International Journal of Mathematics
Vol. 23, No. 9 (2012) 1250094 (28 pages)
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DOI: 10.1142/S0129167X12500942

ON 3-DIMENSIONAL HOMOTOPY QUANTUM FIELD THEORY, I

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Received 27 February 2012
Accepted 4 April 2012
Published 24 July 2012

Given a discrete group G and a spherical G-fusion category whose neutral component has invertible dimension, we use the state-sum method to construct a 3-dimensional Homotopy Quantum Field Theory with target the Eilenberg–MacLane space $K(G, 1)$.

Keywords: State-sum invariants; HQFT; graded categories; fusion categories.

Mathematics Subject Classification 2010: 57M27, 18D10, 57R56

1. Introduction

Homotopy quantum field theory (HQFT) is a branch of quantum topology concerned with maps from manifolds to a fixed target space. The aim is to define and to study homotopy invariants of such maps using methods of quantum topology. A formal notion of an HQFT was introduced in the monograph [12] which treats in detail the 2-dimensional case. The present paper focuses on 3-dimensional HQFTs with target the Eilenberg–MacLane space $K(G, 1)$ where G is a discrete group. These HQFTs generalize more familiar 3-dimensional topological quantum field theories (TQFTs) which correspond to the case $G = 1$.

Two fundamental constructions of 3-dimensional TQFTs are due to Reshetikhin–Turaev and Turaev–Viro. The RT-construction may be viewed as a mathematical realization of Witten's Chern–Simons TQFT. The TV-construction is closely related to the Ponzano–Regge state-sum model for 3-dimensional quantum gravity. The Turaev–Viro [14] state-sum approach, as generalized by Barrett and Westbury [3] (see also [5]), derives TQFTs from spherical fusion categories. In this paper, we extend the state-sum approach to the setting of HQFTs. Specifically, we show that any spherical G-fusion category $\mathcal{C}$ (of invertible dimension) gives rise to a 3-dimensional HQFT $|\cdot|_{\mathcal{C}}$ with target $K(G,1)$.

As in the above mentioned papers, we represent 3-manifolds by their skeletons. The maps to $K(G,1)$ are represented by certain labels on the faces of the skeletons. The HQFTs are obtained by taking appropriate state-sums on the skeletons. In distinction to the earlier papers, we entirely avoid the use of $6j$-symbols and allow non-generic skeletons, i.e. skeletons with edges incident to ≥ 4 regions. In the case $G = 1$ this approach was introduced in [13].

This paper is the first in a series of papers in which we will establish the following further results. We will show that the Reshetikhin–Turaev surgery method also extends to HQFTs: any G-modular category $\mathcal{C}$ determines a 3-dimensional HQFT $\tau_{\mathcal{C}}$. We will generalize the center construction for categories to G-categories and show that the G-center $\mathcal{Z}_G(\mathcal{C})$ of a spherical G-fusion category $\mathcal{C}$ over an algebraically closed field of characteristic zero is a G-modular category. Finally, we will show that under these assumptions on $\mathcal{C}$, the HQFTs $|\cdot|_{\mathcal{C}}$ and $\tau_{\mathcal{Z}_G(\mathcal{C})}$ are isomorphic. This theorem is nontrivial already for $G = 1$. In this case it was first established by the present authors in [13] and somewhat later but independently by Balsam and Kirillov [6, 1, 2]. The case of an arbitrary G is considerably more difficult; it will be treated in the sequel.

The content of this paper is as follows. We recall the notion of a 3-dimensional HQFT in Sec. 2. Then we discuss various classes of monoidal categories and in particular G-fusion categories (Secs. 3 and 4). In Sec. 5 we consider symmetrized multiplicity modules in categories and invariants of planar graphs needed for our state-sums. In Sec. 6 we discuss skeletons of 3-manifolds and presentations of maps to $K(G,1)$ by labelings of skeletons. We use these presentations in Sec. 7 to derive from any G-fusion category a numerical invariant of maps from closed 3-manifolds to $K(G,1)$. In the final Sec. 8 we extend these numerical invariance to an HQFT with target $K(G,1)$. In the appendix we briefly discuss push-forwards of categories and HQFTs.

Throughout the paper, we fix a (discrete) group G and an Eilenberg–MacLane space $\mathbf{X}$ of type $K(G,1)$ with base point x. Thus, $\mathbf{X}$ is a connected aspherical CW-space such that $\pi_1(\mathbf{X},x) = G$. The symbol $\Bbbk$ will denote a commutative ring.

2. 3-Dimensional HQFTs

We recall following [12] the definition of a 3-dimensional homotopy quantum field theory (HQFT) with target $\mathbf{X} = K(G,1)$. Warning: our terminology here is adapted to the 3-dimensional case and differs from that in [12].

2.1. *Preliminaries on G-surfaces and G-manifolds*

A topological space Σ is *pointed* if every connected component of Σ is provided with a base point. The set of base points of Σ is denoted $\Sigma_\bullet$. By a *G-surface* we mean a pair (a pointed closed oriented smooth surface Σ, a homotopy class of maps $f:(\Sigma,\Sigma_\bullet)\to(\mathbf{X},x)$). Reversing orientation in a G-surface (Σ,f), we obtain a G-surface $(-\Sigma,f)$. By convention, an empty set is a G-surface with unique orientation.

By a *G-manifold* we mean a pair (a compact oriented smooth 3-dimensional manifold M with pointed boundary, a homotopy class of maps $f:(M,(\partial M)_\bullet)\to(\mathbf{X},x)$). The manifold M itself is not required to be pointed. The boundary $(\partial M, f|_{\partial M})$ of a G-manifold (M,f) is a G-surface. We use the "outward vector first" convention for the induced orientation of the boundary: at any point of ∂M the given orientation of M is determined by the tuple (a tangent vector directed outward, a basis in the tangent space of ∂M positive with respect to the induced orientation). A G-manifold M is *closed* if $\partial M=\emptyset$ (in this case $(\partial M)_\bullet=\emptyset$).

Any G-surface (Σ,f) determines the *cylinder G-manifold* $(\Sigma\times[0,1],\overline{f})$, where $\overline{f}:\Sigma\times[0,1]\to\mathbf{X}$ is the composition of the projection to Σ with f. Here $\Sigma\times[0,1]$ has the base points $\{(m,0),(m,1)\,|\,m\in\Sigma_\bullet\}$ and is oriented so that its oriented boundary is $(-\Sigma\times\{0\})\amalg(\Sigma\times\{1\})$.

Disjoint unions of G-surfaces (respectively G-manifolds) are G-surfaces (respectively G-manifolds) in the obvious way. A *G-homeomorphism of G-surfaces* $(\Sigma,f)\to(\Sigma',f')$ is an orientation preserving diffeomorphism $g:\Sigma\to\Sigma'$ such that $g(\Sigma_\bullet)=\Sigma'_\bullet$ and $f=f'g$. A *G-homeomorphism of G-manifolds* $(M,f)\to(M',f')$ is an orientation preserving diffeomorphism $g:M\to M'$ such that $g((\partial M)_\bullet)=(\partial M')_\bullet$ and $f=f'g$. In both cases, the equality $f=f'g$ is understood as an equality of homotopy classes of maps.

For brevity, we shall usually omit the maps to $\mathbf{X}$ from the notation for G-surfaces and G-manifolds.

2.2. *The category of G-cobordisms*

We define a category of *3-dimensional G-cobordisms* $\mathrm{Cob}^G=\mathrm{Cob}_3^G$. Objects of Cob^G are G-surfaces. A morphism $\Sigma_0\to\Sigma_1$ in Cob^G is represented by a pair (a G-manifold M, a G-homeomorphism $h:(-\Sigma_0)\sqcup\Sigma_1\simeq\partial M$). We call such pairs *G-cobordisms with bases* Σ_0 *and* Σ_1. Two G-cobordisms $(M,h:(-\Sigma_0)\sqcup\Sigma_1\to\partial M)$ and $(M',h':(-\Sigma_0)\sqcup\Sigma_1\to\partial M')$ represent the same morphism if there is a G-homeomorphism $g:M\to M'$ such that $h'=gh$. The identity morphism of a G-surface Σ is represented by the cylinder G-manifold $\Sigma\times[0,1]$ with tautological identification of the boundary with $(-\Sigma)\sqcup\Sigma$. Composition of morphisms in Cob^G is defined through gluing of G-cobordisms: the composition of morphisms $(M_0,h_0):\Sigma_0\to\Sigma_1$ and $(M_1,h_1):\Sigma_1\to\Sigma_2$ is represented by the G-cobordism (M,h), where M is the G-manifold obtained by gluing M_0 and M_1 along $h_1h_0^{-1}:h_0(\Sigma_1)\to h_1(\Sigma_1)$ and

$$h=h_0|_{\Sigma_0}\sqcup h_1|_{\Sigma_2}:(-\Sigma_0)\sqcup\Sigma_2\simeq\partial M.$$

The given maps $(M_i, (\partial M_i)_\bullet) \to (\mathbf{X}, x)$, $i = 0, 1$ may be chosen in their homotopy classes to agree on $h_0(\Sigma_1) \approx h_1(\Sigma_1)$ and define thus a map $(M, (\partial M)_\bullet) \to (\mathbf{X}, x)$. The asphericity of $\mathbf{X}$ ensures that the homotopy class of this map is well-defined.

The category Cob^G is a symmetric monoidal category with tensor product given by disjoint union of G-surfaces and G-manifolds. The unit object of Cob^G is the empty G-surface $\emptyset$.

2.3. *HQFTs*

Let $\mathrm{vect}_\Bbbk$ be the category of finitely generated projective $\Bbbk$-modules and $\Bbbk$-homomorphisms. It is a symmetric monoidal category with standard tensor product and unit object $\Bbbk$. A (3-dimensional) *homotopy quantum field theory* (*HQFT*) *with target* $\mathbf{X}$ is a symmetric strong monoidal functor $Z\colon \mathrm{Cob}^G \to \mathrm{vect}_\Bbbk$. In particular, $Z(\Sigma \sqcup \Sigma') \simeq Z(\Sigma) \otimes Z(\Sigma')$ for any G-surfaces Σ, Σ', and similarly for morphisms. Also, $Z(\emptyset) \simeq \Bbbk$. We refer to [7] for a detailed definition of a strong monoidal functor.

Every G-manifold M determines two morphisms $\emptyset \to \partial M$ and $-\partial M \to \emptyset$ in Cob^G. The associated homomorphisms $\Bbbk \simeq Z(\emptyset) \to Z(\partial M)$ and $Z(-\partial M) \to Z(\emptyset) \simeq \Bbbk$ are denoted $Z(M, \emptyset, \partial M)$ and $Z(M, -\partial M, \emptyset)$, respectively. If $\partial M = \emptyset$, then $Z(M, \emptyset, \partial M)\colon \Bbbk \to Z(\emptyset) \simeq \Bbbk$ and $Z(M - \partial M, \emptyset)\colon \Bbbk \simeq Z(\emptyset) \to \Bbbk$ are multiplication by the same element of $\Bbbk$ denoted $Z(M)$.

The category of G-cobordisms Cob^G includes as a subcategory the category Homeo^G of G-surfaces and their G-homeomorphisms considered up to isotopy (in the class of G-homeomorphisms). Indeed, a G-homeomorphism of G-surfaces $g\colon \Sigma \to \Sigma'$ determines a morphism $\Sigma \to \Sigma'$ in Cob^G represented by the pair $(C = \Sigma' \times [0,1], h\colon (-\Sigma) \sqcup \Sigma' \simeq \partial C)$, where $h(x) = (g(x), 0)$ for $x \in \Sigma$ and $h(x') = (x', 1)$ for $x' \in \Sigma'$. Isotopic G-homeomorphisms give rise to the same morphism in Cob^G. The category Homeo^G inherits a structure of a symmetric strong monoidal category from that of Cob^G. Restricting an HQFT $Z\colon \mathrm{Cob}^G \to \mathrm{vect}_\Bbbk$ to Homeo^G, we obtain a symmetric monoidal functor $\mathrm{Homeo}^G \to \mathrm{vect}_\Bbbk$. In particular, Z induces a $\Bbbk$-linear representation of the *mapping class group of a G-surface* Σ defined as the group of isotopy classes of G-homeomorphisms $\Sigma \to \Sigma$.

For $G = \{1\}$, the space $\mathbf{X}$ is just a point and without any loss of information we may forget the maps of surfaces and manifolds to $\mathbf{X}$. We recover thus the familiar notion of a 3-dimensional TQFT.

3. Preliminaries on Monoidal Categories

In this section we recall several basic definitions of the theory of monoidal categories needed for the sequel.

3.1. *Conventions*

The symbol $\mathcal{C}$ will denote a monoidal category with unit object $\mathbb{1}$. Notation $X \in \mathcal{C}$ will mean that X is an object of $\mathcal{C}$. To simplify the formulas, we will always pretend that $\mathcal{C}$ is strict. Consequently, we omit brackets in the tenor products and suppress the associativity constraints $(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$ and the unitality constraints $X \otimes \mathbb{1} \cong X \cong \mathbb{1} \otimes X$. By the tensor product $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ of $n \geq 2$ objects $X_1, \ldots, X_n \in \mathcal{C}$ we mean $(\cdots((X_1 \otimes X_2) \otimes X_3) \otimes \cdots \otimes X_{n-1}) \otimes X_n$.

