \otimes M\}_{Y\in\mathcal{D}}$ and carrying any morphism to itself. Then Ψ is a strict monoidal isomorphism, and the composition of Ψ with the forgetful functor $\mathcal{Z}(\mathcal{C};\mathcal{D}) \to \mathcal{C}$ is equal to the forgetful functor $U_Z : \mathcal{C}^Z \to \mathcal{C}$.*

(c) *If $\mathcal{C}$ is $\Bbbk$-additive, then the categories $\mathcal{C}^Z$ and $\mathcal{Z}(\mathcal{C};\mathcal{D})$ are $\Bbbk$-additive and the functors Z and Ψ are $\Bbbk$-linear.*

Proof. The proof of (a) and (b) is obtained from the proof of Theorems 5.6 and 5.12 in [2] by replacing $Y \in \mathcal{C}$ with $Y \in \mathcal{D}$ whenever necessary and in particular by replacing $\int^{Y\in\mathcal{C}} {}^\vee Y \otimes X \otimes Y$ with $\int^{Y\in\mathcal{D}} {}^\vee Y \otimes X \otimes Y$. Claim (c) is obvious. □

We use this lemma to define free objects in $\mathcal{Z}(\mathcal{C};\mathcal{D})$. Recall from Section 7.1 the free module functor $F_Z : \mathcal{C} \to \mathcal{C}^Z$ is left adjoint to the forgetful functor $U_Z\mathcal{C}^Z \to \mathcal{C}$. It is clear that the functor $\Psi F_Z : \mathcal{C} \to \mathcal{Z}(\mathcal{C};\mathcal{D})$ is left adjoint to the forgetful functor $\mathcal{U} : \mathcal{Z}(\mathcal{C};\mathcal{D}) \to \mathcal{C}$. An object of $\mathcal{Z}(\mathcal{C};\mathcal{D})$ is said to be *free* if it is isomorphic to $\Psi F_Z(X)$ for some $X \in \mathcal{C}$.

8.5. The case of G-centers

We shall apply Lemma 8.4 to study the G-centers of G-graded categories. Consider a rigid G-graded category $\mathcal{C}$ such that $\mathcal{C}$ is $\mathcal{C}_1$-centralizable. Lemma 8.4 provides an extension of any $\mathcal{C}_1$-centralizer $Z : \mathcal{C} \to \mathcal{C}$ to a Hopf monad on $\mathcal{C}$ and a $\Bbbk$-linear strict monoidal isomorphism $\Psi : \mathcal{C}^Z \to \mathcal{Z}(\mathcal{C};\mathcal{C}_1) = \mathcal{Z}_G(\mathcal{C})$. We can always choose Z so that $Z(\mathcal{C}_\alpha) \subset \mathcal{C}_\alpha$ for all $\alpha \in G$. Then Z restricts to a monad on $\mathcal{C}_\alpha$, and the corresponding category of modules, denoted $\mathcal{C}^Z_\alpha$, is a full subcategory of $\mathcal{C}^Z$. This turns $\mathcal{C}^Z$ into a G-graded category. It follows from the definitions that Ψ preserves the G-grading. In the terminology above, an object of $\mathcal{Z}_G(\mathcal{C})$ is *free* if it is isomorphic to an object in the image of the functor $\Psi F_Z : \mathcal{C} \to \mathcal{Z}_G(\mathcal{C})$ left adjoint to the forgetful functor $\mathcal{U} : \mathcal{Z}_G(\mathcal{C}) \to \mathcal{C}$.

The $\mathcal{C}_1$-centralizability condition on $\mathcal{C}$ is satisfied, for example, if $\mathcal{C}_1$ is finite split semisimple.

9. Proof of Lemmas 5.3 and 5.4

We begin with a fairly general lemma concerning morphisms between indecomposable objects in abelian categories. Next, we formulate an extension of the graphical calculus allowing to incorporate partitions of objects. Finally, we use these tools and the theory of Hopf monads to prove Lemmas 5.3 and 5.4.

9.1. Indecomposable objects in abelian categories

We recall several standard definitions of the theory of categories. An object X of an additive category is *indecomposable* if X is non-zero and whenever X decomposes as $X = X_1 \oplus X_2$, we have $X_1 = \mathbf{0}$ or $X_2 = \mathbf{0}$. For example, all simple objects of a $\Bbbk$-additive category are indecomposable. An *abelian category* is an additive category such that any morphism f has a kernel and a cokernel and $\mathrm{coker}(\ker f) \simeq \ker(\mathrm{coker} f)$. A *monomorphism* in a category is a morphism $q : X \to Y$ such that any two morphisms $f, g : A \to X$ with $qf = qg$ must be equal. A *retract* of a morphism $q : X \to Y$ is a morphism $p : Y \to X$ such that $pq = \mathrm{id}_X$. Clearly, if q has a retract, then q is a monomorphism.

Lemma 9.1. *Let $\mathcal{A}$ be an abelian category in which any monomorphism has a retract. Then any morphism between indecomposable objects of $\mathcal{A}$ is either zero or an isomorphism.*

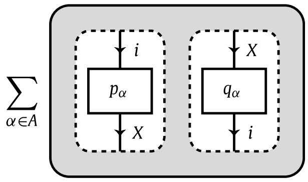
Proof. Let $f : M \to P$ be a morphism in $\mathcal{A}$ where M, P are indecomposable objects. Let $q : N \to M$ be the kernel of f. Then q is a monomorphism in $\mathcal{A}$ and, by assumption, q has a retract. Since $\mathcal{A}$ is abelian, this implies that N is a direct summand of M. The object M being indecomposable, we have $N = \mathbf{0}$ or $M = N \oplus \mathbf{0} \simeq N$. In the latter case, q is an isomorphism, and so $f = 0$ (because $fq = 0$). Assume that $N = \mathbf{0}$. Then f is a monomorphism (since it is a morphism with zero kernel in an abelian category). By assumption, f has a retract $g : P \to M$. In particular $gf = \mathrm{id}_M$ and so $e = \mathrm{id}_P - fg$ is an idempotent of P. Since $\mathcal{A}$ has split idempotents (because it is abelian), there exist an object $A \in \mathcal{A}$ and morphisms $u : A \to P$ and $v : P \to A$ in $\mathcal{A}$ such that $vu = \mathrm{id}_A$ and $e = uv$. In particular, $\mathrm{id}_P = fg + uv$ and so $P = M \oplus A$. Since P is indecomposable, $M = \mathbf{0}$ or $A = \mathbf{0}$. If $M = \mathbf{0}$, then $f = 0$. If $A = \mathbf{0}$, then $uv = 0$, and so $fg = \mathrm{id}_P$, which implies that f is an isomorphism (because $gf = \mathrm{id}_M$). □

9.2. Extension of graphical calculus

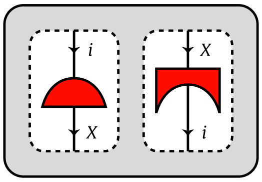
Let $\mathcal{C}$ be a split semisimple pivotal category. Clearly, the Hom spaces in $\mathcal{C}$ are free $\Bbbk$-modules of finite rank. For $X \in \mathcal{C}$ and a simple object $i \in \mathcal{C}$, the modules $\mathrm{Hom}_\mathcal{C}(X, i)$ and $\mathrm{Hom}_\mathcal{C}(i, X)$ have the same rank denoted N^i_X and called the *multiplicity number*.

An *i-decomposition* of X is a family of morphisms $(p_\alpha : X \to i,\ q_\alpha : i \to X)_{\alpha\in A}$ such that $\mathrm{card}(A) = N^i_X$ and $p_\alpha q_\beta = \delta_{\alpha,\beta}\,\mathrm{id}_i$ for all $\alpha, \beta \in A$. Note that if I is a representative set of simple objects of $\mathcal{C}$ and $(p_\alpha, q_\alpha)_{\alpha\in\Lambda}$ is an I-partition of X in the sense of Section 5.2, then for each $i \in I$ the family $(p_\alpha, q_\alpha)_{\alpha\in\Lambda,\, i_\alpha = i}$ is an i-decomposition of X. Conversely, the union of i-decompositions of X over all $i \in I$ is an I-partition of X.

Let i be a simple object of $\mathcal{C}$ and let $(p_\alpha : X \to i, q_\alpha : i \to X)_{\alpha\in A}$ be an i-decomposition of an object X of $\mathcal{C}$. Consider a sum

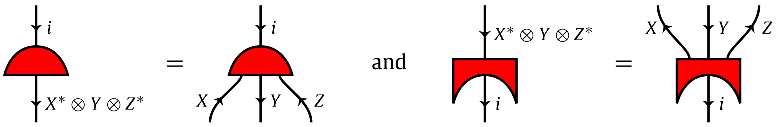


where the gray area contains an oriented planar graph whose edges and vertices are labeled by objects and morphisms of $\mathcal{C}$ not involving p_α, q_α. By the graphical calculus of Section 2.5, this sum represents a morphism in $\mathcal{C}$. Note that the tensor $\sum_{\alpha\in A} p_\alpha \otimes q_\alpha \in \mathrm{Hom}_{\mathcal{C}}(X, i) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(i, X)$ does not depend on the choice of the i-decomposition of X. Therefore the sum above also does not depend on this choice. We graphically present this sum by



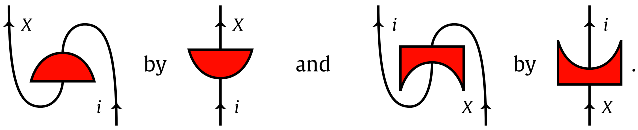
where the gray area contains the same planar graph as before and two curvilinear boxes are endowed with one and the same color. If several such pairs of boxes appear in a picture, they must have different colors.

Note that monoidal products of objects may be depicted as bunches of strands. For example,



where the equality sign means that the pictures represent the same morphism of $\mathcal{C}$.

To simplify the pictures, we will represent



9.3. *Proof of Lemma 5.4*

Fix a (finite) representative set I of simple objects of $\mathcal{D}$ such that $\mathbb{1} \in I$. Consider the associated $\mathcal{D}$-centralizer $Z : \mathcal{C} \to \mathcal{C}$ of $1_{\mathcal{C}}$ as defined in Section 8.3. By Lemma 8.4(a), the functor Z extends to a Hopf monad $((Z, Z_2, Z_0), \mu, \eta)$ on $\mathcal{C}$ with structural morphisms shown in Fig. 2.

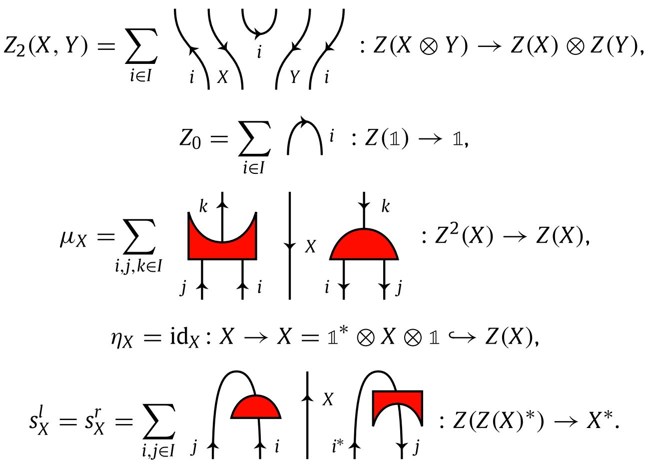
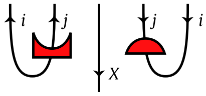


Fig. 2. Structural morphisms of the Hopf monad Z.