3.2. *Pivotal and spherical categories*

Following [8], by a *pivotal* category we mean a monoidal category $\mathcal{C}$ endowed with a rule which assigns to each object $X \in \mathcal{C}$ a *dual object* $X^* \in \mathcal{C}$ and four morphisms

$$\mathrm{ev}_X : X^* \otimes X \to \mathbb{1}, \quad \mathrm{coev}_X : \mathbb{1} \to X \otimes X^*,$$
$$\widetilde{\mathrm{ev}}_X : X \otimes X^* \to \mathbb{1}, \quad \widetilde{\mathrm{coev}}_X : \mathbb{1} \to X^* \otimes X,$$

satisfying the following conditions:

(a) For every $X \in \mathcal{C}$, the triple $(X^*, \mathrm{ev}_X, \mathrm{coev}_X)$ is a left dual of X, i.e.

$$(\mathrm{id}_X \otimes \mathrm{ev}_X)(\mathrm{coev}_X \otimes \mathrm{id}_X) = \mathrm{id}_X \quad \text{and} \quad (\mathrm{ev}_X \otimes \mathrm{id}_{X^*})(\mathrm{id}_{X^*} \otimes \mathrm{coev}_X) = \mathrm{id}_{X^*};$$

(b) For every $X \in \mathcal{C}$, the triple $(X^*, \widetilde{\mathrm{ev}}_X, \widetilde{\mathrm{coev}}_X)$ is a right dual of X, i.e.

$$(\widetilde{\mathrm{ev}}_X \otimes \mathrm{id}_X)(\mathrm{id}_X \otimes \widetilde{\mathrm{coev}}_X) = \mathrm{id}_X \quad \text{and} \quad (\mathrm{id}_{X^*} \otimes \widetilde{\mathrm{ev}}_X)(\widetilde{\mathrm{coev}}_X \otimes \mathrm{id}_{X^*}) = \mathrm{id}_{X^*};$$

(c) For every morphism $f : X \to Y$ in $\mathcal{C}$, the *left dual*

$$f^* = (\mathrm{ev}_Y \otimes \mathrm{id}_{X^*})(\mathrm{id}_{Y^*} \otimes f \otimes \mathrm{id}_{X^*})(\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X) : Y^* \to X^*$$

is equal to the *right dual*

$$f^* = (\mathrm{id}_{X^*} \otimes \widetilde{\mathrm{ev}}_Y)(\mathrm{id}_{X^*} \otimes f \otimes \mathrm{id}_{Y^*})(\widetilde{\mathrm{coev}}_X \otimes \mathrm{id}_{Y^*}) : Y^* \to X^*;$$

(d) For all $X, Y \in \mathcal{C}$, the *left monoidal constraint*

$$(\mathrm{ev}_X \otimes \mathrm{id}_{(Y \otimes X)^*})(\mathrm{id}_{X^*} \otimes \mathrm{ev}_Y \otimes \mathrm{id}_{(Y \otimes X)^*})(\mathrm{id}_{X^* \otimes Y^*} \otimes \mathrm{coev}_{Y \otimes X}) :$$
$$X^* \otimes Y^* \to (Y \otimes X)^*$$

is equal to the *right monoidal constraint*

$$(\mathrm{id}_{(Y \otimes X)^*} \otimes \widetilde{\mathrm{ev}}_Y)(\mathrm{id}_{(Y \otimes X)^*} \otimes \widetilde{\mathrm{ev}}_X \otimes \mathrm{id}_{X^*})(\widetilde{\mathrm{coev}}_{Y \otimes X} \otimes \mathrm{id}_{X^* \otimes Y^*}) :$$
$$X^* \otimes Y^* \to (Y \otimes X)^*;$$

(e) $\mathrm{ev}_{\mathbb{1}} = \widetilde{\mathrm{ev}}_{\mathbb{1}} : \mathbb{1}^* \to \mathbb{1}$ (or, equivalently, $\mathrm{coev}_{\mathbb{1}} = \widetilde{\mathrm{coev}}_{\mathbb{1}} : \mathbb{1} \to \mathbb{1}^*$).

If $\mathcal{C}$ is pivotal, then for any endomorphism f of an object $X \in \mathcal{C}$, one defines the *left* and *right* traces

$$\mathrm{tr}_l(f) = \mathrm{ev}_X(\mathrm{id}_{X^*} \otimes f)\widetilde{\mathrm{coev}}_X \quad \text{and} \quad \mathrm{tr}_r(f) = \widetilde{\mathrm{ev}}_X(f \otimes \mathrm{id}_{X^*})\mathrm{coev}_X.$$

Both traces take values in $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$ and are symmetric: $\mathrm{tr}_l(gh) = \mathrm{tr}_l(hg)$ for any morphisms $g : X \to Y$, $h : Y \to X$ in $\mathcal{C}$ and similarly for tr_r. Also $\mathrm{tr}_l(f) = \mathrm{tr}_r(f^*) = \mathrm{tr}_l(f^{**})$ for any endomorphism f of an object. The *left* and *right* dimensions of an object $X \in \mathcal{C}$ are defined by $\dim_l(X) = \mathrm{tr}_l(\mathrm{id}_X)$ and $\dim_r(X) = \mathrm{tr}_r(\mathrm{id}_X)$. Clearly, $\dim_l(X) = \dim_r(X^*) = \dim_l(X^{**})$ for all X.

When $\mathcal{C}$ is pivotal, we will suppress the duality constraints $\mathbb{1}^* \cong \mathbb{1}$ and $X^* \otimes Y^* \cong (Y \otimes X)^*$. For example, we will write $(f \otimes g)^* = g^* \otimes f^*$ for morphisms f, g in $\mathcal{C}$.

A pivotal category $\mathcal{C}$ is *spherical* if the left and right traces of endomorphisms in $\mathcal{C}$ coincide. Set then $\mathrm{tr}(f) = \mathrm{tr}_l(f) = \mathrm{tr}_r(f)$ for any endomorphism f of an object of $\mathcal{C}$, and $\dim(X) = \dim_l(X) = \dim_r(X) = \mathrm{tr}(\mathrm{id}_X)$ for any $X \in \mathcal{C}$.

3.3. *Additive categories*

A category $\mathcal{C}$ is $\Bbbk$-*additive* if the Hom-sets in $\mathcal{C}$ are modules over the ring $\Bbbk$, the composition of morphisms of $\mathcal{C}$ are $\Bbbk$-bilinear, and any finite family of objects of $\mathcal{C}$ has a direct sum in $\mathcal{C}$. Note that the direct sum of an empty family of objects is a *null object*, that is, an object $\mathbf{0} \in \mathcal{C}$ such that $\mathrm{End}_{\mathcal{C}}(\mathbf{0}) = 0$.

A monoidal category is $\Bbbk$-*additive* if it is $\Bbbk$-additive as a category and the monoidal product of morphisms is $\Bbbk$-bilinear.

3.4. *Semisimple categories*

We call an object U of a $\Bbbk$-additive category $\mathcal{C}$ *simple* if $\mathrm{End}_{\mathcal{C}}(U)$ is a free $\Bbbk$-module of rank 1 (and so has the basis $\{\mathrm{id}_U\}$). It is clear that an object isomorphic to a simple object is itself simple. If $\mathcal{C}$ is pivotal, then the dual of a simple object of $\mathcal{C}$ is simple.

A *split semisimple category* (*over* $\Bbbk$) is a $\Bbbk$-additive category $\mathcal{C}$ such that

(a) each object of $\mathcal{C}$ is a finite direct sum of simple objects;
(b) for any non-isomorphic simple objects i, j of $\mathcal{C}$, we have $\mathrm{Hom}_{\mathcal{C}}(i, j) = 0$.

Clearly, the Hom spaces in such a $\mathcal{C}$ are free $\Bbbk$-modules of finite rank. For $X \in \mathcal{C}$ and a simple object $i \in \mathcal{C}$, the modules $H_X^i = \mathrm{Hom}_{\mathcal{C}}(X, i)$ and $H_i^X = \mathrm{Hom}_{\mathcal{C}}(i, X)$ have same rank denoted N_X^i and called the *multiplicity number*. The bilinear form $H_X^i \times H_i^X \to \Bbbk$ carrying $(p \in H_X^i, q \in H_i^X)$ to $pq \in \mathrm{End}_{\mathcal{C}}(i) = \Bbbk$ is non-degenerate. Note that if i admits a left or right dual $\hat{\imath}$, then $N_{i \otimes X}^{\mathbb{1}} = N_X^{\hat{\imath}} = N_{X \otimes i}^{\mathbb{1}}$.

3.5. *Pre-fusion and fusion categories*

A *pre-fusion category* (*over* $\Bbbk$) is a split semisimple $\Bbbk$-additive pivotal category $\mathcal{C}$ such that the unit object $\mathbb{1}$ is simple. In such a category, the map $\Bbbk \to \mathrm{End}_{\mathcal{C}}(\mathbb{1}), k \mapsto k\,\mathrm{id}_{\mathbb{1}}$ is a $\Bbbk$-algebra isomorphism which we use to identify $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \Bbbk$. The left and right dimensions of any simple object of a pre-fusion category are invertible (see, for example, [13, Lemma 4.1]).

If I is a set of simple objects of pre-fusion category $\mathcal{C}$ such that every simple object of $\mathcal{C}$ is isomorphic to a unique element of I, then for any object X of $\mathcal{C}$,

$N_X^i = 0$ for all but a finite number of $i \in I$, and

$$\dim_l(X) = \sum_{i \in I} \dim_l(i) N_X^i, \quad \dim_r(X) = \sum_{i \in I} \dim_r(i) N_X^i. \tag{3.1}$$

A *fusion category* is a pre-fusion category such that the set of isomorphism classes of simple objects is finite. The *dimension* $\dim(\mathcal{C})$ of a fusion category $\mathcal{C}$ is defined by

$$\dim(\mathcal{C}) = \sum_{i \in I} \dim_l(i) \dim_r(i) \in \Bbbk,$$

where I is a (finite) set of simple objects of $\mathcal{C}$ such that every simple object of $\mathcal{C}$ is isomorphic to a unique element of I. The sum on the right-hand side does not depend on the choice of I. Note that if $\Bbbk$ is an algebraically closed field of characteristic zero, then $\dim(\mathcal{C}) \neq 0$, see [4].

4. *G*-Fusion Categories

4.1. *G-categories*

A *G-graded category* or shorter a *G-category* is a $\Bbbk$-additive monoidal category $\mathcal{C}$ endowed with a system of pairwise disjoint full $\Bbbk$-additive subcategories $\{\mathcal{C}_g\}_{g \in G}$ such that

(a) each object $U \in \mathcal{C}$ splits as a direct sum $\bigoplus_g U_g$ where $U_g \in \mathcal{C}_g$ and g runs over a finite subset of G;
(b) if $U \in \mathcal{C}_g$ and $V \in \mathcal{C}_h$, then $U \otimes V \in \mathcal{C}_{gh}$;
(c) if $U \in \mathcal{C}_g$ and $V \in \mathcal{C}_h$ with $g \neq h$, then $\mathrm{Hom}_{\mathcal{C}}(U, V) = 0$;
(d) the unit object $\mathbb{1}$ of $\mathcal{C}$ belongs to $\mathcal{C}_1$.

Under these assumptions, we write $\mathcal{C} = \bigoplus_g \mathcal{C}_g$. The category $\mathcal{C}_1$ corresponding to the neutral element $1 \in G$ is called the *neutral component* of $\mathcal{C}$. Clearly, $\mathcal{C}_1$ is a $\Bbbk$-additive monoidal category.

An object X of a G-category $\mathcal{C} = \bigoplus_g \mathcal{C}_g$ is *homogeneous* if $X \in \mathcal{C}_g$ for some $g \in G$. Such a g is then uniquely determined by X and denoted $|X|$. If two homogeneous objects $X, Y \in \mathcal{C}$ are isomorphic, then either they are null objects or $|X| = |Y|$.

A G-category $\mathcal{C}$ is *pivotal* (respectively, *spherical*) if it is pivotal (respectively, spherical) as a monoidal category. For such $\mathcal{C}$ and all $X \in \mathcal{C}_g$ with $g \in G$, we can and always do choose X^* to be in $\mathcal{C}_{g^{-1}}$. Note that if $\mathcal{C}$ is pivotal (respectively, spherical), then so is $\mathcal{C}_1$.

A G-category is *pre-fusion* if it is pre-fusion as a monoidal category. In particular, a pre-fusion G-category is supposed to be pivotal. In a pre-fusion G-category $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$, every simple object is isomorphic to a simple object of $\mathcal{C}_g$ for a unique $g \in G$. Moreover, for all $g \in G$, each object of $\mathcal{C}_g$ is a finite direct sum of simple objects of $\mathcal{C}_g$.

A set I of simple objects of a pre-fusion G-category $\mathcal{C}$ is *representative* if $\mathbb{1} \in I$, all elements of I are homogeneous, and every simple object of $\mathcal{C}$ is isomorphic to a unique element of I. Any such set I splits as a disjoint union $I = \coprod_{g\in G} I_g$ where I_g is the set of all elements of I belonging to $\mathcal{C}_g$.

4.2. *G-fusion categories*

A *G-fusion category* is a pre-fusion G-category $\mathcal{C}$ such that the set of isomorphism classes of simple objects of $\mathcal{C}_g$ is finite and nonempty for every $g \in G$. For $G = 1$, we obtain the notion of a fusion category (see Sec. 3.5). The neutral component $\mathcal{C}_1$ of a G-fusion category $\mathcal{C}$ is a fusion category. A G-fusion category is a fusion category if and only if G is finite.

In the next statement we use the multiplicity numbers defined in Sec. 3.4.

Lemma 4.1. *Let $I = \coprod_{g\in G} I_g$ be a representative set of simple objects of a G-fusion category $\mathcal{C}$. Then*

(a) *For all $g \in G$,*

$$\sum_{i\in I_g} \dim_l(i)\dim_r(i) = \dim(\mathcal{C}_1).$$

(b) *For all $a, b, g \in G$, $U \in \mathcal{C}_a$ and $V \in \mathcal{C}_b$,*

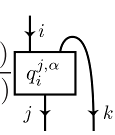
$$\sum_{k\in I_g,\ell\in I_{(agb)^{-1}}} \dim_l(k)\dim_l(\ell) N^{\mathbb{1}}_{U\otimes k\otimes V\otimes \ell} = \dim_r(U)\dim_r(V)\dim(\mathcal{C}_1).$$

Proof. Let us prove (a). Pick $k \in I_g$ and fix it till the end of the argument. For every $i \in I_g$, the object $i \otimes k^* \in \mathcal{C}_1$ splits as a (finite) direct sum of simple objects necessarily belonging to $\mathcal{C}_1$. Every $j \in I_1$ occurs $N^j_{i\otimes k^*}$ times in this sum. Then there is a family of morphisms $(p_i^{j,\alpha} : i \otimes k^* \to j, q_i^{j,\alpha} : j \to i \otimes k^*)_{\alpha\in A_{i,j}}$ such that $A_{i,j}$ has $N^j_{i\otimes k^*}$ elements and $p_i^{j,\alpha} q_i^{j,\alpha} = \delta_{\alpha,\beta}\,\mathrm{id}_j$ for all $\alpha, \beta \in A_{i,j}$. This implies

$$\mathrm{id}_{i\otimes k^*} = \sum_{j\in I_1,\alpha\in A_{i,j}} q_i^{j,\alpha} p_i^{j,\alpha}. \tag{4.1}$$

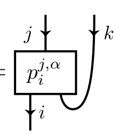
For $i \in I_g$, $j \in I_1$ and $\alpha \in A_{i,j}$, set

$$P_j^{i,\alpha} = \frac{\dim_r(i)}{\dim_r(j)}(\mathrm{id}_i \otimes \mathrm{ev}_k)(q_i^{j,\alpha} \otimes \mathrm{id}_k) = \frac{\dim_r(i)}{\dim_r(j)}$$



$$: j \otimes k \to i,$$

$$Q_j^{i,\alpha} = (p_i^{j,\alpha} \otimes \mathrm{id}_k)(\mathrm{id}_i \otimes \mathrm{coev}_k) =$$



$$: i \to j \otimes k.$$

For any $\alpha, \beta \in A_{i,j}$,

$$P_j^{i,\alpha} Q_j^{i,\beta} = \frac{\mathrm{tr}_r(P_j^{i,\alpha} Q_j^{i,\beta})}{\dim_r(i)}\,\mathrm{id}_i = \frac{\mathrm{tr}_r(p_i^{j,\alpha} q_i^{j,\beta})}{\dim_r(j)}\,\mathrm{id}_i = \delta_{\alpha,\beta}\,\mathrm{id}_i.$$

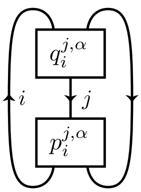
Note that the set $A_{i,j}$ has $N^j_{i\otimes k^*} = N^i_{j\otimes k}$ elements. Thus, for every $j \in I_1$, the family $(P_j^{i,\alpha}, Q_j^{i,\alpha})_{i\in I_g,\alpha\in A_{i,j}}$ encodes a splitting of $j \otimes k$ as a direct sum of simple objects of $\mathcal{C}_g$. Hence

$$\mathrm{id}_{j\otimes k} = \sum_{i\in I_g, \alpha\in A_{i,j}} Q_j^{i,\alpha} P_j^{i,\alpha}. \tag{4.2}$$

Using (4.1), the definition of $P_j^{i,\alpha}$, $Q_j^{i,\alpha}$ and (4.2), we obtain

$$\sum_{i\in I_g} \dim_r(i)\dim_l(i)\dim_l(k)$$

$$= \sum_{\substack{i\in I_g, j\in I_1\\ \alpha\in A_{i,j}}} \dim_r(i)$$



$$= \sum_{i\in I_g, j\in I_1, \alpha\in A_{i,j}} \dim_r(j)\mathrm{tr}_l(P_j^{i,\alpha}Q_j^{i,\alpha}) = \sum_{j\in I_1}\dim_r(j) \sum_{i\in I_g,\alpha\in A_{i,j}} \mathrm{tr}_l(Q_j^{i,\alpha}P_j^{i,\alpha})$$

$$= \sum_{j\in I_1}\dim_r(j)\mathrm{tr}_l(\mathrm{id}_{j\otimes k}) = \sum_{j\in I_1}\dim_r(j)\dim_l(j)\dim_l(k) = \dim(\mathcal{C}_1)\dim_l(k).$$

We conclude using that $\dim_l(k)\in \Bbbk$ is invertible.

Let us prove (b). Using (3.1) and Claim (a) of the lemma, we obtain

$$\sum_{k\in I_g, \ell\in I_{(agb)^{-1}}} \dim_l(k)\dim_l(\ell) N^{\mathbb{1}}_{U\otimes k\otimes V\otimes \ell}$$

$$= \sum_{k\in I_g, \ell\in I_{(agb)^{-1}}} \dim_l(k)\dim_r(\ell^*) N^{\ell^*}_{U\otimes k\otimes V}$$

$$= \sum_{k\in I_g}\dim_l(k)\sum_{m\in I_{agb}}\dim_r(m)N^m_{U\otimes k\otimes V} = \sum_{k\in I_g}\dim_l(k)\dim_r(U\otimes k\otimes V)$$

$$= \dim_r(U)\dim_r(V)\sum_{k\in I_g}\dim_l(k)\dim_r(k) = \dim_r(U)\dim_r(V)\dim(\mathcal{C}_1). \qquad \square$$

4.3. *Example*

Let $\Bbbk^*$ be the group of invertible elements of $\Bbbk$ and let vect_G be the category of G-graded free $\Bbbk$-modules of finite rank. It is well-known that every $\theta\in H^3(G;\Bbbk^*)$ determines associativity constraints on vect_G extending the usual tensor product to a monoidal structure. Denote the resulting monoidal category by vect_G^θ. This category is G-graded: $\mathrm{vect}_G^\theta = \bigoplus_{g\in G}\mathrm{vect}_g^\theta$, where vect_g^θ is the full subcategory of modules of degree g. The module $\Bbbk$ viewed as an object of vect_g^θ is the unique simple object of vect_g^θ, at least up to isomorphism. The usual (co)evaluation morphisms define a pivotal structure on vect_G^θ. It is easy to check that vect_G^θ is spherical G-fusion category. Clearly, $\dim(\mathrm{vect}_1^\theta) = 1\in\Bbbk$.

4.4. *Remark*

Each pre-fusion category $\mathcal{C}$ has a *universal grading* defined as follows. Let $\mathcal{I}$ be the set of isomorphism classes of simple objects of $\mathcal{C}$. Note that gradings of $\mathcal{C}$ by a group (in the sense of Sec. 4.1) bijectively correspond to maps ϱ from $\mathcal{I}$ to this group verifying $\varrho(X)\,\varrho(Y) = \varrho(Z)$ for all $X, Y, Z \in \mathcal{I}$ such that Z is a direct summand of $X \otimes Y$. A grading $\partial\colon \mathcal{I} \to \Gamma$ of $\mathcal{C}$ by a group Γ is *universal* if any grading of $\mathcal{C}$ by any group can be uniquely expressed as the composition of ∂ with a homomorphism from Γ to that group. To construct a universal grading of $\mathcal{C}$ set $\Gamma = \mathcal{I}/\!\sim$, where $\sim$ is the weakest equivalence relation on $\mathcal{I}$ such that any two simple objects appearing as direct summands of the same (finite) monoidal product of simple objects are equivalent. The product, the unit, and the inverses in Γ are induced by the pivotal structure of $\mathcal{C}$. The projection $\mathcal{I} \to \Gamma = \mathcal{I}/\!\sim$ is a universal grading of $\mathcal{C}$. The group Γ is called the *graduator* of $\mathcal{C}$. For instance, the grading of vect_G^θ in Example 4.3 is universal, and the graduator is G.

5. Multiplicity Modules and Graphs

We recall here symmetrized multiplicity modules and invariants of colored plane graphs introduced in [13]. Throughout this section, we fix a pivotal $\Bbbk$-additive monoidal category $\mathcal{C}$ such that $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \Bbbk$.

5.1. *Multiplicity modules*

A *signed object* of $\mathcal{C}$ is a pair (U, ε) where $U \in \mathcal{C}$ and $\varepsilon \in \{+,-\}$. For a signed object (U,ε) of $\mathcal{C}$, we set $U^\varepsilon = U$ if $\varepsilon = +$ and $U^\varepsilon = U^*$ if $\varepsilon = -$. A *cyclic $\mathcal{C}$-set* is a triple $(E, c\colon E \to \mathcal{C}, \varepsilon\colon E \to \{+,-\})$, where E is a totally cyclically ordered finite set. In other words, a cyclic $\mathcal{C}$-set is a totally cyclically ordered finite set whose elements are labeled by signed objects of $\mathcal{C}$. For shortness, we sometimes write E for (E, c, ε).

Given a cyclic $\mathcal{C}$-set $E = (E, c, \varepsilon)$ and $e \in E$, we can order $e = e_1 < e_2 < \cdots < e_n$ the elements of E starting from e and using the given cyclic order in E (here $n = \#E$ is the number of elements of E). Set

$$Z_e = c(e_1)^{\varepsilon(e_1)} \otimes \cdots \otimes c(e_n)^{\varepsilon(e_n)} \in \mathcal{C} \quad \text{and} \quad H_e = \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, Z_e).$$

By [13], the structure of a pivotal category in $\mathcal{C}$ determines a projective system of $\Bbbk$-module isomorphisms $\{H_e \approx H_{e'}\}_{e,e' \in E}$. For example, if in the notation above $\varepsilon(e_1) = -$, then the isomorphism $H_{e_1} \to H_{e_2}$ carries any $f \in H_{e_1}$ to

$$(\widetilde{\mathrm{ev}}_{c(e_1)} \otimes \mathrm{id}_{Z_{e_2}})(\mathrm{id}_{c(e_1)} \otimes f \otimes \mathrm{id}_{c(e_1)^*})\mathrm{coev}_{c(e_1)} \in H_{e_2}.$$

If $\varepsilon(e_1) = +$, then the isomorphism $H_{e_1} \to H_{e_2}$ carries $f \in H_{e_1}$ to

$$(\mathrm{ev}_{c(e_1)} \otimes \mathrm{id}_{Z_{e_2}})(\mathrm{id}_{c(e_1)^*} \otimes f \otimes \mathrm{id}_{c(e_1)})\widetilde{\mathrm{coev}}_{c(e_1)} \in H_{e_2}.$$

The projective limit of this system of isomorphisms $H(E) = \varprojlim H_e$ is a $\Bbbk$-module depending only on E. It is equipped with isomorphisms $\{H(E) \to H_e\}_{e \in E}$, called the *cone isomorphisms.*

An isomorphism of cyclic $\mathcal{C}$-sets $\phi\colon E \to E'$ is a bijection preserving the cyclic order and commuting with the maps to $\mathcal{C}$ and $\{+,-\}$. Such a ϕ induces a $\Bbbk$-isomorphism $H(\phi)\colon H(E) \to H(E')$ in the obvious way.

The *dual* of a cyclic $\mathcal{C}$-set $E = (E, c, \varepsilon)$ is the cyclic $\mathcal{C}$-set $(E^{\mathrm{op}}, c, -\varepsilon)$, where E^{op} is E with opposite cyclic order. The pivotality of $\mathcal{C}$ determines a $\Bbbk$-bilinear pairing $\omega_E\colon H(E^{\mathrm{op}}) \otimes H(E) \to \Bbbk$ where $\otimes = \otimes_\Bbbk$ is the tensor product over $\Bbbk$. A *duality* between cyclic $\mathcal{C}$-sets E and E' is an isomorphism of cyclic $\mathcal{C}$-sets $\phi\colon E' \to E^{\mathrm{op}}$. Such a ϕ induces a $\Bbbk$-bilinear pairing $\omega_E \circ (H(\phi) \otimes \mathrm{id})\colon H(E') \otimes H(E) \to \Bbbk$.

5.2. *Colored graphs*

By a *graph*, we mean a finite graph without isolated vertices. We allow multiple edges with the same endpoints and loops (edges with both endpoints in the same vertex). A graph is *oriented*, if all its edges are oriented.