For $X \in \mathcal{C}$, set

$$\gamma_X = \sum_{i,j\in I} \frac{\dim_r(i)}{\dim(\mathcal{D})}$$

$$: X \to Z^2(X) = Z(Z(X)).$$

It is clear that γ_X is natural in $X \in \mathcal{C}$.

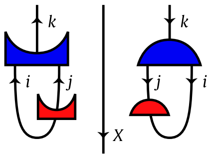
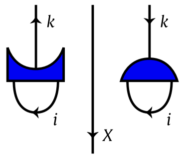
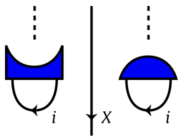
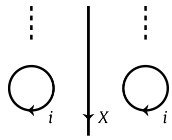
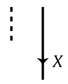
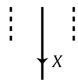
Lemma 9.2. *The natural transformation $\gamma = \{\gamma_X : X \to Z^2(X)\}_{X\in\mathcal{C}}$ satisfies*

$$\mu_X\gamma_X = \eta_X \quad \text{and} \quad Z(\mu_X)\gamma_{Z(X)} = \mu_{Z(X)}Z(\gamma_X)$$

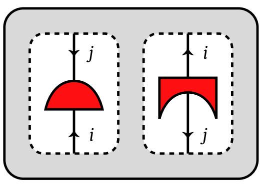
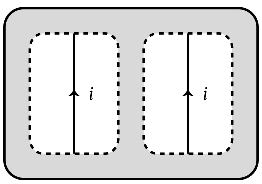
for any $X \in \mathcal{C}$.

In terminology of [1, Section 6], Lemma 9.2 may be reformulated by saying that the Hopf monad Z is separable.

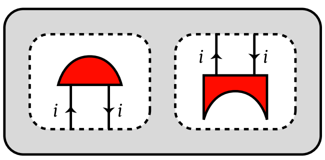
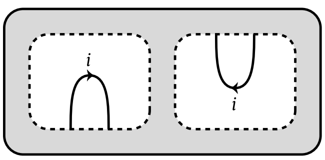
Proof. For $X \in \mathcal{C}$,

$$\mu_X\gamma_X = \sum_{i,j,k\in I} \frac{\dim_r(i)}{\dim(\mathcal{D})}$$

$$\overset{(i)}{=} \sum_{i\in I} \frac{\dim_r(i)}{\dim(\mathcal{D})}$$

$$\overset{(ii)}{=} \sum_{i\in I} \frac{\dim_r(i)}{\dim(\mathcal{D})}$$

$$\overset{(iii)}{=} \sum_{i\in I} \frac{\dim_r(i)}{\dim(\mathcal{D})\dim_l(i)}$$

$$= \sum_{i\in I} \frac{\dim_r(i)\dim_l(i)}{\dim(\mathcal{D})}$$

$$\overset{(iv)}{=}$$

$$\overset{(v)}{=} \eta_X,$$

where the dotted lines represent $\mathrm{id}_{\mathbb{1}}$ and can be removed without changing the morphisms (we depicted them in order to remember which factor of $Z(X)$ is concerned). In the above, the equality *(i)* follows from the fact that there are no non-zero morphisms between non-isomorphic simple objects, and so


$$= \delta_{j,i^*}$$

(34)

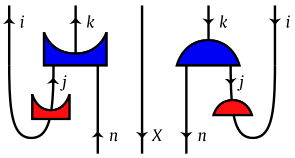
where $\delta_{j,i^*} = 1$ if j is isomorphic to i^* and $\delta_{j,i^*} = 0$ otherwise. (If j is isomorphic to i^*, then the right picture implicitly involves a box labeled with an isomorphism $i^* \to j$ attached to the top of the left string and a box labeled with the inverse isomorphism $j \to i^*$ attached to the bottom of the right string.) The equality (ii) follows from the fact that for $k \in I$, a morphism $\mathbb{1} \to k$ is zero unless $k = \mathbb{1}$. The equality (iii) follows from the equality


$$= \frac{1}{\dim_l(i)}$$

,
(35)

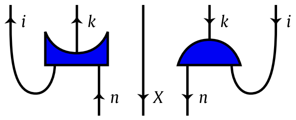
which is a consequence of the duality and the fact that $\mathrm{Hom}_{\mathcal{C}}(i, i) = \Bbbk$. Finally the equalities (iv) and (v) follow from the definitions of $\dim(\mathcal{D})$ and η_X, respectively.

Let us prove the second equality of the lemma. We have:

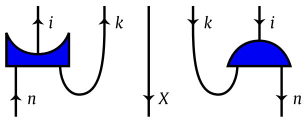
$$Z(\mu_X)\gamma_{Z(X)} = \sum_{i,j,k,n\in I} \frac{\dim_r(i)}{\dim(\mathcal{D})}$$



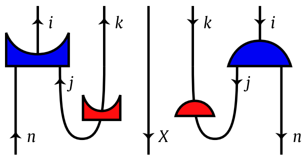
$$\overset{(i)}{=} \sum_{i,k,n\in I} \frac{\dim_r(i)}{\dim(\mathcal{D})}$$



$$\overset{(ii)}{=} \sum_{i,k,n\in I} \frac{\dim_l(k)}{\dim(\mathcal{D})}$$

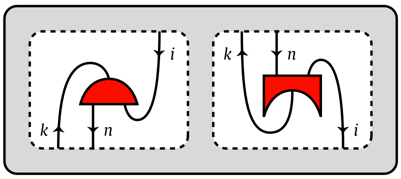


$$\overset{(iii)}{=} \sum_{i,j,k,n\in I} \frac{\dim_r(j)}{\dim(\mathcal{D})}$$

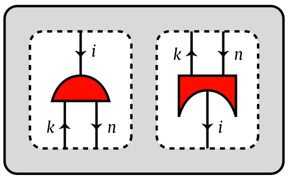


$$= \mu_{Z(X)} Z(\gamma_X).$$

In the above, the equality *(i)* follows from (34), *(iii)* follows from (34) and the fact that if $j \simeq k^*$ then $\dim_l(k) = \dim_l(j^*) = \dim_r(j)$, and *(ii)* follows from the following equality: for any $i, k, n \in I$,

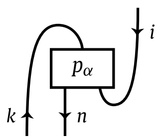
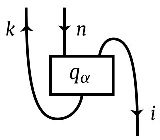


$$= \frac{\dim_l(k)}{\dim_r(i)}$$



. (36)

It remains to prove (36). Let $(p_\alpha : n \otimes i^* \to k,\ q_\alpha : k \to n \otimes i^*)_{\alpha \in A}$ be a k-decomposition of $n \otimes i^*$. For $\alpha \in A$, set

$P_\alpha =$  and $Q_\alpha =$  .

For $\alpha, \beta \in A$,

$$P_\alpha Q_\beta = \frac{\mathrm{tr}_r(P_\alpha Q_\beta)}{\dim_r(i)}\mathrm{id}_i = \frac{\mathrm{tr}_l(p_\alpha q_\beta)}{\dim_r(i)}\mathrm{id}_i = \frac{\mathrm{tr}_l(\delta_{\alpha,\beta}\,\mathrm{id}_k)}{\dim_r(i)}\mathrm{id}_i = \delta_{\alpha,\beta}\frac{\dim_l(k)}{\dim_r(i)}\mathrm{id}_i.$$

Therefore, since $\mathrm{card}(A) = N^k_{n\otimes i^*} = N^i_{k^*\otimes n}$, we obtain that the family

$$\left(\frac{\dim_r(i)}{\dim_l(k)}P_\alpha, Q_\alpha\right)_{\alpha\in A}$$

is an i-decomposition of $k^* \otimes n$, from which we deduce (36) and the lemma. $\square$

Lemma 9.3. *Any monomorphism in the category $\mathcal{C}^Z$ of Z-modules has a retract.*

Proof. Let $q: (N, s) \to (M, r)$ be a monomorphism in $\mathcal{C}^Z$. The forgetful functor $\mathcal{C}^Z \to \mathcal{C}$ is a right adjoint of a functor $\mathcal{C} \to \mathcal{C}^Z$ and so carries monomorphisms in $\mathcal{C}^Z$ to monomorphisms in $\mathcal{C}$. Thus, $q: N \to M$ is a monomorphism in $\mathcal{C}$. Expanding N and M as direct sums of simple objects of $\mathcal{C}$ we can represent q by a matrix over the field $\Bbbk$. Using standard arguments of linear algebra, we conclude that there is a morphism $v\colon M \to N$ in $\mathcal{C}$ such that $vq = \mathrm{id}_N$. Set $p = sZ(vr)\gamma_M\colon M \to N$ where γ is the natural transformation of Lemma 9.2. Then

$$\begin{aligned} sZ(p) &= sZ(s)Z^2(vr)Z(\gamma_M)\\ &= s\mu_N Z^2(vr)Z(\gamma_M) \quad \text{since } s \text{ is a } Z\text{-action},\\ &= sZ(vr)\mu_{Z(M)}Z(\gamma_M) \quad \text{by the naturality of } \mu,\\ &= sZ(vr\mu_M)\gamma_{Z(M)} \quad \text{by Lemma 9.2},\\ &= sZ(vr)Z^2(r)\gamma_{Z(M)} \quad \text{since } r \text{ is a } Z\text{-action},\\ &= sZ(vr)\gamma_M r \quad \text{by the naturality of } \gamma,\\ &= pr. \end{aligned}$$

Thus, p is a morphism $(M, r) \to (N, s)$ in $\mathcal{C}^Z$. Also,

$$\begin{aligned} pq &= sZ(vr)\gamma_M q\\ &= sZ(vrZ(q))\gamma_N \quad \text{by the naturality of } \gamma,\\ &= sZ(vqs)\gamma_N \quad \text{since } q \text{ is } Z\text{-linear},\\ &= sZ(s)\gamma_N \quad \text{since } vq = \mathrm{id}_N,\\ &= s\mu_N\gamma_N \quad \text{since } s \text{ is a } Z\text{-action},\\ &= s\eta_N \quad \text{by Lemma 9.2},\\ &= \mathrm{id}_N \quad \text{since } s \text{ is a } Z\text{-action}. \end{aligned}$$

Hence, p is a retract of q. $\square$

We can now complete the proof of Lemma 5.4. Since $\mathcal{C}$ is a split semisimple category over $\Bbbk$ which is assumed to be a field, $\mathcal{C}$ is an abelian $\Bbbk$-additive category with finite-dimensional Hom-spaces. Since the Hopf monad Z is $\Bbbk$-linear and preserves cokernels (because Z has a right adjoint by [1, Corollary 3.12]), we deduce that $\mathcal{C}^Z$ is an abelian $\Bbbk$-additive category with finite-dimensional Hom-spaces.