Let Σ be an oriented surface. By a *graph* in Σ, we mean a graph embedded in Σ. A vertex v of a graph $A \subset \Sigma$ determines a cyclically ordered set A_v consisting of the half-edges of A incident to v with cyclic order induced by the opposite orientation of Σ. If A is oriented, then we have a map $\varepsilon_v\colon A_v \to \{+,-\}$ assigning $+$ to the half-edges oriented towards v and $-$ to the half-edges oriented away from v.

A *$\mathcal{C}$-colored graph* in Σ is an oriented graph in Σ whose every edge is labeled with an object of $\mathcal{C}$ called the *color* of this edge. A vertex v of a $\mathcal{C}$-colored graph $A \subset \Sigma$ determines a cyclic $\mathcal{C}$-set $A_v = (A_v, c_v, \varepsilon_v)$, where c_v assigns to each half-edge the corresponding color. Set $H_v(A) = H(A_v)$ and $H(A) = \bigotimes_v H_v(A)$, where v runs over all vertices of A. To stress the role of Σ, we sometimes write $H_v(A;\Sigma)$ for $H_v(A)$ and $H(A;\Sigma)$ for $H(A)$.

Consider now the case $\Sigma = \mathbb{R}^2$ with counterclockwise orientation. Every $\mathcal{C}$-colored graph A in $\mathbb{R}^2$ determines a vector $\mathbb{F}_\mathcal{C}(A) \in H(A)^\star = \mathrm{Hom}_\Bbbk(H(A), \Bbbk)$, see [13]. The idea behind the definition of $\mathbb{F}_\mathcal{C}(A)$ is to deform A in the plane so that in a neighborhood of every vertex v all half-edges incident to v lie above v with respect to the second coordinate in the plane. For each v, pick any $\alpha_v \in H_v(A)$ and replace v by a box colored with the image of α_v under the corresponding cone isomorphism. This transforms A into a planar diagram formed by colored edges and colored boxes. Such a diagram determines, through composition and tensor product of morphisms in $\mathcal{C}$ (and the use of the left and right evaluation/co-evaluation morphisms), an element of $\mathrm{End}_\mathcal{C}(\mathbb{1}) = \Bbbk$. By linear extension, this procedure defines a vector $\mathbb{F}_\mathcal{C}(A) \in H(A)^\star$. The key property of this vector is the independence from the auxiliary choices. Moreover, both $H(A)$ and $\mathbb{F}_\mathcal{C}(A)$ are preserved under color-preserving isotopies of A in $\mathbb{R}^2$.

For example, if $A = S^1 \subset \mathbb{R}^2$ is the unit circle with one vertex $v = (1,0)$ and one edge oriented clockwise and colored with $U \in \mathcal{C}$, then A_v consists of two elements labeled by $(U,+)$, $(U,-)$ and $H(A) = H_v(A) \cong \mathrm{Hom}_\mathcal{C}(\mathbb{1}, U^* \otimes U)$. Here $\mathbb{F}_\mathcal{C}(A)(\alpha) = \mathrm{ev}_U \alpha \in \mathrm{End}_\mathcal{C}(\mathbb{1}) = \Bbbk$ for all $\alpha \in H(A)$.

If $\mathcal{C}$ is pre-fusion, then the $\Bbbk$-modules associated in Sec. 5.1 with cyclic $\mathcal{C}$-sets are free of finite rank and the pairings defined at the end of that section are non-degenerate (cf. [13, Lemmas 2.3 and 4.1(a)]). Note also that every $\mathcal{C}$-colored graph A in an oriented surface Σ determines a $\mathcal{C}$-colored graph A^{op} in $-\Sigma$ obtained by reversing orientation in Σ and in all edges of A while keeping the colors of the edges. The cyclic $\mathcal{C}$-sets determined by a vertex v of A and A^{op} are dual. If $\mathcal{C}$ is pre-fusion, then $H_v(A^{\mathrm{op}}; -\Sigma) = H_v(A; \Sigma)^\star$ and $H(A^{\mathrm{op}}; -\Sigma) = H(A; \Sigma)^\star$.

5.3. *The spherical case*

If $\mathcal{C}$ is spherical, then the invariant $\mathbb{F}_\mathcal{C}$ of $\mathcal{C}$-colored graphs in $\mathbb{R}^2$ generalizes to $\mathcal{C}$-colored graphs in the 2-sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$ (the orientation of S^2 extends the counterclockwise orientation in $\mathbb{R}^2$). Indeed, pushing a $\mathcal{C}$-colored graph $A \subset S^2$ away from ∞, we obtain a $\mathcal{C}$-colored graph A_0 in $\mathbb{R}^2$. The sphericity of $\mathcal{C}$ ensures that

$$\mathbb{F}_\mathcal{C}(A) = \mathbb{F}_\mathcal{C}(A_0) \in H(A_0)^\star = H(A)^\star$$

is a well-defined isotopy invariant of A. We state a few simple properties of $\mathbb{F}_\mathcal{C}$.

(i) (Naturality) If a $\mathcal{C}$-colored graph $A' \subset S^2$ is obtained from a $\mathcal{C}$-colored graph $A \subset S^2$ through the replacement of the color U of an edge by an isomorphic object U', then any isomorphism $\phi: U \to U'$ induces an isomorphism $\hat{\phi}: H(A) \to H(A')$ in the obvious way and $\hat{\phi}^\star(\mathbb{F}_\mathcal{C}(A')) = \mathbb{F}_\mathcal{C}(A)$.

(ii) If an edge e of a $\mathcal{C}$-colored graph $A \subset S^2$ is colored by $\mathbb{1}$ and the endpoints of e are also endpoints of other edges of A, then $\mathbb{F}_\mathcal{C}(A) = \mathbb{F}_\mathcal{C}(A \backslash \mathrm{Int}(e))$.

(iii) If a $\mathcal{C}$-colored graph $A' \subset S^2$ is obtained from a $\mathcal{C}$-colored graph $A \subset S^2$ through the replacement of the color U of an edge e by U^* and the reversion of the orientation of e, then there is an isomorphism $H(A) \to H(A')$ such that the dual isomorphism carries $\mathbb{F}_\mathcal{C}(A')$ to $\mathbb{F}_\mathcal{C}(A)$.

(iv) We have $H(A \amalg A') = H(A) \otimes H(A')$ and $\mathbb{F}_\mathcal{C}(A \amalg A') = \mathbb{F}_\mathcal{C}(A) \otimes \mathbb{F}_\mathcal{C}(A')$ for any disjoint $\mathcal{C}$-colored graphs $A, A' \subset S^2$.

6. Skeletons and Maps

We recall the notions of stratified 2-polyhedra and skeletons of 3-manifolds following [13, Sec. 6]. Then we explain how to encode maps from closed 3-manifolds to $\mathbf{X} = K(G, 1)$ in terms of labelings of skeletons.

6.1. *Stratified 2-polyhedra*

By an *arc* in a topological space P, we mean the image of a path $\alpha: [0, 1] \to P$ which is an embedding except that possibly $\alpha(0) = \alpha(1)$ (i.e. arcs may be loops.) The points $\alpha(0)$, $\alpha(1)$ are the *endpoints* and the set $\alpha((0, 1))$ is the *interior* of the arc. By a 2-*polyhedron*, we mean a compact topological space that can be triangulated using only simplices of dimensions 0, 1 and 2. For a 2-polyhedron P, denote by $\mathrm{Int}(P)$ the subspace of P consisting of all points having a neighborhood homeomorphic to $\mathbb{R}^2$. Clearly, $\mathrm{Int}(P)$ is an (open) 2-manifold.

Consider a 2-polyhedron P endowed with a finite set of arcs E such that

(a) different arcs in E may meet only at their endpoints;
(b) $P \backslash \bigcup_{e \in E} e \subset \mathrm{Int}(P)$;
(c) $P \backslash \bigcup_{e \in E} e$ is dense in P.

The arcs of E are called *edges* of P and their endpoints are called *vertices* of P. The vertices and edges of P form a graph $P^{(1)} = \bigcup_{e \in E} e$. Cutting P along $P^{(1)}$, we obtain a compact surface $\widetilde{P}$ with interior $P \backslash P^{(1)}$. The polyhedron P can be recovered by gluing $\widetilde{P}$ to $P^{(1)}$ along a map $p : \partial \widetilde{P} \to P^{(1)}$. Condition (c) ensures the surjectivity of p. We call the pair (P, E) (or, shorter, P) a *stratified* 2*-polyhedron* if the set p^{-1}(the set of vertices of P) is finite and each component of the complement of this set in $\partial \widetilde{P}$ is mapped homeomorphically onto the interior of an edge of P.

For a stratified 2-polyhedron P, the connected components of $\widetilde{P}$ are called *regions* of P. The set $\mathrm{Reg}(P)$ of the regions of P is finite. For a vertex x of P, a *branch* of P at x is a germ at x of a region of P adjacent to x. The set of branches of P at x is finite and nonempty. The branches of P at x bijectively correspond to the elements of the set $p^{-1}(x)$, where $p : \partial \widetilde{P} \to P^{(1)}$ is the map above. Similarly, a *branch* of P at an edge e of P is a germ at e of a region of P adjacent to e. The set of branches of P at e is denoted P_e. This set is finite and nonempty. There is a natural bijection between P_e and the set of connected components of p^{-1}(interior of e). The number of elements of P_e is the *valence* of e. The edges of P of valence 1 and their vertices form a graph called the *boundary* of P and denoted ∂P. We say that P is *orientable* (respectively, *oriented*) if all regions of P are orientable (respectively, oriented).

6.2. *Skeletons of* 3*-manifolds*

Let M be a closed oriented 3-dimensional manifold. A *skeleton* of M is an oriented stratified 2-polyhedron $P \subset M$ such that $\partial P = \emptyset$ and $M \backslash P$ is a disjoint union of open 3-balls. The components of $M \backslash P$ are called P*-balls*. An example of a skeleton of M is provided by the (oriented) 2-skeleton $t^{(2)}$ of a triangulation t of M, where the edges of $t^{(2)}$ are the edges of t.

We now analyze regular neighborhoods of edges and vertices of a skeleton $P \subset M$. Pick an edge e of P and orient it in an arbitrary way. The orientations of e and M determine a positive direction on a small loop in $M \backslash e$ encircling e so that the linking number of this loop with e is $+1$. This direction induces a cyclic order on the set P_e of branches of P at e. For a branch $b \in P_e$, set $\varepsilon_e(b) = +$ if the orientation of e is compatible with the orientation of b induced by that of the ambient region of P and set $\varepsilon_e(b) = -$ otherwise. This gives a map $\varepsilon_e : P_e \to \{+, -\}$. When orientation of e is reversed, the cyclic order on P_e is reversed and ε_e is multiplied by $-$.

Any vertex v of P has a closed ball neighborhood $B_v \subset M$ such that $\Gamma_v = P \cap \partial B_v$ is a nonempty graph and $P \cap B_v$ is the cone over Γ_v. The vertices of Γ_v

are the intersections of ∂B_v with the half-edges of P incident to v. Similarly, each edge α of Γ_v is the intersection of ∂B_v with a branch b_α of P at v. We endow α with orientation induced by that of b_α restricted to $b_\alpha \backslash \mathrm{Int}(B_v)$. We endow $\partial B_v \approx S^2$ with orientation induced by that of M restricted to $M \backslash \mathrm{Int}(B_v)$. In this way, Γ_v becomes an oriented graph in the oriented 2-sphere ∂B_v. The pair $(\partial B_v, \Gamma_v)$ is the *link* of v in (M, P). It is well-defined up to orientation preserving homeomorphism. Note that the condition $\partial P = \emptyset$ implies that every vertex of Γ_v is incident to at least two half-edges of Γ_v.

6.3. *G-labelings*

Let P be a skeleton of a closed oriented 3-dimensional manifold M. A *G-labeling* of P is a map $\ell : \mathrm{Reg}(P) \to G$ such that for every edge e of P the labels of the adjacent branches $b_1, \ldots, b_n$ of P, enumerated in the cyclic order determined by an orientation of e, satisfy the following product condition:

$$\prod_{i=1}^{n} \ell(b_i)^{\varepsilon_e(b_i)} = 1, \tag{6.1}$$

where the map $\varepsilon_e : P_e \to \{+, -\}$ is determined by the same orientation of e. Note that if (6.1) holds for one orientation of e, then it holds also for the opposite orientation.

The G-labelings of P determine homotopy classes of maps $M \to \mathbf{X} = K(G, 1)$ as follows. Pick a point, called the *center* in every P-ball (i.e. in every component of $M \backslash P$). Each region r of $\mathrm{Reg}(P)$ is adjacent to two (possibly coinciding) P-balls. Pick an arc α_r in M whose endpoints are the centers of these balls and whose interior meets P transversely in a single point lying in r. We choose the arcs $\{\alpha_r\}_r$ so that they meet only in the endpoints and orient them so that the intersection number $\alpha_r \cdot r = r \cdot \alpha_r$ is $+1$ for all r.