Combining Lemmas 9.1 and 9.3, we obtain that the Hom-spaces between non-isomorphic indecomposable Z-modules are zero, and the algebra of endomorphisms of an indecomposable Z-module is a finite-dimensional division $\Bbbk$-algebra. Since the field $\Bbbk$ is algebraically closed, such an algebra is isomorphic to $\Bbbk$. Thus, all indecomposable Z-modules are simple. The finite-dimensionality of the End-spaces in $\mathcal{C}^Z$ implies that any Z-module is a finite direct sum of indecomposable Z-modules. Hence, $\mathcal{C}^Z$ is split semisimple. By Lemma 8.4(b), (c), the $\Bbbk$-additive categories $\mathcal{Z}(\mathcal{C}; \mathcal{D})$ and $\mathcal{C}^Z$ are isomorphic. Therefore, $\mathcal{Z}(\mathcal{C}; \mathcal{D})$ is split semisimple. This concludes the proof of Lemma 5.4.

9.4. Proof of Lemma 5.3

Since $\mathcal{C}$ is a G-fusion category, it is split semisimple, its unit object $\mathbb{1}$ is simple, and each $\mathcal{C}_\alpha$ with $\alpha \in G$ is finite split semisimple. Lemma 5.4 applied to $\mathcal{D} = \mathcal{C}_1$ yields that $\mathcal{Z}_G(\mathcal{C}) = \mathcal{Z}(\mathcal{C}; \mathcal{C}_1)$ is split semisimple. The unit object of $\mathcal{Z}_G(\mathcal{C})$ is simple because $\mathbb{1}_\mathcal{C}$ is simple. It remains to prove that the set of isomorphism classes of simple objects of $\mathcal{Z}_\alpha(G)$ is finite for all $\alpha \in G$. In the notation of Section 8.5, it is enough to prove that the set of isomorphism classes of simple objects of the category $\mathcal{C}^Z_\alpha$ is finite. By Lemma 8.2, it suffices to prove that the coend

$$\int^{(N,s)\in\mathcal{C}^Z_\alpha} (N,s)^* \otimes (N,s)$$

exists in $\mathcal{C}^Z$. Lemma 8.3 and the equality $1_{\mathcal{C}^Z} \rtimes Z = Z$ imply that it is enough to establish the existence of a coend $\int^{Y\in\mathcal{C}_\alpha} Z(Y)^* \otimes Y$ in $\mathcal{C}$. The latter follows from Lemma 8.1 because $\mathcal{C}_\alpha$ is a finite split semisimple subcategory of $\mathcal{C}$.

10. Crossing and G-braiding via free functors

In this section, $\mathcal{C}$ is a rigid G-graded category which is *G-centralizable* in the sense that $\mathcal{C}$ is $\mathcal{C}_\alpha$-centralizable for all $\alpha \in G$ (see Section 8.1). By Section 8.5, the forgetful functor $\mathcal{Z}_G(\mathcal{C}) \to \mathcal{C}$ has a left adjoint $\mathcal{F}\colon \mathcal{C} \to \mathcal{Z}_G(\mathcal{C})$, and the objects of $\mathcal{Z}_G(\mathcal{C})$ isomorphic to objects in the image of $\mathcal{F}$ are said to be free. We introduce here a larger class of G-free objects of $\mathcal{Z}_G(\mathcal{C})$ and compute for them the crossing and the G-braiding in $\mathcal{Z}_G(\mathcal{C})$.

10.1. Free functors

By assumption, for all $\alpha \in G$ and $X \in \mathcal{C}$, the coend $Z_\alpha(X) = \int^{Y \in \mathcal{C}_\alpha} {}^\vee Y \otimes X \otimes Y$ exists in $\mathcal{C}$. Let $Z_\alpha = Z_{1_\mathcal{C}}^{\mathcal{C}_\alpha} : \mathcal{C} \to \mathcal{C}$ be a $\mathcal{C}_\alpha$-centralizer of $1_\mathcal{C}$ with universal dinatural transformation

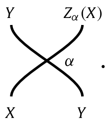
$$\rho^\alpha = \{\rho^\alpha_{X,Y} : {}^\vee Y \otimes X \otimes Y \to Z_\alpha(X)\}_{X \in \mathcal{C}, Y \in \mathcal{C}_\alpha}. \tag{37}$$

If $X \in \mathcal{C}_\beta$ with $\beta \in G$, then we always choose $Z_\alpha(X)$ in $\mathcal{C}_{\alpha^{-1}\beta\alpha}$.

For any $X \in \mathcal{C}$ and $Y \in \mathcal{C}_\alpha$, set

$$\partial^\alpha_{X,Y} = (\mathrm{id}_Y \otimes \rho^\alpha_{X,Y})(\mathrm{coev}_Y \otimes \mathrm{id}_{X \otimes Y}) : X \otimes Y \to Y \otimes Z_\alpha(X), \tag{38}$$

which we depict as

$\partial^\alpha_{X,Y} =$  .

For any $X_1, X_2 \in \mathcal{C}$, the parameter theorem and the Fubini theorem for coends (see [5]) imply the existence of unique morphisms

$$(Z_\alpha)_2(X_1, X_2) : Z_\alpha(X_1 \otimes X_2) \to Z_\alpha(X_1) \otimes Z_\alpha(X_2), \qquad (Z_\alpha)_0 : Z_\alpha(\mathbb{1}) \to \mathbb{1}$$

such that the first two equalities of Fig. 3 are satisfied for all $Y \in \mathcal{C}_\alpha$. Similarly, for any $\alpha, \beta \in G$ and $X \in \mathcal{C}$ there is a unique morphism $Z_2(\alpha, \beta)_X : Z_\alpha Z_\beta(X) \to Z_{\beta\alpha}(X)$ such that the third equality of Fig. 3 is satisfied for all $Y \in \mathcal{C}_\alpha$ and $Y' \in \mathcal{C}_\beta$. Finally, for all $X \in \mathcal{C}$, set $(Z_0)_X = \partial^1_{X,\mathbb{1}} : X \to Z_1(X)$.

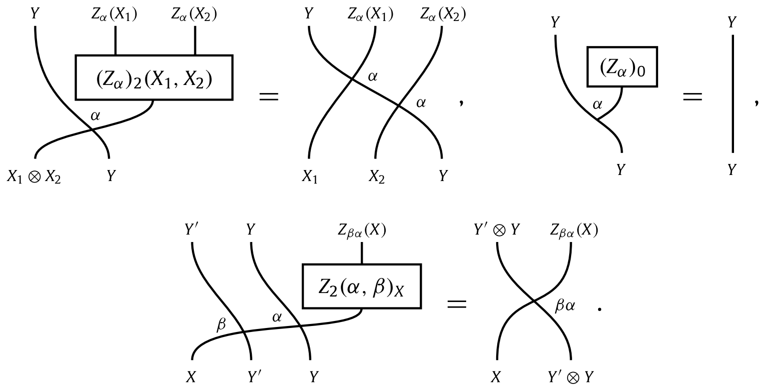


Fig. 3. The structural morphisms of Z.

Lemma 10.1. *The endofunctor $Z_\alpha = (Z_\alpha, (Z_\alpha)_2, (Z_\alpha)_0)$ of $\mathcal{C}$ is comonoidal for all $\alpha \in G$. The formula $\alpha \mapsto Z_\alpha$ defines a monoidal functor*

$$Z = (Z, Z_2, Z_0) : \overline{G} \to \mathrm{End}_{\mathrm{co}\otimes}(\mathcal{C})$$

such that $Z_\alpha(\mathcal{C}_\beta) \subset \mathcal{C}_{\alpha^{-1}\beta\alpha}$ for all $\alpha, \beta \in G$.

Proof. The proof is obtained from that of Theorems 5.6 in [2], by replacing $Y \in \mathcal{C}$ with $Y \in \mathcal{C}_\alpha$ and in particular by replacing $\int^{Y \in \mathcal{C}}$ with $\int^{Y \in \mathcal{C}_\alpha}$. $\square$

For $\alpha = 1$, the functor $Z_\alpha = Z_1 : \mathcal{C} \to \mathcal{C}$, endowed with the product $\mu = Z_2(1, 1)$ and unit $\eta = Z_0$, is nothing but the Hopf monad produced by Lemma 8.4(a) for $\mathcal{D} = \mathcal{C}_1$. Let $F_1 = F_{Z_1} : \mathcal{C} \to \mathcal{C}^{Z_1}$ be the free module functor; see Section 7.1. Recall that $F_1(X) = (Z_1(X), Z_2(1, 1)_X)$ for any object X of $\mathcal{C}$ and $F_1(f) = Z_1(f)$ for any morphism f in $\mathcal{C}$. The functor F_1 is comonoidal with comonoidal structure $(F_1)_2 = (Z_1)_2$ and $(F_1)_0 = (Z_1)_0$. The forgetful functor $U_1 = U_{Z_1} : \mathcal{C}^{Z_1} \to \mathcal{C}$ is strict monoidal and therefore is comonoidal in the canonical way. One can check that $U_1 F_1 = Z_1$ as comonoidal functors. Our immediate aim is to introduce α-analogues of F_1 for all $\alpha \in G$.

Pick any $\alpha \in G$. For all $X, Y \in \mathcal{C}$ and any morphism f in $\mathcal{C}$, set

$$F_\alpha(X) = (Z_\alpha(X), Z_2(1,\alpha)_X), \qquad F_\alpha(f) = Z_\alpha(f),$$
$$(F_\alpha)_2(X,Y) = (Z_\alpha)_2(X,Y), \qquad (F_\alpha)_0 = (Z_\alpha)_0.$$

Lemma 10.2. *$F_\alpha = (F_\alpha, (F_\alpha)_2, (F_\alpha)_0) \colon \mathcal{C} \to \mathcal{C}^{Z_1}$ is a comonoidal functor such that $U_1 F_\alpha = Z_\alpha$ as comonoidal functors.*

Proof. The monoidality of Z implies that $Z_2(1,\alpha)_X (Z_0)_{Z_\alpha(X)} = \mathrm{id}_{Z_\alpha(X)}$ and

$$Z_2(1,\alpha)_X Z_1(Z_2(1,\alpha)_X) = Z_2(1,\alpha)_X Z_2(1,1)_{Z_\alpha(X)}.$$

Therefore $F_\alpha(X) \in \mathcal{C}^{Z_1}$. The naturality of $Z_2(1,\alpha)$ implies that

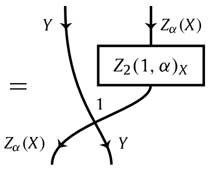
$$Z_\alpha(f) Z_2(1,\alpha)_X = Z_2(1,\alpha)_Y Z_1(Z_\alpha(f)).$$

This means that $Z_\alpha(f)$ is Z_1-linear. Hence F_α is a well-defined functor. Now the comonoidality of the natural transformation $Z_2(1,\alpha)$ gives

$$Z_\alpha(X,Y) Z_2(1,\alpha)_{X\otimes Y} = (Z_2(1,\alpha)_X \otimes Z_2(1,\alpha)_Y)(Z_1)_2(Z_\alpha(X), Z_\alpha(Y)) Z_1((Z_\alpha)_2(X,Y))$$

and $(Z_\alpha)_0 Z_2(1,\alpha)_{\mathbb{1}} = (Z_1)_0 Z_1((Z_\alpha)_0)$. Thus $(Z_\alpha)_2(X,Y)$ and $(Z_\alpha)_0$ are Z_1-linear. Hence F_α is comonoidal. Clearly $U_1 F_\alpha = Z_\alpha$ as comonoidal functors because U_1 is strict monoidal. $\square$

By Section 8.5 (where Z should be replaced with Z_1), the category $\mathcal{C}^{Z_1}$ is G-graded with $\mathcal{C}^{Z_1}_\alpha = (\mathcal{C}_\alpha)^{Z_1}$ for all $\alpha \in G$, and we have a $\Bbbk$-linear strict monoidal isomorphism of G-graded categories $\Psi \colon \mathcal{C}^{Z_1} \to \mathcal{Z}_G(\mathcal{C})$. For $\alpha \in G$, set $\mathcal{F}_\alpha = \Psi F_\alpha \colon \mathcal{C} \to \mathcal{Z}_G(\mathcal{C})$. By definition, for $X \in \mathcal{C}$,

$\mathcal{F}_\alpha(X) = (Z_\alpha(X), \sigma^\alpha_X)$ with $\sigma^\alpha_{X,Y} =$  for all $Y \in \mathcal{C}_1$.

Clearly, $\mathcal{F}_\alpha(\mathcal{C}_\beta) \subset \mathcal{Z}_\beta(\mathcal{C})$ for all $\beta \in G$. By Lemma 10.2, $\mathcal{F}_\alpha$ is a comonoidal functor and $\mathcal{U}\mathcal{F}_\alpha = Z_\alpha$ as comonoidal functors, where $\mathcal{U} \colon \mathcal{Z}_G(\mathcal{C}) \to \mathcal{C}$ is the (strict monoidal) forgetful functor.

We call $\mathcal{F}_\alpha \colon \mathcal{C} \to \mathcal{Z}_G(\mathcal{C})$ the *α-free functor*, and we call the objects of $\mathcal{Z}_G(\mathcal{C})$ isomorphic to objects in the image of $\mathcal{F}_\alpha$ *α-free*. For $\alpha = 1$, this definition is equivalent to the definition of free objects given in Section 8.5. Collectively, the α-free objects of $\mathcal{Z}_G(\mathcal{C})$ with $\alpha \in G$ are said to be *G-free*.

The following lemma will be used in Appendix.

Lemma 10.3. *$Z_2(\alpha,\beta)$ is a comonoidal natural transformation from $\mathcal{F}_\alpha Z_\beta$ to $\mathcal{F}_{\beta\alpha}$ for all $\alpha, \beta \in G$.*

Proof. Since the isomorphism $\Psi \colon \mathcal{C}^{Z_1} \to \mathcal{Z}_G(\mathcal{C})$ is strict monoidal and acts as the identity on morphisms, it is enough to prove that $Z_2(\alpha,\beta)$ is a comonoidal natural transformation from $F_\alpha Z_\beta$ to $F_{\beta\alpha}$. Since $U_1 F_\gamma = Z_\gamma$ as comonoidal functors for all $\gamma \in G$ and $Z_2(\alpha,\beta)$ is a comonoidal natural transformation from $Z_\alpha Z_\beta$ to $Z_{\beta\alpha}$, we only need to check that for $X \in \mathcal{C}$, $Z_2(\alpha,\beta)_X$ is a morphism in C^{Z_1} from $F_\alpha Z_\beta(X)$ to $F_{\beta\alpha}(X)$, i.e., that

$$Z_2(\alpha,\beta)_X Z_2(1,\alpha)_{Z_\beta(X)} = Z_2(1,\beta\alpha)_X Z_1(Z_2(\alpha,\beta)_X).$$

This equality holds by the monoidality of Z (see Lemma 10.1). $\square$

10.2. Computations on G-free objects

Assume that, as above, $\mathcal{C}$ is a G-centralizable rigid G-graded category, and assume additionally that $\mathcal{C}$ is pivotal and non-singular. Then $\mathcal{Z}_G(\mathcal{C})$ is G-braided by Theorem 4.1. The next theorem computes the action of the crossing $\varphi \colon \overline{G} \to \mathrm{Aut}(\mathcal{Z}_G(\mathcal{C}))$ of $\mathcal{Z}_G(\mathcal{C})$ on the G-free objects. Note that each auto-equivalence φ_α of $\mathcal{Z}_G(\mathcal{C})$, being strong monoidal, can be seen as a strong comonoidal endofunctor $(\varphi_\alpha, (\varphi_\alpha)_2^{-1}, (\varphi_\alpha)_0^{-1})$ of $\mathcal{Z}_G(\mathcal{C})$.

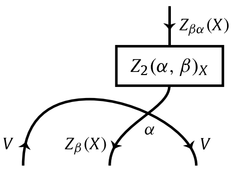
Theorem 10.4. *For $\alpha, \beta \in G$, there is a canonical comonoidal isomorphism*

$$\omega^{\alpha,\beta} = \{\omega_X^{\alpha,\beta} : \varphi_\alpha \mathcal{F}_\beta(X) \to \mathcal{F}_{\beta\alpha}(X)\}_{X \in \mathcal{C}}$$

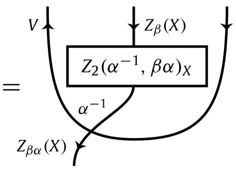
from $\varphi_\alpha \mathcal{F}_\beta$ to $\mathcal{F}_{\beta\alpha}$. Moreover, the family $\omega = \{\omega^{\alpha,\beta}\}_{\alpha,\beta \in G}$ is compatible with the monoidal structure of the crossing φ in the following sense: $\omega_X^{1,\alpha} = (\varphi_0)^{-1}_{\mathcal{F}_\alpha(X)}$ and for all $\alpha, \beta, \gamma \in G$ and $X \in \mathcal{C}$, the following diagram commutes

$$\begin{tikzcd}
\varphi_\alpha \varphi_\beta \mathcal{F}_\gamma(X) \arrow[r, "(\varphi_2)(\alpha,\beta)_{\mathcal{F}_\gamma(X)}"] \arrow[d, "\varphi_\alpha(\omega_X^{\beta,\gamma})"'] & \varphi_{\beta\alpha} \mathcal{F}_\gamma(X) \arrow[d, "\omega_X^{\beta\alpha,\gamma}"] \\
\varphi_\alpha \mathcal{F}_{\gamma\beta}(X) \arrow[r, "\omega_X^{\alpha,\gamma\beta}"'] & \mathcal{F}_{\beta\alpha}(X).
\end{tikzcd}$$

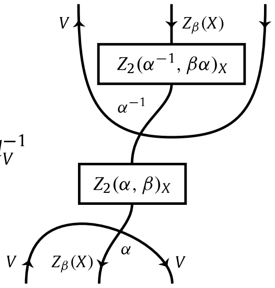
Proof. Let $\alpha, \beta \in G$ and $X \in \mathcal{C}$. Pick $V \in \mathcal{E}_\alpha$, where $\mathcal{E}_\alpha$ denotes the class of objects of $\mathcal{C}_\alpha$ with invertible left dimension (recall that $\mathcal{E}_\alpha \neq \emptyset$ since $\mathcal{C}$ is non-singular). Set $d_V = \dim_l(V)$ and

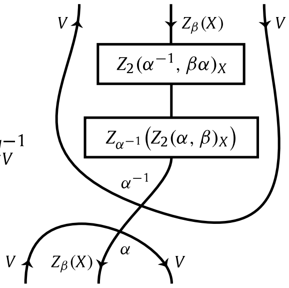
$a_X^{V,\beta} = d_V^{-1}$  $: V^* \otimes Z_\beta(X) \otimes V \to Z_{\beta\alpha}(X)$

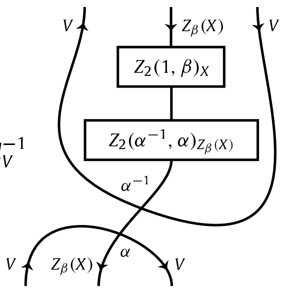
and

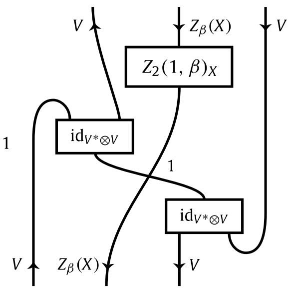
$b_X^{V,\beta} =$  $: Z_{\beta\alpha}(X) \to V^* \otimes Z_\beta(X) \otimes V.$

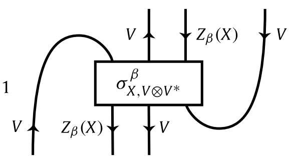
Observe that

$b_X^{V,\beta} a_X^{V,\beta} = d_V^{-1}$ 

$\overset{(i)}{=} d_V^{-1}$ 

$\overset{(ii)}{=} d_V^{-1}$ 

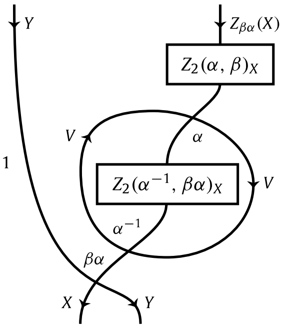
$\overset{(iii)}{=} d_V^{-1}$ 

$\overset{(iv)}{=} d_V^{-1}$  $= \pi^V_{\mathcal{F}_\beta(X)}.$

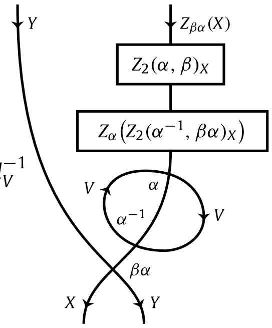
Here, *(i)* follows from the naturality of $\partial^{\alpha^{-1}}$, *(ii)* from the monoidality of Z, *(iii)* from the definition of $Z_2(\alpha^{-1}, \alpha)$, and *(iv)* from the definition of $\sigma^{\beta}_{X,V\otimes V^*}$.