Lemma 6.1. *For each G-labeling ℓ of P there is a map $f_\ell : M \to \mathbf{X}$ carrying the centers of the P-balls to the base point $x \in \mathbf{X}$ and carrying α_r to a loop in $\mathbf{X}$ representing $\ell(r) \in G = \pi_1(\mathbf{X}, x)$ for all $r \in \mathrm{Reg}(P)$. The homotopy class of f_ℓ depends only on P and ℓ. Any map $M \to \mathbf{X}$ is homotopic to f_ℓ for some G-coloring ℓ of P.*

Proof. Consider first the case where all regions of P are disks. Then P determines a dual CW-decomposition P^* of M, whose 0-cells are the centers of the P-balls, the 1-cells are the arcs $\{\alpha_r\}_r$, the 2-cells are meridional disks of the edges of $P^{(1)}$, and the 3-cells are ball neighborhoods in M of the vertices of P. Consider a map from the 1-skeleton of P^* to $\mathbf{X}$ carrying all 0-cells to x and carrying each α_r to a loop in $\mathbf{X}$ representing $\ell(r)$. The product condition (6.1) ensures that this map extends to the 2-skeleton of P^*. The equality $\pi_2(\mathbf{X}) = 0$ ensures that there is a further extension to a map $f_\ell : M \to \mathbf{X}$. The independence of f_ℓ of the choice of $\{\alpha_r\}_r$ follows from the fact that any two such systems of arcs are homotopic in M

relative to the centers of the P-balls. The independence of the choice of the centers and the last claim of the lemma are straightforward.

In the general case, each region r of P is a disk with holes and may be collapsed onto a wedge of circles $W_r \subset r \backslash \partial r$ based at the point $\alpha_r \cap r$. Let S_r be obtained from $W_r \times [-1, 1]$ by contracting the sets $W_r \times \{-1\}$ and $W_r \times \{1\}$ into points called the vertices of S_r. (If the two P-balls adjacent to r are equal, then we identify the two vertices of S_r.) We embed S_r into M as a union of two cones with base W_r and with the cone points in the centers of the P-balls adjacent to r. We can assume that $\alpha_r \subset S_r$ and that S_r does not meet $S_{r'}$ for $r \neq r'$ except possibly in the vertices. Then the pair $(M, \bigcup_r S_r)$ has a relative CW-decomposition whose only cells are meridional disks of the edges of P and ball neighborhoods in M of the vertices of P. The projection $W_r \times [-1, 1] \to [-1, 1]$ induces a retraction $S_r \to \alpha_r$. Composing with loops $\alpha_r \to \mathbf{X}$ representing $\ell(r)$ we obtain a map $\bigcup_r S_r \to \mathbf{X}$ carrying the centers of the P-balls to x and carrying each α_r to a loop in $\mathbf{X}$ representing $\ell(r)$. The rest of the proof goes as in the previous paragraph. To prove the last claim of the lemma it is useful to note that the inclusion $W_r \to M$ is null-homotopic for all $r \in \mathrm{Reg}(P)$. $\square$

The maps from the set $\pi_0(M \backslash P)$ of P-balls to G form a group with respect to pointwise multiplication. This group is called the *gauge group of* P and denoted by $\mathcal{G}_P$. The group $\mathcal{G}_P$ acts (on the left) on the set of G-labelings of P: for $\lambda \in \mathcal{G}_P$, a G-labeling $\ell : \mathrm{Reg}(P) \to G$, and a region r of P,

$$(\lambda\ell)(r) = \lambda(r_-)\ell(r)\lambda(r_+)^{-1}$$

where $r_\pm$ are the P-balls (possibly coinciding) adjacent to r and indexed so that $\alpha_r(0) \in r_-$ and $\alpha_r(1) \in r_+$. It is easy to see that $\lambda\ell$ is a G-labeling of P determining the same homotopy class of maps $M \to \mathbf{X}$ as ℓ. Moreover, two G-labelings of P determine the same homotopy class of maps $M \to \mathbf{X}$ if and only if these G-labelings belong to the same $\mathcal{G}_P$-orbit.

6.4. *G-skeletons*

A *G-skeleton* of a G-manifold M is a pair (a skeleton P of M, a G-labeling of P representing the given homotopy class of maps $M \to \mathbf{X}$). For brevity, such a pair will be often denoted by the same letter P as the underlying skeleton. Lemma 6.1 shows that any skeleton of M extends to a G-skeleton.

7. State-Sum Invariants of Closed G-Manifolds

7.1. *The state-sum invariant*

Fix a spherical G-fusion category $\mathcal{C}$ such that $\dim(\mathcal{C}_1) \in \Bbbk$ is invertible in the ground ring $\Bbbk$. For any closed G-manifold M, we define a topological invariant $|M|_\mathcal{C} \in \Bbbk$ of M. This invariant is obtained as a state-sum on a G-skeleton $P = (P, \ell)$ of M as follows. Pick a representative set $I = \coprod_{g \in G} I_g$ of simple objects of $\mathcal{C}$. Let $\mathrm{Col}(P)$

be the set of maps $c\colon \mathrm{Reg}(P) \to I$ such that $c(r) \in I_{\ell(r)}$ for all regions r of P. For $c \in \mathrm{Col}(P)$ and an oriented edge e of P, we have a $\Bbbk$-module $H_c(e) = H(P_e)$, where P_e is the set of branches of P at e turned into a cyclic $\mathcal{C}$-set as follows: the cyclic order and the map to $\{\pm\}$ are as in Sec. 6.2 and the $\mathcal{C}$-color of a branch $b \in P_e$ is the value of c on the region of P containing b. If e^{op} is the same edge with opposite orientation, then $P_{e^{\mathrm{op}}} = (P_e)^{\mathrm{op}}$. This induces a duality between the modules $H_c(e)$, $H_c(e^{\mathrm{op}})$ and a contraction homomorphism $*_e\colon H_c(e)^\star \otimes H_c(e^{\mathrm{op}})^\star \to \Bbbk$.

By Sec. 6.2, the link of a vertex $v \in P$ is an oriented graph $\Gamma_v \subset \partial B_v \approx S^2$. Given $c \in \mathrm{Col}(P)$, we transform Γ_v into a $\mathcal{C}$-colored graph by coloring each edge of Γ_v with the value of c on the region of P containing this edge. Section 5.3 yields a tensor $\mathbb{F}_\mathcal{C}(\Gamma_v) \in H(\Gamma_v)^*$. By definition, $H(\Gamma_v) = \bigotimes_e H_c(e)$, where e runs over all edges of P incident to v and oriented away from v (an edge with both endpoints in v appears in this tensor product twice with opposite orientations). The tensor product $\bigotimes_v \mathbb{F}_\mathcal{C}(\Gamma_v)$ over all vertices v of P is a vector in $\bigotimes_e H_c(e)^\star$, where e runs over all oriented edges of P. Set $*_P = \bigotimes_e *_e\colon \bigotimes_e H_c(e)^\star \to \Bbbk$.

Theorem 7.1. *Set*

$$|M|_\mathcal{C} = (\dim(\mathcal{C}_1))^{-|P|} \sum_{c\in \mathrm{Col}(P)} \left(\prod_{r \in \mathrm{Reg}(P)} (\dim c(r))^{\chi(r)} \right) *_P\Big(\bigotimes_v \mathbb{F}_\mathcal{C}(\Gamma_v)\Big) \in \Bbbk, \tag{7.1}$$

where $|P|$ is the number of P-balls and χ is the Euler characteristic. Then $|M|_\mathcal{C}$ is a topological invariant of the G-manifold M independent of the choice of I and P.

This theorem generalizes [13, Theorems 5.1 and 6.1] which produce a 3-manifold invariant $|\cdot|_\mathcal{D}$ from any spherical fusion category $\mathcal{D}$ whose dimension is invertible in the ground ring (the case $G = \{1\}$).

We illustrate Theorem 7.1 with two examples. First, if the given homotopy class of maps $M \to \mathbf{X}$ includes the constant map, then $|M|_\mathcal{C} = |M|_{\mathcal{C}_1}$. This follows from the definitions because the constant map is represented by the constant labeling $1 \in G$. In particular, $|S^3|_\mathcal{C} = |S^3|_{\mathcal{C}_1} = (\dim(\mathcal{C}_1))^{-1}$. Secondly, $|(S^1 \times S^2, f)|_\mathcal{C} = 1$ for any map $f\colon S^1 \times S^2 \to \mathbf{X}$. Indeed, pick a point $s \in S^1$ and a circle $L \subset S^2$. The set $P = (\{s\} \times S^2) \cup (S^1 \times L)$ is a skeleton of $S^1 \times S^2$ with one edge $\{s\} \times L$ and three regions. We orient the two disk regions of P lying in S^2 counterclockwise and orient the annulus region of P in an arbitrary way. We label the annulus region with $1 \in G$ and label both disk regions with an element g of G represented (up to conjugation and inversion) by the restriction of f to $S^1 \times \{pt\}$. This G-labeling of P represents f. Formula (7.1) gives

$$\begin{aligned} &|(S^1 \times S^2, f)|_\mathcal{C} \\ &\quad = (\dim(\mathcal{C}_1))^{-2} \sum_{j \in I_1, k,l \in I_g} \dim(k)\dim(l)\mathrm{rank}_\Bbbk \mathrm{Hom}_\mathcal{C}(\mathbb{1}, j \otimes k \otimes j^* \otimes l^*). \end{aligned}$$

The right-hand side is equal to 1 as easily follows from Lemma 4.1.

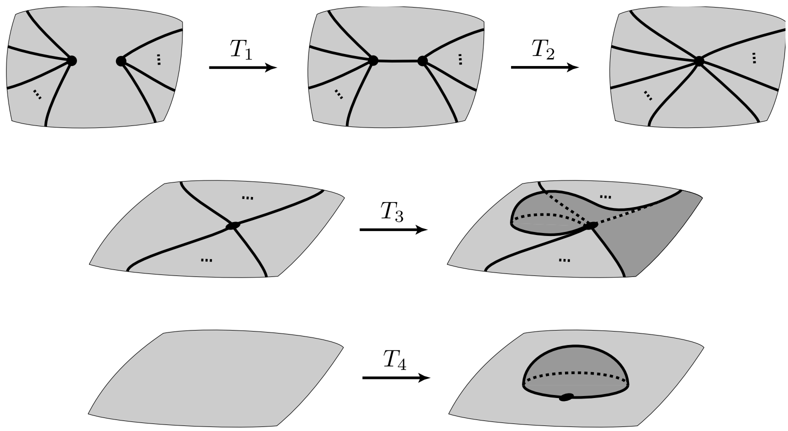


Fig. 1. Local moves on skeletons.

Theorem 7.1 will be proved at the end of the section. We first recall the moves on skeletons introduced in [13] and lift them to G-skeletons.

7.2. *Moves on skeletons*

Let M be a closed oriented 3-dimensional manifold. We consider four moves $T_1 - T_4$ on a skeleton $P \subset M$ transforming it into a new skeleton P' of M, see Fig. 1. The "phantom edge move" T_1 keeps P as a polyhedron and adds one new edge connecting distinct vertices of P; this edge is an arc in P meeting $P^{(1)}$ solely at the endpoints and has valence 2 in P'. The "contraction move" T_2 collapses into a point an edge e of P. This move is allowed only when the endpoints of e are distinct and at least one of them is the endpoint of some other edge. The "percolation move" T_3 pushes a branch b of P through a vertex v of P. The branch b is pushed across a small disk D lying in another branch of P at v so that $D \cap P^{(1)} = \partial D \cap P^{(1)} = \{v\}$ and both these branches are adjacent to the same component of $M \backslash P$. The disk D becomes a region of the resulting skeleton P'; all other regions of P' correspond bijectively to the regions of P in the obvious way. The "bubble move" T_4 adds to P an embedded disk $D_+ \subset M$ such that $D_+ \cap P = \partial D_+ \subset P \backslash P^{(1)}$, the circle ∂D_+ bounds a disk D_- in $P \backslash P^{(1)}$, and the 2-sphere $D_+ \cup D_-$ bounds a ball in M meeting P precisely at D_-. A point of the circle ∂D_+ is chosen as a vertex and the circle itself is viewed as an edge of the resulting skeleton P'. The disks D_+ and D_- become regions of P'; all other regions of P' correspond bijectively to the regions of P.

In the pictures of $T_1 - T_4$ and in similar pictures below we distinguish the "small" regions entirely contained in the 3-ball where the move proceeds and the "big" regions not entirely contained in the 3-ball where the move proceeds. The moves T_1, T_2 have no small regions, T_3 creates one small region D, and T_4 creates two small regions D_+ and D_-. In the definition of $T_1 - T_4$ we use the following orientation convention: orientations of the big regions are preserved under the move while orientations of the small regions may be arbitrary.

The moves $T_1 - T_4$ have obvious inverses. The move T_1^{-1} deletes a 2-valent edge e with distinct endpoints; this move is allowed only when both endpoints of e are endpoints of some other edges and the orientations of the two regions adjacent to e are compatible.

The moves $T_1 - T_4$ lift to G-labelings of skeletons of M by requiring that the labels of all big regions are preserved under the moves. We impose no conditions on the labels of the small regions except the product condition (6.1) which must hold both before and after the move. The labelings transform in a unique way under $T_1^{\pm 1}, T_2^{\pm 1}, T_3^{\pm 1}, T_4^{-1}$. Under T_4, the label of D_+ may be an arbitrary element of G and the label of D_- is then determined uniquely. If M is a G-manifold, then each of these moves transforms a G-skeleton of M into a G-skeleton of M. These transformations of G-skeletons are denoted by the same symbols $T_1^{\pm 1} - T_4^{\pm 1}$ and called *primary moves*. Label-preserving ambient isotopies of G-skeletons in M are also viewed as primary moves.