We claim that $a_X^{V,\beta} b_X^{V,\beta} = \mathrm{id}_{Z_{\beta\alpha}(X)}$. Indeed, for any $Y \in \mathcal{C}_{\beta\alpha}$,

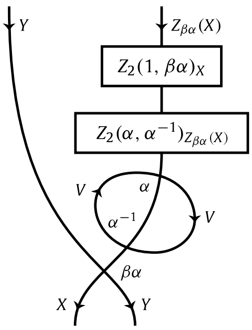
$(\mathrm{id}_Y \otimes a_X^{V,\beta} b_X^{V,\beta})\partial^{\beta\alpha}_{X,Y} = d_V^{-1}$



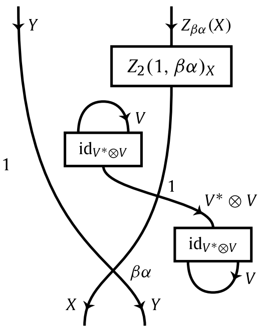
$\overset{(i)}{=} d_V^{-1}$



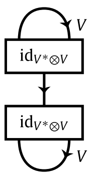
$\overset{(ii)}{=} d_V^{-1}$

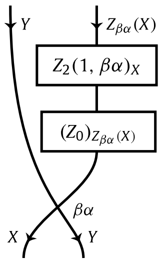


$\overset{(iii)}{=} d_V^{-1}$



$\overset{(iv)}{=} d_V^{-1}$



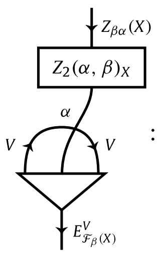


$\overset{(v)}{=} \partial^{\beta\alpha}_{X,Y}$.

Here the equality *(i)* follows from the naturality of $\partial^{\beta\alpha}$, *(ii)* and *(v)* are consequences of the monoidality of Z, *(iii)* follows from the definition of $Z_2(\alpha, \alpha^{-1})$, and *(iv)* is a consequence of the naturality of ∂^1 and the definition of Z_0. Now, the universal property of $\rho^{\beta\alpha}_{X,Y} = (\mathrm{ev}_Y \otimes \mathrm{id}_{Z_{\beta\alpha}(X)})(\mathrm{id}_{Y^*} \otimes \partial^{\beta\alpha}_{X,Y})$ implies that $a_X^{V,\beta} b_X^{V,\beta} = \mathrm{id}_{Z_{\beta\alpha}(X)}$.

The latter equality and the formula $b_X^{V,\beta} a_X^{V,\beta} = \pi^V_{\mathcal{F}_\beta(X)}$ mean that $Z_{\beta\alpha}(X)$ is the image of the idempotent $\pi^V_{\mathcal{F}_\beta(X)}$. Set

$\omega_X^{V,\beta} = a_X^{V,\beta}\, q^V_{\mathcal{F}_\beta(X)} = d_V^{-1}$



$: E^V_{\mathcal{F}_\beta(X)} \to Z_{\beta\alpha}(X).$ (39)

By Section 4.1, $\omega_X^{V,\beta}$ is the unique isomorphism $E^V_{\mathcal{F}_\beta(X)} \to Z_{\beta\alpha}(X)$ such that

$$p^V_{\mathcal{F}_\beta(X)} = \big(\omega_X^{V,\beta}\big)^{-1} a_X^{V,\beta} \quad \text{and} \quad q^V_{\mathcal{F}_\beta(X)} = b_X^{V,\beta}\omega_X^{V,\beta}. \tag{40}$$

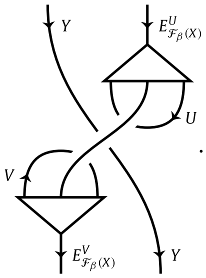
Note that $\left(\omega_X^{V,\beta}\right)^{-1} = p^V_{\mathcal{F}_\beta(X)}\, b_X^{V,\beta}$. We can now state the key lemma of the proof.

Lemma 10.5. (a) *$\omega_X^{V,\beta} \colon \varphi_V(\mathcal{F}_\beta(X)) \to \mathcal{F}_{\beta\alpha}(X)$ is an isomorphism in $\mathcal{Z}_G(\mathcal{C})$.*

(b) *$\omega_X^U \delta^{U,V}_{\mathcal{F}_1(X)} = \omega_X^V$ for all $U, V \in \mathcal{E}_\alpha$,*

Proof. We use the pictorial formalism of Section 4.6. For $U, V \in \mathcal{E}_\alpha$ and $Y \in \mathcal{C}_1$, set

$$\lambda^{U,V,\beta}_{X,Y} = d_V^{-1}$$



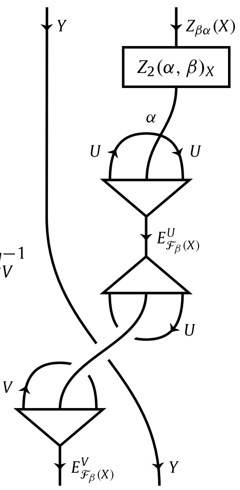
.

Then

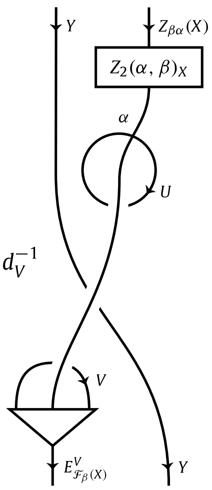
$$(\mathrm{id}_Y \otimes \omega_X^{U,\beta})\lambda^{U,V,\beta}_{X,Y} = \sigma^{\beta\alpha}_{X,Y}(\omega_X^{V,\beta} \otimes \mathrm{id}_Y). \tag{41}$$

Indeed,

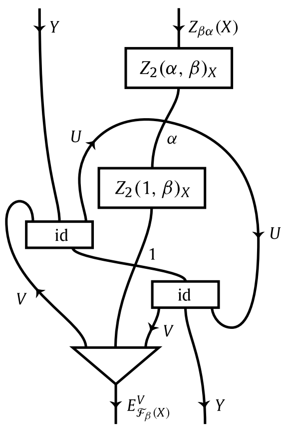
$$(\mathrm{id}_Y \otimes \omega_X^{U,\beta})\lambda^{U,V,\beta}_{X,Y} = d_U^{-1} d_V^{-1}$$



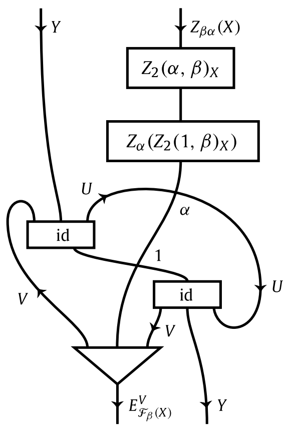
$$\overset{(i)}{=} d_U^{-1} d_V^{-1}$$

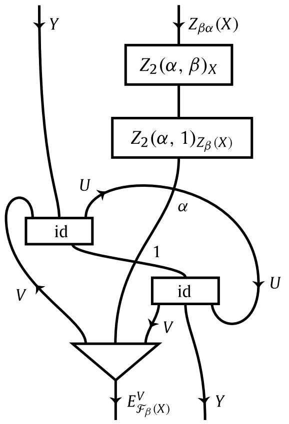


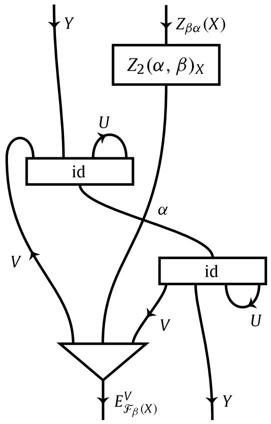
$$\overset{(ii)}{=} d_U^{-1} d_V^{-1}$$

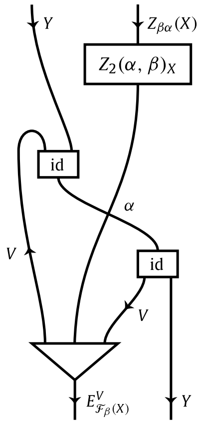


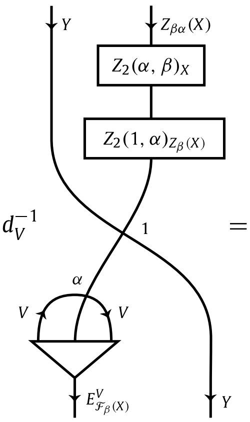
$$\overset{(iii)}{=} d_U^{-1} d_V^{-1}$$



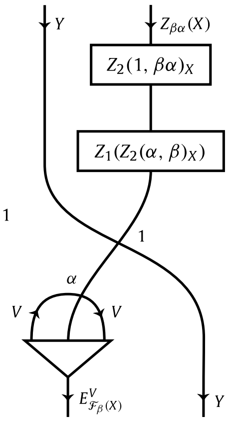
$\overset{(iv)}{=} d_U^{-1} d_V^{-1}$ 

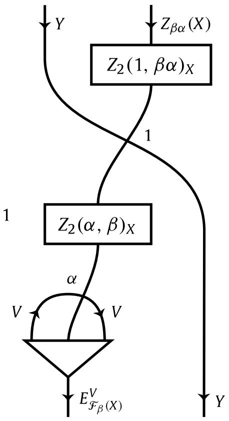
$\overset{(v)}{=} d_U^{-1} d_V^{-1}$ 

$\overset{(vi)}{=} d_V^{-1}$ 

$\overset{(vii)}{=} d_V^{-1}$ 

$= \sigma^{\beta\alpha}_{X,Y}(\omega^{V,\beta}_X \otimes \mathrm{id}_Y)$

$\overset{(viii)}{=} d_V^{-1}$ 

$\overset{(ix)}{=} d_V^{-1}$ 

$= \sigma^{\beta\alpha}_{X,Y}(\omega^{V,\beta}_X \otimes \mathrm{id}_Y).$

Here, the equality *(i)* is obtained by applying (23) and then (22), *(ii)* follows from the definition of σ^β_X, *(iii)* and *(vi)* from the naturality of ∂^α, *(iv)* and *(viii)* from the monoidality of Z, *(v)* from the definition of $Z_2(\alpha, 1)$, *(vii)* from the definition of $Z_2(1, \alpha)$, and (*ix*) from the naturality of ∂^1.