Lemma 7.2. *Any two G-skeletons of a closed G-manifold M can be related by a finite sequence of primary moves.*

Proof. We first define several further moves on G-skeletons of M. In these definitions, we apply the same orientation and labeling conventions as above. For any non-negative integers m, n with $m + n \geq 1$, we define a move $T^{m,n}$ on G-skeletons, see Fig. 2. The move $T^{m,n}$ destroys $\max(m-1, 0)$ small regions and creates $\max(n-1, 0)$ small regions. The labelings transform in a unique way under $T^{m,n}$. For $n = 0$, this move is allowed only when the orientations of the top and bottom regions on the left are compatible. It is shown in [13] that $T^{m,n}$ is a composition of primary moves (and the same argument works in the G-labeled case). The move inverse to $T^{m,n}$ is $T^{n,m}$.

In particular, the moves $T^{2,0}$ and $T^{0,2}$ push a branch of a G-skeleton P across a segment of $P^{(1)}$ containing a vertex of valence 2. One can consider similar moves $\widetilde{T}^{2,0}$ and $\widetilde{T}^{0,2}$ pushing a branch of P across a segment of $P^{(1)}$ containing no vertices. These moves decrease/increase the number of vertices of the skeleton by 2. The moves $\widetilde{T}^{2,0}$ and $\widetilde{T}^{0,2}$ may be expanded as compositions of primary moves. Indeed, applying T_2^{-1}, we can transform any point of $P^{(1)}$ into a vertex (keeping P and $P^{(1)}$) and then use $T^{2,0}$ and $T^{0,2}$.

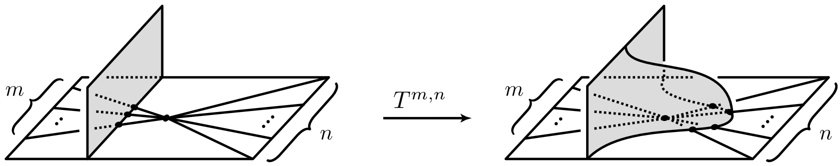


Fig. 2. The move $T^{m,n}$ on G-skeletons.

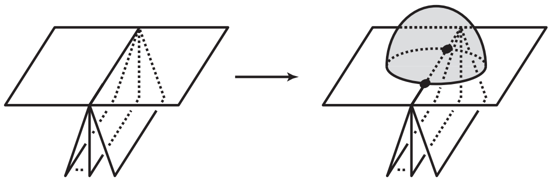


Fig. 3. A modified bubble moves on G-skeletons.

We need the modified version of the bubble move shown in Fig. 3. It is easy to show that this move is a composition of primary moves (cf. [13, Sec. 7.2]).

We now prove the following special case of the lemma.

Claim 1. Let $P \subset M$ be a G-skeleton of M and $r \in \mathrm{Reg}(P)$. Let P_- be the same G-skeleton with both the orientation and the label of r inverted. Then there is a finite sequence of primary moves transforming P into P_- (we write $P \rightsquigarrow P_-$).

Proof. It is enough to prove this claim in the case where r is a disk. Indeed, if r is not a disk, then we can use T_1 to add new edges to P lying in r and splitting r into disks. Consecutively reversing the orientation and the labels of these disks and then removing the newly added edges by T_1^{-1} we obtain P_-. Therefore if our claim holds for disk regions of G-skeletons, then it also holds for r.

A similar argument shows that it is enough to consider the case where r satisfies the following condition: $(*)$ the restriction of the gluing map $\partial\widetilde{P} \to P^{(1)}$ (see Sec. 6.1) to ∂r is injective and all edges of r (i.e. all edges of P adjacent to r) have valence ≥ 3. Indeed, adding a bubble on each edge of r as in Fig. 3 and then pushing these bubbles along the edges of r and further across the vertices, we transform r into a smaller region $r_0 \subset r$ satisfying $(*)$. Reversing the orientation and the label of r_0 and then removing the newly added bubbles we obtain P_-. Therefore if our claim holds for r_0, then it holds for r.

Let now r be a disk region of P satisfying $(*)$ and let $f \in G$ be the label of r. Since M and r are oriented, we may speak about positive and negative normal vectors on r. Let $r_- \subset M\backslash r$ be a 2-disk obtained by pushing r in the positive normal direction so that ∂r sweeps a narrow annulus $A \subset P - \mathrm{Int}(r)$. It is understood that $\partial A = \partial r \amalg \partial r_-$ and $\partial r_- = r_- \cap P$. Then $\widehat{P} = P \cup r_-$ is a skeleton of M where the regions of $\widehat{P}$ contained in P receive the induced orientation and the orientation of r_- is opposite to that of r. Every region R of $\widehat{P}$ distinct from r, r_- and not lying in A is contained in a unique region of P. We take the label of the latter region as the label of R. We endow r, r_- with labels $h, h_- \in G$, respectively, such that $hh_-^{-1} = f$. The labels of the regions of $\widehat{P}$ lying in A are determined uniquely by the product condition. This turns $\widehat{P}$ into a G-skeleton of M. We say that $\widehat{P}$ is obtained by *doubling* r. We can transform $\widehat{P}$ into P by moves of type $T^{m,n}, \widetilde{T}^{0,2}, \widetilde{T}^{2,0}$ pushing ∂r_- inside r and an inverse bubble move eliminating the bubble resulting from r_-.

A similar elimination of r transforms $\widehat{P}$ into P_-. Therefore P and P_- are related by primary moves. This completes the proof of our claim. $\square$

We can now finish the proof of Lemma 7.2. By [13, Lemma 7.1], any two skeletons of M can be related by a finite sequence of primary moves. This lifts to a sequence of primary moves on G-skeletons. Therefore we need only to prove that if $P_1 = (P, \ell_1)$ and $P_2 = (P, \ell_2)$ are two G-skeletons of M with the same underlying skeleton P, then there is a sequence of primary moves $P_1 \rightsquigarrow P_2$.

By Sec. 6.3, the G-labelings ℓ_1 and ℓ_2 of P lie in the same $\mathcal{G}_P$-orbit, where $\mathcal{G}_P = \mathrm{Map}(\pi_0(M\backslash P), G)$ is the gauge group of P. This group is generated by the maps $\lambda_{g,b} : \pi_0(M\backslash P) \to G$, with $g \in G$ and b a P-ball, where $\lambda_{g,b}$ carries b to g and carries all other P-balls to 1. To prove Lemma 7.2, it is enough to produce for any G-labeling ℓ of P, any $g \in G$, and any P-ball b, a sequence of primary moves $(P, \ell) \rightsquigarrow (P, \ell_{g,b} = \lambda_{g,b}\ell)$.

Consider the bubble move $(P, \ell) \mapsto (P', \ell')$ attaching a small disk inside b to a region of P adjacent to b. Under this move the ball b splits into two balls: the small one (the bubble) and the complementary one, b'. The same bubble move transforms $(P, \ell_{g,b})$ into $(P', \ell'_{g,b'})$. Therefore if there are primary moves transforming (P', ℓ') into $(P', \ell'_{g,b'})$, then there are primary moves transforming (P, ℓ) into $(P, \ell_{g,b})$. Similar arguments work for the modified bubble move shown in Fig. 3 and for the moves $T^{m,n}$. Applying such moves inside b, we can replace P and b with another pair still denoted P, b such that the new P-ball b has one additional property: its closure $\overline{b}$ is a closed embedded 3-ball in M with interior b. Then each region of P is adjacent to b on one side or not at all.

Suppose there is a region r_0 of P adjacent to b such that the positive normal vectors on r_0 look inside b. Consider the G-skeleton (P_-, ℓ_-) obtained from (P, ℓ) by inverting the orientation and the label of r_0. By the claim above, there are sequences of primary moves $(P, \ell) \rightsquigarrow (P_-, \ell_-)$ and $(P, \ell_{g,b}) \rightsquigarrow (P_-, (\ell_{g,b})_-)$. Observe that $(\ell_{g,b})_- = (\ell_-)_{g,b}$. Therefore if there is a sequence of primary moves $(P_-, \ell_-) \rightsquigarrow (P_-, (\ell_-)_{g,b})$, then there is a sequence of primary moves $(P, \ell) \rightsquigarrow (P, \ell_{g,b})$. Continuing by induction, we can reduce ourselves to the case where the orientation of all regions of P lying in the 2-sphere $\partial\overline{b}$ is induced by that of M restricted to $\overline{b}$.

We now produce a sequence of primary moves $(P, \ell) \rightsquigarrow (P, \ell_{g,b})$. Pick a region r_0 of P adjacent to b. We first apply a bubble move $(P, \ell) \mapsto (P', \ell')$ which adds to P a 2-disk $D_+ \subset \overline{b}$ such that the circle ∂D_+ lies in r_0 and bounds a 2-disk $D_- \subset r_0$. We endow D_- with the orientation induced by that of r_0 and orient D_+ so that $\partial D_+ = \partial D_-$ in the category of oriented manifolds. We label D_- with $g^{-1}\ell(r_0)$ and D_+ with g (the orientations and the labels of all "big" regions of P' are the same as in P). Next, we isotop the disk D_+ in b so that it sweeps b almost entirely while its boundary slides along $\partial\overline{b}$. We arrange that in the terminal position, D'_+, of the moving disk its boundary circle lies in $r_0\backslash D_-$ and bounds there a 2-disk. This isotopy of D_+ transforms the G-skeleton (P', ℓ') into a new G-skeleton (P'', ℓ'') via a sequence of moves $T^{m,n}$, $\widetilde{T}^{2,0}$ and $\widetilde{T}^{0,2}$. Under these moves, all regions of the intermediate skeletons lying in $\partial\overline{b}$ are provided with orientation induced by that of M restricted to $\overline{b}$. This ensures that there are no orientation obstructions to the moves $T^{2,0}$ and $\widetilde{T}^{2,0}$ that may appear in our sequence. Finally, the inverse bubble move, removing D'_+, transforms (P'', ℓ'') into $(P, \ell_{g,b})$. □

7.3. *Proof of Theorem* 7.1

The state-sum $|M|_\mathcal{C}$ does not depend on the choice of the representative set I by the naturality of $\mathbb{F}_\mathcal{C}$ and of the contraction maps. Lemma 7.2 shows that to prove the rest of the theorem, we need only to prove that $|M|_\mathcal{C}$ is preserved under the primary moves $P \mapsto P'$. This follows from the "local invariance" which says that the contribution of any $c \in \mathrm{Col}(P)$ to the state-sum is equal to the sum of the contributions of all $c' \in \mathrm{Col}(P')$ equal to c on all big regions. For the primary moves T_1, T_2, T_3, this local invariance was proved in [13, Sec. 7.5]. For the bubble move T_4, the local invariance follows from Lemma 4.1 where U is the value of c on the region of P where the bubble is attached, $V = \mathbb{1}$, and k, l are the values of c' on the disks D_+, D_- created by the move. The factor $\dim(U)\dim(\mathcal{C}_1)$ is compensated by the change in the number of components of $M \backslash P$ and in the Euler characteristic. We use here the equality $*_e(\mathbb{F}_\mathcal{C}(\Gamma_v)) = N^{\mathbb{1}}_{U \otimes k \otimes l}$, where e and v are respectively the edge and the vertex forming the circle $\partial D_+ = \partial D_-$.

7.4. *Remark*

The right-hand side of formula (7.1) is the product of $(\dim(\mathcal{C}_1))^{-|P|}$ and a certain sum which we denote $\Sigma_\mathcal{C}(P)$. The definition of $\Sigma_\mathcal{C}(P) \in \Bbbk$ does not use the assumption that $\dim(\mathcal{C}_1)$ is invertible in $\Bbbk$ and applies to an arbitrary spherical G-fusion category $\mathcal{C}$. This allows us to generalize the invariant $|M|_\mathcal{C}$ of a closed G-manifold M to any such $\mathcal{C}$. We use the theory of spines, see [9]. By a spine of M, we mean an oriented stratified 2-polyhedron $P \subset M$ such that P has at least 2 vertices, P is locally homeomorphic to the cone over the 1-skeleton of a tetrahedron, and $M \backslash P$ is an open ball. By [9], M has a spine P and any two spines of M can be related by the moves $T^{1,2}$, $T^{2,1}$ in the class of spines. The arguments above imply that $\Sigma_\mathcal{C}(P)$ is preserved under these moves. Therefore $\|M\|_\mathcal{C} = \Sigma_\mathcal{C}(P)$ is a topological invariant of M. If $\dim(\mathcal{C}_1)$ is invertible, then $\|M\|_\mathcal{C} = \dim(\mathcal{C}_1)|M|_\mathcal{C}$.

8. The State-Sum HQFT

In generalization of the state-sum invariant introduced in the previous section, we derive from any spherical G-fusion category $\mathcal{C}$ such that $\dim(\mathcal{C}_1) \in \Bbbk$ is invertible an HQFT $|\cdot|_\mathcal{C}$ with target $\mathbf{X} = K(G, 1)$.

8.1. *Skeletons of G-surfaces*

Let Σ be a pointed closed oriented surface. Recall that each component of Σ has a base point and $\Sigma_\bullet$ is the set of the base points. A *skeleton* of Σ is an oriented graph

$A \subset \Sigma$ such that all components of $\Sigma \backslash A$ are open disks, all vertices of A have valence ≥ 2, and $A \cap \Sigma_\bullet = \emptyset$. For example, the vertices and the edges of a triangulation of Σ (with an arbitrary orientation of the edges) form a skeleton provided the base points of Σ lie inside the 2-faces.