We now prove the claims of the lemma. Remark first that

$$\lambda^{V,V,\beta}_{X,Y} = \gamma^V_{\mathcal{F}_\beta(X),Y}, \qquad \lambda^{U,V,\beta}_{X,\mathbb{1}} = \delta^{U,V}_{\mathcal{F}_\beta(X)}, \quad \text{and} \quad \sigma^{\beta\alpha}_{X,\mathbb{1}} = \mathrm{id}_{Z_{\beta\alpha}(X)}.$$

Thus (41) for $U = V$ gives that $\omega_X^{U,\beta}$ is a morphism in $\mathcal{Z}_G(\mathcal{C})$. Since it is an isomorphism in $\mathcal{C}$ and the forgetful functor $\mathcal{U} \colon \mathcal{Z}_G(\mathcal{C}) \to \mathcal{C}$ reflects isomorphisms, we obtain that $\omega_X^{U,\beta}$ is a isomorphism in $\mathcal{Z}_G(\mathcal{C})$, which is claim (a). Finally, Equation (41) for $Y = \mathbb{1}$ is nothing but claim (b). This completes the proof of Lemma 10.5. □

By Lemma 10.5, the system $(\omega_X^V)_{V\in\mathcal{E}_\alpha}$ induces an isomorphism in $\mathcal{Z}_G(\mathcal{C})$,

$$\omega_X^{\alpha,\beta} \colon \varphi_\alpha \mathcal{F}_\beta(X) \to \mathcal{F}_{\beta\alpha}(X),$$

related to the universal cone ι^α of (20) by

$$\begin{tikzcd}
\varphi_\alpha \mathcal{F}_\beta(X) \arrow[rr, "(\iota_V^\alpha)_{\mathcal{F}_\beta(X)}"] \arrow[dr, "\omega_X^{\alpha,\beta}"'] & & \varphi_V \mathcal{F}_\beta(X) \arrow[dl, "\omega_X^{V,\beta}"] \\
& \mathcal{F}_{\beta\alpha}(X) &
\end{tikzcd}$$

It remains to check that the family

$$\omega^{\alpha,\beta} = \{\omega_X^{\alpha,\beta} \colon \varphi_\alpha \mathcal{F}_\beta(X) \to \mathcal{F}_{\beta\alpha}(X)\}_{X\in\mathcal{C}}$$

is a comonoidal natural transformation and that the family $\{\omega^{\alpha,\beta}\}_{\alpha,\beta\in G}$ is compatible with the monoidal structure of φ . We only need to verify that for all $\alpha, \beta, \gamma \in G$, $U \in \mathcal{E}_\alpha$, $V \in \mathcal{E}_\beta$, $W \in \mathcal{E}_{\beta\alpha}$, $R \in \mathcal{E}_1$, and any morphism $f \colon X \to Y$ in $\mathcal{C}$, the following equalities hold:

$$Z_{\beta\alpha}(f)\omega_X^{U,\beta} = \omega_Y^{U,\beta}\varphi_U V(Z_\beta(f)),$$

$$(Z_{\beta\alpha})_2(X,Y)\omega_{X\otimes Y}^{U,\beta} = (\omega_X^{U,\beta} \otimes \omega_Y^{U,\beta})(\varphi_U)_2(Z_\beta(X), Z_\beta(Y))^{-1}\varphi_U((Z_\beta)_2(X,Y)),$$

$$(Z_{\beta\alpha})_0\omega_{\mathbb{1}}^{U,\beta} = (\varphi_U)_0^{-1}\varphi_U((Z_\beta)_0),$$

$$\omega_X^{W,\gamma}\xi_{\mathcal{F}_\gamma(X)}^{U,V,W} = \omega_X^{U,\gamma\beta}\varphi_U(\omega_X^{V,\gamma}),$$

$$\omega_X^{R,1} = \eta_{\mathcal{F}_\alpha(X)}^R.$$

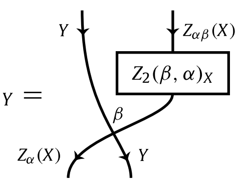
These equalities are verified via graphical computations similar to those above using the definitions of $\varphi_V(f)$, $(\varphi_V)_2^{-1}$, $(\varphi_V)_0^{-1}$, $\xi^{U,V,W}$, η^R given in Sections 4.4 and 4.6. □

We next compute the enhanced G-braiding $\{\tau_{(A,\sigma),Y}\}_{(A,\sigma)\in\mathcal{Z}_G(\mathcal{C}),Y\in\mathcal{C}_{\text{hom}}}$ of $\mathcal{Z}_G(\mathcal{C})$ on the G-free objects.

Theorem 10.6. *Let $\alpha, \beta \in G$, $X \in \mathcal{C}$, and $Y \in \mathcal{C}_\beta$. Then*

$$\tau_{\mathcal{F}_\alpha(X),Y} = \big(\mathrm{id}_Y \otimes \big(\omega_X^{\beta,\alpha}\big)^{-1} Z_2(\beta,\alpha)_X\big)\partial^\beta_{Z_\alpha(X),Y},$$

where ∂^β is defined in (38), that is,

$$\big(\mathrm{id}_Y \otimes \omega_X^{\beta,\alpha}\big)\tau_{\mathcal{F}_\alpha(X),Y} =$$ 

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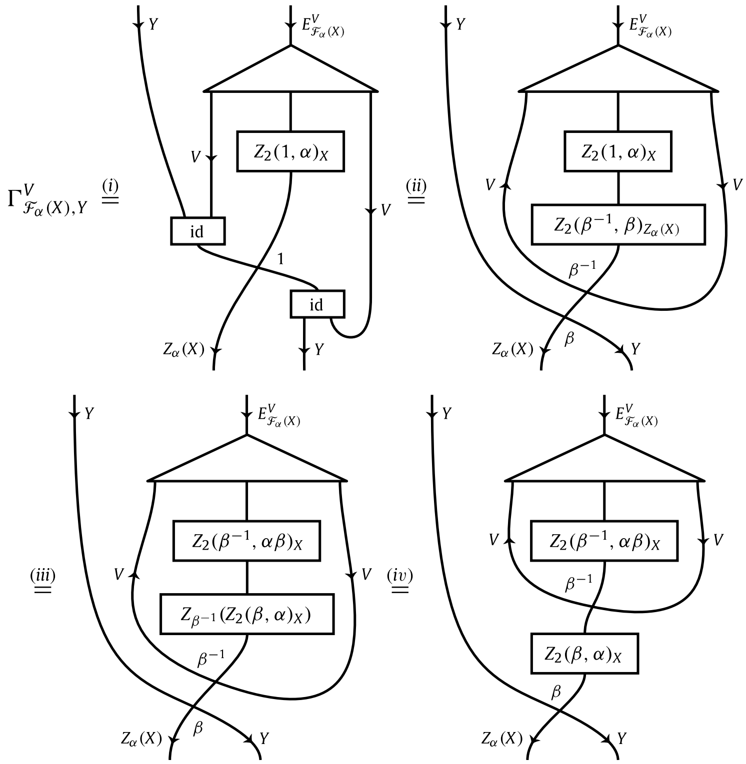
Proof. For $V \in \mathcal{E}_\beta$, recall notation Γ^V from Section 4.5. We have



$$= \big(\mathrm{id}_Y \otimes \big(\omega_X^{V,\alpha}\big)^{-1} Z_2(\beta,\alpha)_X\big)\partial^{\beta}_{Z_\alpha(X),Y}.$$

Here, the equality *(i)* follows from the definition of Γ^V, *(ii)* from the definition of $Z_2(\beta^{-1},\beta)$, *(iii)* from the monoidality of Z, and *(iv)* from the naturality of $\partial^{\beta^{-1}}$. Composing the above equality on the left with $(\mathrm{id}_Y \otimes (\iota_V^\beta)_{\mathcal{F}_\alpha(X)})$, where ι^β is the universal cone (20), we obtain the claim of the theorem. $\square$

10.3. Remark

Let ∇ be the following composition of monoidal functors:

$$\mathrm{Aut}(\mathcal{Z}_G(\mathcal{C})) \xrightarrow{\ \simeq\ } \mathrm{Aut}(\mathcal{C}^{Z_1}) \longrightarrow \mathrm{End}_{\mathrm{co}\otimes}(\mathcal{C}^{Z_1}) \xrightarrow{?\rtimes Z_1} \mathrm{End}_{\mathrm{co}\otimes}(\mathcal{C}).$$

Here the first arrow is the strict monoidal isomorphism induced by $\Psi\colon \mathcal{C}^{Z_1} \to \mathcal{Z}_G(\mathcal{C})$, the second arrow is the strict monoidal functor acting as $(F, F_2, F_0) \mapsto (F, F_2^{-1}, F_0^{-1})$ on the objects and as the identity on the morphisms, and the third arrow is the functor (32) with $T = Z_1$. Both $\nabla\varphi$ and Z are monoidal functors $\overline{G} \to \mathrm{End}_{\mathrm{co}\otimes}(\mathcal{C})$, and $\nabla\varphi(\alpha) = \mathcal{U}\varphi_\alpha\mathcal{F}_1$ for any $\alpha \in G$. The comonoidal isomorphism $\omega^{\alpha,1}\colon \varphi_\alpha\mathcal{F}_1 \to \mathcal{F}_\alpha$ of Theorem 10.4 induces a comonoidal isomorphism $\mathcal{U}(\omega^{\alpha,1})\colon \nabla\varphi(\alpha) = \mathcal{U}\varphi_\alpha\mathcal{F}_1 \to \mathcal{U}\mathcal{F}_\alpha = Z_\alpha$. The second statement of Theorem 10.4 implies that the family $\{\mathcal{U}(\omega^{\alpha,1})\}_{\alpha\in G}$ is a monoidal natural isomorphism $\nabla\varphi \simeq Z$.

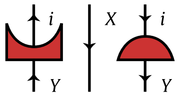
10.4. The case of G-fusion categories

Let $\mathcal{C}$ be a G-fusion category (over $\Bbbk$). It is clear that $\mathcal{C}$ satisfies all the assumptions of this section. We give here explicit formulas for the functor Z of Lemma 10.1.

Fix a G-representative set I of simple objects of $\mathcal{C}$ such that $\mathbb{1} \in I$. Note that $I = \amalg_{\alpha\in G} I_\alpha$ where I_α is the set of all elements of I belonging to $\mathcal{C}_\alpha$. By Lemma 8.1,

$$Z_\alpha(X) = \bigoplus_{i\in I_\alpha} i^* \otimes X \otimes i,$$

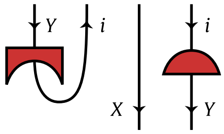
with universal dinatural transformation

$$\rho^{\alpha}_{X,Y} = \sum_{i\in I_\alpha}$$ 

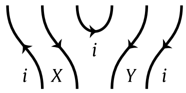
$$: Y^* \otimes X \otimes Y \to Z_\alpha(X) \quad \text{for} \quad Y \in \mathcal{C}_\alpha.$$

For any morphism f in $\mathcal{C}$, we have $Z_\alpha(f) = \sum_{i\in I_\alpha} \mathrm{id}_{i^*} \otimes f \otimes \mathrm{id}_i$.