A *G-labeling* ℓ of a skeleton A of Σ is a map from the set of edges of A to G such that for any vertex v of A,

$$\prod_{a \in A_v} \ell(a)^{\varepsilon_v(a)} = 1, \tag{8.1}$$

where the product is determined by the cyclic order in A_v and $\ell(a) \in G$ is the value of ℓ on the edge of A containing the half-edge a (see Sec. 5.2 for the definition of A_v and ε_v). A G-labeling ℓ of A determines a homotopy class of maps $f_\ell : (\Sigma, \Sigma_\bullet) \to (\mathbf{X}, x)$ as follows (cf. Sec. 6.3). Pick a *central point* in each component of $\Sigma \backslash A$ so that all base points of Σ are among these centers. Choose oriented arcs in Σ dual to the edges of A and connecting the central points of the adjacent regions (the interiors of the arcs must be disjoint and the intersection number of each edge with the dual arc is $+1$). The map f_ℓ carries all the central points to x and carries the arcs in question to loops in $(\mathbf{X}, x)$ representing the values of ℓ on the corresponding edges. Formula (8.1) ensures that such a map f_ℓ exists. It is clear that the homotopy class of f_ℓ depends only on A and ℓ. We call the pair (A, ℓ) a *G-skeleton* of the G-surface (Σ, f_ℓ).

An appropriate choice of a G-labeling turns any skeleton of a G-surface into a G-skeleton of this G-surface.

8.2. *Skeletons in dimension* 3

We now extend the theory of skeletons of closed G-manifolds to G-manifolds with boundary. Given a compact oriented 3-manifold M with pointed boundary and a skeleton $A \subset \partial M$, we define a *skeleton* of (M, A) to be an oriented stratified 2-polyhedron $P \subset M$ such that $P \cap \partial M = \partial P$ and

(i) $\partial P = A$ as graphs, i.e. ∂P and A have the same vertices and edges;
(ii) for every vertex v of A, there is a unique edge e_v of P such that v is an endpoint of e_v and $e_v \not\subseteq \partial M$; the edge e_v is not a loop and $e_v \cap \partial M = \{v\}$;
(iii) every edge a of A is an edge of P of valence 1; the only region D_a of P adjacent to a is a closed 2-disk meeting ∂M precisely along a; the orientation of D_a is compatible with that of a;
(iv) all components of $M \backslash P$ are open or half-open 3-balls.

Note that the boundary disks of the half-open components of $M \backslash P$ are precisely the components of $\partial M \backslash A$. Conditions (i)–(iii) imply that the intersection of P with a tubular neighborhood of ∂M in M is homeomorphic to $A \times [0, 1]$. If an edge e of P has both endpoints in ∂M, then $e \subset A$ is an edge of A.

Let now $M = (M, f : (M, (\partial M)_\bullet) \to (\mathbf{X}, x))$ be a G-manifold and let $A = (A, \ell)$ be a G-skeleton of the G-surface ∂M. Given a skeleton P of (M, A), consider a map $\tilde{\ell}: \mathrm{Reg}(P) \to G$ such that Formula (6.1) holds for every edge of P not lying in A and $\tilde{\ell}(D_a) = \ell(a)$ for every edge a of A. The map $\tilde{\ell}$ determines a homotopy class of maps $f_{\tilde{\ell}}: (M, (\partial M)_\bullet) \to (\mathbf{X}, x)$ as in Sec. 6.3 where all points of $(\partial M)_\bullet$ are chosen as the centers of the corresponding components of $M \backslash P$. We say that $\tilde{\ell}$ is a *G-labeling* of P and $(P, \tilde{\ell})$ is a *G-skeleton of* (M, A) if $f_{\tilde{\ell}} = f$ is the given homotopy class of maps. It is easy to see that (M, A) has a skeleton (cf. [13, Lemma 8.1]) and every skeleton of (M, A) has a G-labeling turning it into a G-skeleton.

The primary moves $T_1^{\pm 1} - T_4^{\pm 1}$ defined above for closed G-manifolds extend to G-skeletons of (M, A) in the obvious way. All these moves proceed inside 3-balls in $\mathrm{Int}(M)$ and do not modify the boundary of the skeletons. In particular, the move T_1 adds an edge with both endpoints in $\mathrm{Int}(M)$, the move T_2 collapses an edge contained in $\mathrm{Int}(M)$, etc. The action of the moves on the G-labelings is determined by the requirement that the labels of the big regions are preserved under the moves. As in the case of closed G-manifolds, the labelings transform uniquely under $T_1^{\pm 1}, T_2^{\pm 1}, T_3^{\pm 1}, T_4^{-1}$ and non-uniquely under T_4, where the label of D_+ may be an arbitrary element of G. These moves as well as label-preserving ambient isotopies of G-skeletons of (M, A) keeping the boundary pointwise are called *primary moves*. All primary moves transform G-skeletons of (M, A) into G-skeletons of (M, A).

Lemma 8.1. *Any two G-skeletons of (M, A) can be related by a finite sequence of primary moves in the class of G-skeletons of (M, A).*

Proof. The proof reproduces the proof of Lemma 7.2 with obvious changes. Instead of [13, Lemma 7.1] we should use [13, Lemma 8.1] which says that any two skeletons of (M, A) can be related by primary moves in M. □

8.3. *Invariants of pairs (M, A)*

Fix up to the end of Sec. 8 a spherical G-fusion category $\mathcal{C}$ over $\Bbbk$ such that $\dim(\mathcal{C}_1)$ is invertible in $\Bbbk$. We shall derive from $\mathcal{C}$ a 3-dimensional HQFT $|\cdot|_{\mathcal{C}}$ with target $\mathbf{X} = K(G, 1)$. The construction proceeds in three steps described in this and the next two sections.

Fix a representative set $I = \coprod_{g \in G} I_g$ of simple objects of $\mathcal{C}$. By an *I-coloring* of a G-skeleton (A, ℓ) of a G-surface, we mean a map c from the set of edges of A to I such that $c(a) \in I_{\ell(a)}$ for all edges a of A. Note that an I-colored G-skeleton is $\mathcal{C}$-colored in the sense of Sec. 5.2 so that the definitions and notation of that section apply.

For a G-manifold M and an I-colored G-skeleton $A = (A, \ell, c)$ of ∂M, we define a topological invariant $|M, A| \in \Bbbk$ as follows. Pick a G-skeleton $P = (P, \tilde{\ell})$ of (M, A). Let $\mathrm{Col}(P, c)$ be the set of all maps $\tilde{c}: \mathrm{Reg}(P) \to I$ such that $\tilde{c}(r) \in I_{\tilde{\ell}(r)}$ for all $r \in \mathrm{Reg}(P)$ and $\tilde{c}(D_a) = c(a)$ for all edges a of A. For every $\tilde{c} \in \mathrm{Col}(P, c)$ and every oriented edge e of P, consider the $\Bbbk$-module $H_{\tilde{c}}(e) = H(P_e)$, where P_e is the set of branches of P at e turned into a cyclic $\mathcal{C}$-set as in Sec. 7.1 (with c replaced by $\tilde{c}$). Let E_0 be the set of oriented edges of P with both endpoints in $\mathrm{Int}(M)$, and let E_∂ be the set of edges of P with exactly one endpoint in ∂M oriented towards this endpoint. Every vertex v of A is incident to a unique edge e_v belonging to E_∂ and $H_{\tilde{c}}(e_v) = H_v(A^{\mathrm{op}}; -\partial M)$. Therefore

$$\bigotimes_{e \in E_\partial} H_{\tilde{c}}(e)^\star = \bigotimes_v H_v(A^{\mathrm{op}}; -\partial M)^\star = H(A^{\mathrm{op}}; -\partial M)^\star.$$

For $e \in E_0$, the equality $P_{e^{\mathrm{op}}} = (P_e)^{\mathrm{op}}$ induces a duality between the modules $H_{\tilde{c}}(e)$, $H_{\tilde{c}}(e^{\mathrm{op}})$ and a contraction homomorphism $H_{\tilde{c}}(e)^\star \otimes H_{\tilde{c}}(e^{\mathrm{op}})^\star \to \Bbbk$. This contraction does not depend on the orientation of e up to permutation of the factors. Applying these contractions, we obtain a homomorphism

$$*_P : \bigotimes_{e \in E_0 \cup E_\partial} H_{\tilde{c}}(e)^\star \to \bigotimes_{e \in E_\partial} H_{\tilde{c}}(e)^\star = H(A^{\mathrm{op}}; -\partial M)^\star.$$

As in Sec. 6.2, any vertex v of P lying in $\mathrm{Int}(M)$ determines an oriented graph Γ_v on S^2, and $\tilde{c}$ turns Γ_v into a $\mathcal{C}$-colored graph. Section 5.3 yields a tensor $\mathbb{F}_{\mathcal{C}}(\Gamma_v) \in H_{\tilde{c}}(\Gamma_v)^*$. Here $H_{\tilde{c}}(\Gamma_v) = \bigotimes_e H_{\tilde{c}}(e)$, where e runs over all edges of P incident to v and oriented away from v. The tensor product $\bigotimes_v \mathbb{F}_{\mathcal{C}}(\Gamma_v)$ over all vertices v of P lying in $\mathrm{Int}(M)$ is a vector in $\bigotimes_{e \in E_0 \cup E_\partial} H_{\tilde{c}}(e)^\star$.

Theorem 8.2. *Set*

$$|M, A| = (\dim(\mathcal{C}_1))^{-|P|} \sum_{\tilde{c} \in \mathrm{Col}(P,c)} \left(\prod_{r \in \mathrm{Reg}(P)} (\dim \tilde{c}(r))^{\chi(r)} \right) *_P \left(\bigotimes_v \mathbb{F}_{\mathcal{C}}(\Gamma_v) \right),$$

where $|P|$ is the number of components of $M \backslash P$ and χ is the Euler characteristic. Then $|M, A| \in H(A^{\mathrm{op}}; -\partial M)^\star$ does not depend on the choice of I and P.

Proof. Since any two G-skeletons of (M, A) are related by primary moves, we need only to verify the invariance of $|M, A|$ under these moves. This is done exactly as in the proof of Theorem 7.1. $\square$

Though there is a canonical isomorphism $H(A^{\mathrm{op}}; -\partial M)^\star \simeq H(A; \partial M)$ (see Sec. 5.2), it is convenient to view $|M, A|$ as a vector in $H(A^{\mathrm{op}}; -\partial M)^\star$.

8.4. *Functoriality*

Consider a G-manifold M whose boundary is a disjoint union of two G-surfaces Σ_0 and Σ_1. More precisely, we assume that $\partial M = (-\Sigma_0) \amalg \Sigma_1$ (as G-surfaces). Given an I-colored G-skeleton A_i of Σ_i for $i = 0, 1$, we form the I-colored G-skeleton $A_0^{\mathrm{op}} \cup A_1$ of ∂M. Theorem 8.2 yields a vector

$$|M, A_0^{\mathrm{op}} \cup A_1| \in H(A_0 \cup A_1^{\mathrm{op}}, -\partial M)^\star = H(A_0, \Sigma_0)^\star \otimes H(A_1^{\mathrm{op}}, -\Sigma_1)^\star.$$

The canonical isomorphism $H(A_1^{\mathrm{op}}, -\Sigma_1)^\star \simeq H(A_1, \Sigma_1)$ induces an isomorphism

$$\Upsilon : H(A_0, \Sigma_0)^\star \otimes H(A_1^{\mathrm{op}}, -\Sigma_1)^\star \to \mathrm{Hom}_{\Bbbk}(H(A_0, \Sigma_0), H(A_1, \Sigma_1)).$$

Set

$$|M, \Sigma_0, A_0, \Sigma_1, A_1| = \frac{(\dim(\mathcal{C}_1))^{|A_1|}}{\dim(A_1)} \Upsilon(|M, A_0^{\mathrm{op}} \cup A_1|) : H(A_0; \Sigma_0) \to H(A_1; \Sigma_1),$$

where $|A_1|$ is the number of components of $\Sigma_1 \backslash A_1$ and $\dim(A_1)$ is the product of the dimensions of the simple objects of $\mathcal{C}$ associated with the edges of A_1 by the given I-coloring of A_1. By definition, if $\Sigma_1 = \emptyset$, then $A_1 = \emptyset$, $|A_1| = 0$ and $\dim(A_1) = 1$.

Lemma 8.3. *Let M_i be a G-manifold with $\partial M_i = (-\Sigma_i) \amalg \Sigma_{i+1}$, where Σ_i, Σ_{i+1} are G-surfaces and $i = 0, 1$. Let M be the G-manifold obtained by gluing M_0 and M_1 along Σ_1 so that $\partial M = (-\Sigma_0) \amalg \Sigma_2$. For any I-colored G-skeletons $A_0 \subset \Sigma_0$, $A_2 \subset \Sigma_2$ and any G-skeleton A_1 of Σ_1,*

$$|M, \Sigma_0, A_0, \Sigma_2, A_2| = \sum_c |M, \Sigma_1, (A_1, c), \Sigma_2, A_2| \circ |M, \Sigma_0, A_0, \Sigma_1, (A_1, c)|,$$

where c runs over all I-colorings of A_1.