The natural transformations ∂^α of (38) are given, for $X \in \mathcal{C}$ and $Y \in \mathcal{C}_\alpha$, by

$$\partial^{\alpha}_{X,Y} = \sum_{i\in I_\alpha}$$ 

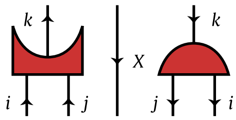
Then we deduce from Fig. 3 that the comonoidal structure of Z_α and the monoidal structure of Z are given, for $\alpha, \beta \in G$ and $X, Y \in \mathcal{C}$, by

$$(Z_\alpha)_2(X,Y) = \sum_{i\in I_\alpha}$$ 

$$: Z_\alpha(X\otimes Y) \to Z_\alpha(X) \otimes Z_\alpha(Y),$$

$$(Z_\alpha)_0 = \sum_{i\in I_\alpha}$$ 

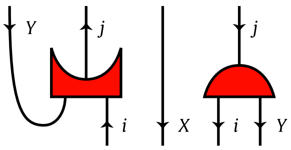
$$: Z_\alpha(\mathbb{1}) \to \mathbb{1},$$

$$Z_2(\alpha,\beta)_X = \sum_{\substack{i\in I_\alpha\\ j\in I_\beta\\ k\in I_{\beta\alpha}}}$$ 

$$: Z_\alpha Z_\beta(X) \to Z_{\beta\alpha}(X),$$

$$(Z_0)_X = \mathrm{id}_X : X \to \mathbb{1}^* \otimes X \otimes \mathbb{1} \hookrightarrow Z_1(X).$$

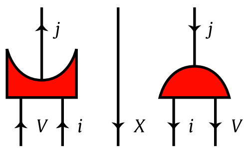
From these formulas, we deduce that for $\alpha \in G$, the α-free functor $\mathcal{F}_\alpha : \mathcal{C} \to \mathcal{Z}_G(\mathcal{C})$ carries any $X \in \mathcal{C}$ to $\mathcal{F}_\alpha(X) = (Z_\alpha(X), \sigma^\alpha_X)$ where for $Y \in \mathcal{C}_1$,

$$\sigma^{\alpha}_{X,Y} = \sum_{i,j\in I_\alpha}$$ 

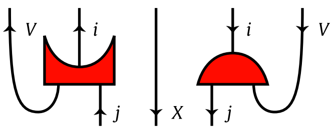
$$: Z_\alpha(X) \otimes Y \to Y \otimes Z_\alpha(X).$$

The functor $\mathcal{F}_\alpha$ carries any morphism f in $\mathcal{C}$ to $\mathcal{F}_\alpha(f) = Z_\alpha(f)$.

Let $\alpha, \beta \in G$ and $X \in \mathcal{C}$. For any $V \in \mathcal{C}_\alpha$ with invertible left dimension, we always may choose $E^V_{\mathcal{F}_\alpha(X)}$ to be the object $Z_{\alpha\beta}(X)$ and the morphisms $p^V_{\mathcal{F}_\alpha(X)}, q^V_{\mathcal{F}_\alpha(X)}$ of (19) to be

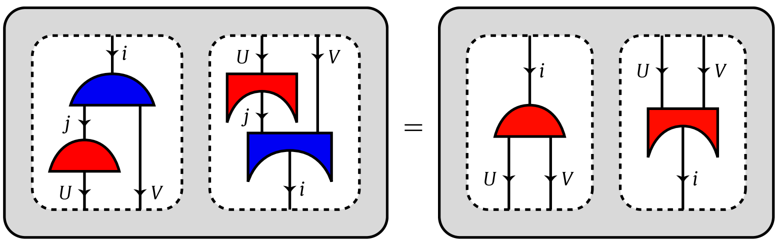
$$p^V_{\mathcal{F}_\alpha(X)} = d_V^{-1} \sum_{\substack{i\in I_\alpha\\ j\in I_{\alpha\beta}}}$$ 

$$,$$

$$q^V_{\mathcal{F}_\alpha(X)} = \sum_{\substack{i\in I_\alpha\\ j\in I_{\alpha\beta}}}$$ 

$$.$$

This can be verified using (35) and the following equality: for all $U, V \in \mathcal{C}$ and $i \in I$,

$$\sum_{j\in I}$$ 

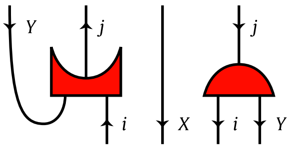
$$, \tag{42}$$

which graphically reflects the fact that

$$\mathrm{Hom}_{\mathcal{C}}(U \otimes V, i) = \bigoplus_{j \in I} \mathrm{Hom}_{\mathcal{C}}(X, j) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(j \otimes V, i).$$

With the above choices, we obtain that $\varphi_\beta \mathcal{F}_\alpha = \mathcal{F}_{\alpha\beta}$ as comonoidal functors. Also, the isomorphism $\omega_X^{\beta,\alpha}$ of Theorem 10.4 is the identity, and for any $Y \in \mathcal{C}_\beta$,

$$\tau_{\mathcal{F}_\alpha(X),Y} = \sum_{\substack{i \in I_\alpha \\ j \in I_{\alpha\beta}}}$$



.

Appendix. The object $(C_{\alpha,\beta}, \sigma^{\alpha,\beta})$

Let $\mathcal{C}$ be a non-singular G-centralizable G-graded pivotal category. Let $\varphi\colon \overline{G} \to \mathrm{Aut}(\mathcal{Z}_G(\mathcal{C}))$ be the crossing provided by Theorem 4.1 and $Z\colon \overline{G} \to \mathrm{End}_{\mathrm{co}\otimes}(\mathcal{C})$ be the monoidal functor of Lemma 10.1. Consider $\alpha, \beta \in G$ such that the following coend exists in $\mathcal{C}$:

$$C_{\alpha,\beta} = \int^{Y \in \mathcal{C}_\beta} Z_\alpha(Y)^* \otimes Y.$$

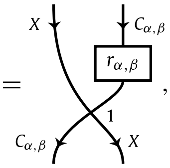
Here we lift $C_{\alpha,\beta}$ to an object of $\mathcal{Z}_G(\mathcal{C})$ in a canonical way. The latter object plays an important role in 3-dimensional HQFT, as will be discussed elsewhere.

Let $\varrho^{\alpha,\beta} = \{\varrho_Y^{\alpha,\beta}\colon Z_\alpha(Y)^* \otimes Y \to C_{\alpha,\beta}\}_{Y \in \mathcal{C}_\beta}$ be the universal dinatural transformation associated with $C_{\alpha,\beta}$. By [2, Corollary 3.8], since Z_1 is a Hopf monad, the dinatural transformation $Z_1(\varrho^{\alpha,\beta})$ is universal. Therefore there is a unique morphism $r_{\alpha,\beta}\colon Z_1(C_{\alpha,\beta}) \to C_{\alpha,\beta}$ such that for all $Y \in \mathcal{C}_\beta$,

$$r_{\alpha,\beta} Z_1(\varrho_Y^{\alpha,\beta}) = \varrho_{Z_1(Y)}^{\alpha,\beta}\big(Z_2(\alpha,1)_Y^* s^l_{Z_\alpha(Y)} Z_1(Z_2(1,\alpha)_Y^*) \otimes \mathrm{id}_{Z_1(Y)}\big)(Z_1)_2(Z_\alpha(Y)^*, Y), \tag{43}$$

where s^l is the left antipode of Z_1 (given by Lemma 8.4(a) for $\mathcal{D} = \mathcal{C}_1$). Define $\sigma^{\alpha,\beta} = \{\sigma_X^{\alpha,\beta}\colon C_{\alpha,\beta} \otimes X \to X \otimes C_{\alpha,\beta}\}_{X \in \mathcal{C}_1}$ by

$$\sigma_X^{\alpha,\beta} = (\mathrm{id}_X \otimes r_{\alpha,\beta})\partial^1_{C_{\alpha,\beta},X} =$$



,

where ∂^1 is defined in (38).

Theorem A.1. *$(C_{\alpha,\beta}, \sigma^{\alpha,\beta})$ is an object of $\mathcal{Z}_G(\mathcal{C})$ lying in $\mathcal{Z}_{\alpha^{-1}\beta^{-1}\alpha\beta}(\mathcal{C})$ and*

$$(C_{\alpha,\beta}, \sigma^{\alpha,\beta}) = \int^{(A,\sigma) \in \mathcal{Z}_\beta(\mathcal{C})} (\varphi_\alpha(A,\sigma))^* \otimes (A,\sigma).$$

Proof. Let F_α, Ψ, and $\mathcal{F}_\alpha = \Psi F_\alpha$ be as in Section 10.1. Denote by $U_1\colon \mathcal{C}^{Z_1} \to \mathcal{C}$ and $\mathcal{U}\colon \mathcal{Z}_G(\mathcal{C}) \to \mathcal{C}$ the forgetful functors. Set

$$Q = \Psi^{-1}\varphi_\alpha\Psi\colon \mathcal{C}^{Z_1} \to \mathcal{C}^{Z_1}.$$

Observe that $Q \rtimes Z_1 = U_1\Psi^{-1}\varphi_\alpha\Psi F_1 = \mathcal{U}\varphi_\alpha\mathcal{F}_1$. Theorem 10.4 provides a comonoidal natural isomorphism $\omega^{\alpha,1} = \{\omega_Y^{\alpha,1}\colon \varphi_\alpha\mathcal{F}_1(Y) \to \mathcal{F}_\alpha(Y)\}_{Y \in \mathcal{C}}$. Since

$$\{\omega_Y^{\alpha,1} = \mathcal{U}(\omega_Y^{\alpha,1})\colon (Q \rtimes Z_1)(Y) \to Z_\alpha(Y)\}_{Y \in \mathcal{C}_\beta}$$

is then a natural isomorphism,

$$C_{\alpha,\beta} = \int^{Y \in \mathcal{C}_\beta} Z_\alpha(Y)^* \otimes Y = \int^{Y \in \mathcal{C}_\beta} (Q \rtimes Z_1)(Y)^* \otimes Y,$$

with associated dinatural transformation $j = \{j_Y\colon (Q \rtimes Z_1)(Y)^* \otimes Y \to C_{\alpha,\beta}\}_{Y \in \mathcal{C}_\beta}$ given by

$$j_Y = \varrho_Y^{\alpha,\beta}\big(((\omega_Y^{\alpha,1})^{-1})^* \otimes \mathrm{id}_Y\big).$$

Therefore, by Lemma 8.3 applied to $\mathcal{D} = \mathcal{C}_\beta$ and $T = Z_1$, we deduce that

$$(C_{\alpha,\beta}, c) = \int^{(A,r)\in \mathcal{C}_\beta^{Z_1}} (Q(A,r))^* \otimes (A,r),$$

where c is some Z_1-action on $C_{\alpha,\beta}$. Since $\Psi : \mathcal{C}^{Z_1} \to \mathcal{Z}_G(\mathcal{C})$ is a grading-preserving strict monoidal isomorphism of G-graded categories,

$$\int^{(A,\sigma)\in \mathcal{Z}_\beta(\mathcal{C})} (\varphi_\alpha(A,\sigma))^* \otimes (A,\sigma) = \Psi\left(\int^{(A,r)\in \mathcal{C}_\beta^{Z_1}} (\Psi^{-1}\varphi_\alpha\Psi(A,r))^* \otimes (A,r)\right)$$

$$= \Psi\left(\int^{(A,r)\in \mathcal{C}_\beta^{Z_1}} (Q(A,r))^* \otimes (A,r)\right) = \Psi(C_{\alpha,\beta}, c).$$

Since $\Psi(C_{\alpha,\beta}, r_{\alpha,\beta}) = (C_{\alpha,\beta}, \sigma^{\alpha,\beta})$, we only need to prove that $c = r_{\alpha,\beta}$. The dinatural transformation $Z_1(\varrho^{\alpha,\beta})$ being universal, this is equivalent to proving that $cZ_1(\varrho_Y^{\alpha,\beta}) = r_{\alpha,\beta}Z_1(\varrho_Y^{\alpha,\beta})$ for all $Y \in \mathcal{C}_\beta$. Fix $Y \in \mathcal{C}_\beta$. Let a_Y is the Z_1-action of $QF_1(Y) = \Psi^{-1}\varphi_\alpha\Psi F_1(Y) = \Psi^{-1}\varphi_\alpha\mathcal{F}_1(Y)$. Since $\omega_Y^{\alpha,1}$ is a morphism is $\mathcal{Z}_G(\mathcal{C})$, the morphism $\Psi^{-1}(\omega_Y^{\alpha,1}) = \omega_Y^{\alpha,1} : QF_1(Y) \to F_\alpha(Y)$ is a morphism in $\mathcal{C}^{Z_1}$, that is,

$$\omega_Y^{\alpha,1} a_Y = Z_2(1,\alpha)_Y Z_1(\omega_Y^{\alpha,1}), \quad \text{and so} \quad a_Y = \left(\omega_Y^{\alpha,1}\right)^{-1} Z_2(1,\alpha)_Y Z_1(\omega_Y^{\alpha,1}).$$

By Lemma 8.3, the Z_1-action $c : Z_1(C_{\alpha,\beta}) \to C_{\alpha,\beta}$ satisfies

$$cZ_1(j_Y) = \iota_{Z_1(Y)}\big(Q(Z_2(1,1)_Y)^* s^l_{Q\rtimes Z_1(Y)} Z_1(a_Y^*) \otimes \mathrm{id}_{Z_1(Y)}\big) Z_2((Q \rtimes T)(Y)^*, Y).$$

Composing this equality on the right with $Z_1((\omega_Y^{\alpha,1})^* \otimes \mathrm{id}_Y)$, using the above expression of a_Y and the naturality of s^l, we obtain that

$$cZ_1(\varrho_Y^{\alpha,\beta}) = \varrho^{\alpha,\beta}_{Z_1(Y)}\big(\Delta^* s^l_{Z_\alpha(Y)} Z_1(Z_2(1,\alpha)_Y^*) \otimes \mathrm{id}_{Z_1(Y)}\big)(Z_1)_2(Z_\alpha(Y)^*, Y)$$

where $\Delta = \omega_Y^{\alpha,1} Q(Z_2(1,1)_Y)\big(\omega^{\alpha,1}_{Z_1(Y)}\big)^{-1}$. Pick $V \in \mathcal{E}_\alpha$. Since Ψ acts as the identity on morphisms and using that $\omega^{\alpha,1} = (\iota^\alpha)^{-1}\omega^{V,1}$, and $\varphi_\alpha = (\iota^\alpha)^{-1}\varphi_V\, \iota^\alpha$, where ι^α is the universal cone (20), we obtain

$$\Delta = \omega_Y^{\alpha,1}\varphi_\alpha(Z_2(1,1)_Y)\big(\omega^{\alpha,1}_{Z_1(Y)}\big)^{-1} = \omega_Y^{V,1}\varphi_V(Z_2(1,1)_Y)\big(\omega^{V,1}_{Z_1(Y)}\big)^{-1}.$$

In view of the definition of $r_{\alpha,\beta}$ given in (43), to prove that $cZ_1(\varrho_Y^{\alpha,\beta}) = r_{\alpha,\beta}Z_1(\varrho_Y^{\alpha,\beta})$, it is enough to prove that $\Delta = Z_2(\alpha,1)_Y$, i.e., that $\omega_Y^{V,1}\varphi_V(Z_2(1,1)_Y) = Z_2(\alpha,1)_Y\omega^{V,1}_{Z_1(Y)}$. Denote by ρ^α the universal dinatural transformation (37) and let π^V, p^V, q^V be as in (19). Recall from (39) that $\omega_X^{V,1} = d_V^{-1} Z_2(\alpha,1)_X \rho^\alpha_{Z_1(X),V} q^V_{\mathcal{F}_1(X)}$ for any $X \in \mathcal{C}$. Now, by Lemma 10.3, $Z_2(1,1)_Y$ is a morphism in $\mathcal{Z}_G(\mathcal{C})$ from $\mathcal{F}_1(Z_1(Y))$ to $\mathcal{F}_1(Y)$. This implies that

$$\pi^V_{\mathcal{F}_1(Y)}(\mathrm{id}_{V^*} \otimes Z_2(1,1)_Y \mathrm{id}_Y) = (\mathrm{id}_{V^*} \otimes Z_2(1,1)_Y\mathrm{id}_Y)\pi^V_{\mathcal{F}_1(Z_1(Y))}. \tag{44}$$

Then

$$\begin{aligned}
\omega_Y^{V,1}\varphi_V(Z_2(1,1)_Y) &\overset{(i)}{=} d_V^{-1}Z_2(\alpha,1)_Y\rho^\alpha_{Z_1(Y),V} q^V_{\mathcal{F}_1(Y)} p^V_{\mathcal{F}_1(Y)}(\mathrm{id}_{V^*} \otimes Z_2(1,1)_Y\mathrm{id}_Y) q^V_{\mathcal{F}_1(Z_1(Y))}\\
&\overset{(ii)}{=} d_V^{-1}Z_2(\alpha,1)_Y\rho^\alpha_{Z_1(Y),V}\pi^V_{\mathcal{F}_1(Y)}(\mathrm{id}_{V^*} \otimes Z_2(1,1)_Y\mathrm{id}_Y) q^V_{\mathcal{F}_1(Z_1(Y))}\\
&\overset{(iii)}{=} d_V^{-1}Z_2(\alpha,1)_Y\rho^\alpha_{Z_1(Y),V}(\mathrm{id}_{V^*} \otimes Z_2(1,1)_Y\mathrm{id}_Y)\pi^V_{\mathcal{F}_1(Z_1(Y))} q^V_{\mathcal{F}_1(Z_1(Y))}\\
&\overset{(iv)}{=} d_V^{-1}Z_2(\alpha,1)_Y Z_\alpha(Z_2(1,1)_Y)\rho^\alpha_{Z_1^2(Y),V} q^V_{\mathcal{F}_1(Z_1(Y))}\\
&\overset{(v)}{=} d_V^{-1}Z_2(\alpha,1)_Y Z_2(\alpha,1)_{Z_1(Y)}\rho^\alpha_{Z_1^2(Y),V} q^V_{\mathcal{F}_1(Z_1(Y))}\\
&= Z_2(\alpha,1)_Y\omega^{V,1}_{Z_1(Y)}.
\end{aligned}$$

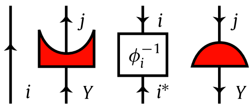
Here, the equality *(i)* follows from the definition of φ_V (see Section 4.4), *(ii)* from (19), *(iii)* from (44), *(iv)* from (19) and the naturality of ρ^α, and *(v)* from the monoidality of Z. This completes the proof of the theorem. □

We can explicitly compute $(C_{\alpha,\beta}, \sigma^{\alpha,\beta})$ in the case where $\mathcal{C}$ is a G-fusion category. Fix a G-representative set $I = \amalg_{\alpha \in G} I_\alpha$ of simple objects of $\mathcal{C}$. By Lemma 8.1,

$$C_{\alpha,\beta} = \bigoplus_{\substack{i \in I_\alpha \\ j \in I_\beta}} i^* \otimes j^* \otimes i \otimes j \tag{45}$$

with universal dinatural transformation given, for $Y \in \mathcal{C}_\beta$, by

$$\varrho_Y^{\alpha,\beta} = \sum_{\substack{i \in I_\alpha \\ j \in I_\beta}}$$



$$: Z_\alpha(Y)^* \otimes Y \to C_{\alpha,\beta},$$

where $\phi = \{\phi_X : X \to X^{**}\}_{X \in \mathcal{C}}$ is the pivotal structure of $\mathcal{C}$; see (3). For $X \in \mathcal{C}$, set

$$\zeta_X^\alpha = \sum_{i \in I_\alpha} \mathrm{id}_{i^* \otimes X^*} \otimes \phi_i : Z_\alpha(X^*) \to Z_\alpha(X)^*,$$

so that

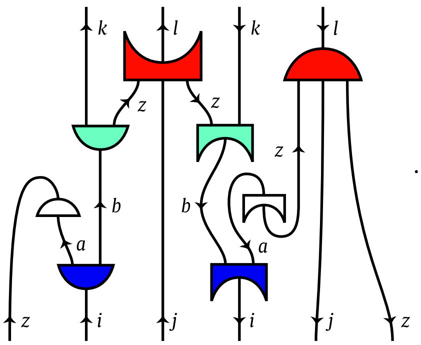
$$\mathrm{id}_{C_{\alpha,\beta}} = \sum_{j \in I_\beta} \varrho_j^{\alpha,\beta}(\zeta_j^\alpha \otimes \mathrm{id}_j).$$

Then using (43), the above description of Z, and the description of s^l given in Fig. 2, we obtain that

$$r_{\alpha,\beta} = r_{\alpha,\beta} Z_1(\mathrm{id}_{C_{\alpha,\beta}}) = \sum_{j \in I_\beta} r_{\alpha,\beta} Z_1(\varrho_j^{\alpha,\beta}) Z_1(\zeta_j^\alpha \otimes \mathrm{id}_j)$$

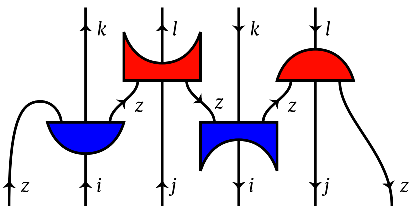
$$= \sum_{j \in I_\beta} \varrho^{\alpha,\beta}_{Z_1(j)} \big(Z_2(\alpha,1)^*_j s^l_{Z_\alpha(j)} Z_1(Z_2(1,\alpha)^*_j) \otimes \mathrm{id}_{Z_1(j)}\big)(Z_1)_2(Z_\alpha(j)^*, j) Z_1(\zeta_j^\alpha \otimes \mathrm{id}_j)$$

$$= \sum_{\substack{i,k,b \in I_\alpha \\ j,l \in I_\beta \\ z,a \in I_1}}$$



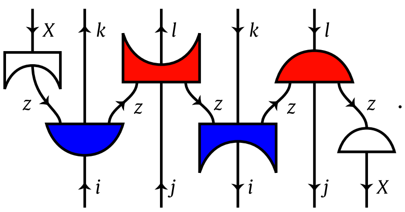
Finally, using (34) and (42), we obtain

$$r_{\alpha,\beta} = \sum_{\substack{i,k, \in I_\alpha \\ j,l \in I_\beta \\ z \in I_1}}$$



Hence $\sigma^{\alpha,\beta} = \{\sigma_X^{\alpha,\beta} = (\mathrm{id}_X \otimes r_{\alpha,\beta})\partial^1_{C_{\alpha,\beta},X} : C_{\alpha,\beta} \otimes X \to X \otimes C_{\alpha,\beta}\}_{X \in \mathcal{C}_1}$ is given by

$$\sigma_X^{\alpha,\beta} = \sum_{\substack{i,k \in I_\alpha \\ j,l \in I_\beta \\ z \in I_1}}$$



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