Proof. This follows from the definitions since the union of a G-skeleton, P_0, of $(M_0, A_0^{\mathrm{op}} \cup A_1)$ with a G-skeleton, P_1, of $(M_1, A_1^{\mathrm{op}} \cup A_2)$ is a G-skeleton, P, of $(M, A_0^{\mathrm{op}} \cup A_2)$ and $|P| = |P_0| + |P_1| - |A_1|$. The term $-|A_1|$ explains the need for the factor $(\dim(\mathcal{C}_1))^{|A_1|}$ in the definition of $|M, \Sigma_0, A_0, \Sigma_1, A_1|$. Similarly, given a region r_1 of P_1 and a region r_2 of P_2 adjacent to the same edge e of A_1, the union $r = r_1 \cup r_2 \cup e$ is a region of P and $\chi(r) = \chi(r_1) + \chi(r_2) - 1$. The term -1 explains the need for the factor $(\dim(A_1))^{-1}$ in the definition of $|M, \Sigma_0, A_0, \Sigma_1, A_1|$. $\square$

8.5. *The HQFT* $|\cdot|_{\mathcal{C}}$

For a G-skeleton A of a G-surface Σ, denote by $\mathrm{Col}(A)$ the set of all I-colorings of A. Set

$$|A; \Sigma|^\circ = \bigoplus_{c \in \mathrm{Col}(A)} H((A, c); \Sigma).$$

Given a G-cobordism $(M, h : (-\Sigma_0) \sqcup \Sigma_1 \simeq \partial M)$ between G-surfaces Σ_0 and Σ_1, we now define for any G-skeletons $A_0 \subset \Sigma_0$ and $A_1 \subset \Sigma_1$ a homomorphism

$$|M, \Sigma_0, A_0, \Sigma_1, A_1|^\circ : |A_0; \Sigma_0|^\circ \to |A_1; \Sigma_1|^\circ. \tag{8.2}$$

For $i = 0, 1$ denote by Σ_i' the G-surface $h(\Sigma_i) \subset \partial M$ with orientation induced by the one in Σ_i. Then $A_i' = h(A_i)$ with the G-labeling induced by that of A_i is a G-skeleton of Σ_i'. Consider the homomorphism

$$\sum_{\substack{c_0 \in \mathrm{Col}(A_0') \\ c_1 \in \mathrm{Col}(A_1')}} |M, \Sigma_0', (A_0', c_0), \Sigma_1', (A_1', c_1)| : |A_0'; \Sigma_0'|^\circ \to |A_1'; \Sigma_1'|^\circ, \tag{8.3}$$

where

$$|M, \Sigma_0', (A_0', c_0), \Sigma_1', (A_1', c_1)| : H((A_0', c_0); \Sigma_0') \to H((A_1', c_1); \Sigma_1').$$

Conjugating (8.3) by the obvious isomorphisms $\{|A_i;\Sigma_i|^\circ \cong |A'_i;\Sigma'_i|^\circ\}_{i=0,1}$ induced by h, we obtain the homomorphism (8.2). Lemma 8.3 implies that for any G-cobordisms M_0, M_1, M as in this lemma and for any G-skeletons $\{A_i \subset \Sigma_i\}_{i=0}^2$,

$$|M,\Sigma_0,A_0,\Sigma_2,A_2|^\circ = |M,\Sigma_1,A_1,\Sigma_2,A_2|^\circ \circ |M,\Sigma_0,A_0,\Sigma_1,A_1|^\circ. \tag{8.4}$$

These constructions assign a finitely generated free $\Bbbk$-module to every G-surface with distinguished G-skeleton and a homomorphism of these modules to every G-cobordism whose bases are endowed with G-skeletons. This data satisfies an appropriate version of the axioms of an HQFT except one: the homomorphism associated with the cylinder over a G-surface, generally speaking, is not the identity. There is a standard procedure which transforms such a "pseudo-HQFT" into a genuine HQFT and also gets rid of the skeletons. This procedure is described in a similar situation in [11, Sec. VII.3]; we outline it in our setting.

Observe that if A_0, A_1 are two G-skeletons of a G-surface Σ, then the cylinder G-cobordism $\Sigma\times[0,1]$ gives a homomorphism

$$\begin{aligned} &p(A_0,A_1)\\ &\quad = |\Sigma\times[0,1],\Sigma\times\{0\},A_0\times\{0\},\Sigma\times\{1\},A_1\times\{1\}|^\circ : |A_0;\Sigma|^\circ \to |A_1;\Sigma|^\circ. \end{aligned}$$

Formula (8.4) implies that $p(A_0,A_2)=p(A_1,A_2)\,p(A_0,A_1)$ for any G-skeletons A_0, A_1, A_2 of Σ. Taking $A_0=A_1=A_2$ we obtain that $p(A_0,A_0)$ is a projector onto a direct summand $|A_0;\Sigma|$ of $|A_0;\Sigma|^\circ$. Moreover, $p(A_0,A_1)$ maps $|A_0;\Sigma|$ isomorphically onto $|A_1;\Sigma|$. The finitely generated projective $\Bbbk$-modules $\{|A;\Sigma|\}_A$, where A runs over all G-skeletons of Σ, and the homomorphisms $\{p(A_0,A_1)\}_{A_0,A_1}$ form a projective system. The projective limit of this system is a $\Bbbk$-module independent of the choice of a G-skeleton of Σ, and we denote it by $|\Sigma|_{\mathcal{C}}$. For each G-skeleton A of Σ, we have a "cone isomorphism" $|A;\Sigma|\cong|\Sigma|_{\mathcal{C}}$. By convention, the empty surface $\emptyset$ has a unique (empty) skeleton and $|\emptyset|_{\mathcal{C}}=\Bbbk$.

Any G-cobordism (M,Σ_0,Σ_1) splits as a product of a G-cobordism with a G-cylinder over Σ_1. Using this splitting and Formula (8.4), we obtain that the homomorphism (8.3) carries $|\Sigma_0|_{\mathcal{C}}\cong|A_0;\Sigma_0|\subset|A_0;\Sigma_0|^\circ$ into $|\Sigma_1|_{\mathcal{C}}\cong|A_1;\Sigma_1|\subset|A_1;\Sigma_1|^\circ$ for any G-skeletons A_0, A_1 of Σ_0, Σ_1, respectively. This gives a homomorphism $|M,\Sigma_0,\Sigma_1|_{\mathcal{C}}: |\Sigma_0|_{\mathcal{C}}\to|\Sigma_1|_{\mathcal{C}}$ independent of the choice of A_0 and A_1. Moreover, two G-cobordisms representing the same morphism $\varphi:\Sigma_0\to\Sigma_1$ in Cob_3^G give rise to the same homomorphism $|\varphi|_{\mathcal{C}}:|\Sigma_0|_{\mathcal{C}}\to|\Sigma_1|_{\mathcal{C}}$. By construction, $|\mathrm{id}_\Sigma|_{\mathcal{C}}=\mathrm{id}_{|\Sigma|_{\mathcal{C}}}$. The assignment $\Sigma\mapsto|\Sigma|_{\mathcal{C}}$, $\varphi\mapsto|\varphi|_{\mathcal{C}}$ defines a functor $|\cdot|_{\mathcal{C}}:\mathrm{Cob}_3^G\to\mathrm{vect}_\Bbbk$. The results above imply the following theorem.

Theorem 8.4. *The functor $|\cdot|_{\mathcal{C}}$ is a 3-dimensional HQFT with target* $\mathbf{X}$.

The HQFT $|\cdot|_{\mathcal{C}}$ is called the *state-sum HQFT* derived from $\mathcal{C}$. Considered up to isomorphism, the HQFT $|\cdot|_{\mathcal{C}}$ does not depend on the choice of the representative set I of simple objects of $\mathcal{C}$. For a closed G-manifold M, the scalar $|M|_{\mathcal{C}}\in\Bbbk$ produced by this HQFT is precisely the invariant of Sec. 7.

8.6. *Example*

By [12, Sec. I.2.1], every $\theta \in H^3(G, \Bbbk^*) = H^3(\mathbf{X}; \Bbbk^*)$ defines a 3-dimensional HQFT τ^θ with target $\mathbf{X}$, called a primitive cohomological HQFT. In particular, $\tau^\theta(\Sigma) \cong \Bbbk$ for any G-surface Σ and $\tau^\theta(M, g) = g^*(\theta)([M])$ for any closed G-manifold $(M, g \colon M \to \mathbf{X})$, where $[M]$ is the fundamental class of M. On the other hand, the spherical G-fusion category vect_G^θ of Example 4.3 defines a state-sum HQFT $|\cdot|_{\mathrm{vect}_G^\theta}$ with target $\mathbf{X}$. It can be shown that these HQFTs are isomorphic: $\tau^\theta \cong |\cdot|_{\mathrm{vect}_G^\theta}$. This shows that the HQFT τ^θ may be fully computed via a state-sum on G-skeletons.

Appendix. Push-Forwards of Categories and HQFTs

Let $\phi \colon H \to G$ be a group epimorphism with finite kernel. Every H-category $\mathcal{C} = \bigoplus_{h \in H} \mathcal{C}_h$ determines a G-category $\phi_*(\mathcal{C}) = \bigoplus_{g \in G} \phi_*(\mathcal{C})_g$ called the *push-forward of* $\mathcal{C}$. By definition, $\phi_*(\mathcal{C}) = \mathcal{C}$ as pivotal categories and $\phi_*(\mathcal{C})_g = \bigoplus_{h \in \phi^{-1}(g)} \mathcal{C}_h$ for all $g \in G$. If $\mathcal{C}$ is H-fusion, then $\phi_*(\mathcal{C})$ is G-fusion and $\dim(\phi_*(\mathcal{C})_1) = \gamma \dim(\mathcal{C}_1)$ where γ is the order of $\mathrm{Ker}(\phi)$. If $\mathcal{C}$ is spherical, then so is $\phi_*(\mathcal{C})$.

Suppose that $\mathcal{C}$ is a spherical H-fusion category and $\gamma \dim(\mathcal{C}_1) \in \Bbbk^*$. Then $\mathcal{C}$ and $\phi_*(\mathcal{C})$ determine HQFTs $|\cdot|_{\mathcal{C}}$ and $|\cdot|_{\phi_*(\mathcal{C})}$ with targets $\mathbf{Y} = K(H, 1)$ and $\mathbf{X} = K(G, 1)$ respectively. One can directly compute $|\cdot|_{\phi_*(\mathcal{C})}$ from $|\cdot|_{\mathcal{C}}$ in terms of the map $\widetilde{\phi} \colon \mathbf{Y} \to \mathbf{X}$ inducing ϕ in π_1. In particular, for any G-surface $(\Sigma, f \colon \Sigma \to \mathbf{X})$ and for any closed connected G-manifold $(M, g \colon M \to \mathbf{X})$,

$$|\Sigma, f|_{\phi_*(\mathcal{C})} = \bigoplus_{\widetilde{f} \in [\Sigma, \mathbf{Y}], \widetilde{\phi}\widetilde{f} = f} |\Sigma, \widetilde{f}|_{\mathcal{C}} \quad \text{and} \quad |M, g|_{\phi_*(\mathcal{C})} = \gamma^{-1} \sum_{\widetilde{g} \in [M, \mathbf{Y}], \widetilde{\phi}\widetilde{g} = g} |M, \widetilde{g}|_{\mathcal{C}},$$

where $[\Sigma, \mathbf{Y}]$ (respectively, $[M, \mathbf{Y}]$) denotes the set of homotopy classes of maps $\Sigma \to \mathbf{Y}$ (respectively, $M \to \mathbf{Y}$). It is understood that the maps carry the set of base points of Σ (respectively, a distinguished point of M) to the base point of $\mathbf{Y}$ and the homotopies are constant on this set. The sums above are finite because $\mathrm{Ker}(\phi)$ is finite. Similar formulas hold for G-cobordisms.

We point out a special case of this construction. Any spherical fusion category $\mathcal{C}$ can be viewed as a spherical Γ-fusion category where Γ is the graduator of $\mathcal{C}$, see Sec. 4.4. Denote the resulting spherical Γ-fusion category by $\widetilde{\mathcal{C}}$. Clearly, the group Γ is finite and $\mathcal{C} = \phi_*(\widetilde{\mathcal{C}})$ for the trivial homomorphism $\phi \colon \Gamma \to \{1\}$. If $\dim(\mathcal{C}) \in \Bbbk^*$, then $\mathcal{C}$ gives rise to a state-sum TQFT $|\cdot|_{\mathcal{C}}$ and $\widetilde{\mathcal{C}}$ gives rise to a state-sum HQFT $|\cdot|_{\widetilde{\mathcal{C}}}$ with target $K(\Gamma, 1)$. For any closed connected oriented surface Σ and any closed connected oriented 3-manifold M,

$$|\Sigma|_{\mathcal{C}} = \bigoplus_{f \colon \pi_1(\Sigma) \to \Gamma} |\Sigma, f|_{\widetilde{\mathcal{C}}} \quad \text{and} \quad |M|_{\mathcal{C}} = |\Gamma|^{-1} \sum_{g \colon \pi_1(M) \to \Gamma} |M, g|_{\widetilde{\mathcal{C}}}.$$

For more on this, see [10].

Acknowledgments

The work of V. Turaev on this paper was partially supported by the NSF grants DMS-0707078 and DMS-0904262.

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