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Quantum invariants of 3-manifolds, TQFTs, and Hopf monads

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Contents

Introduction

My work concerns quantum topology. Quantum Topology has its inception in the discovery by Jones (1984) of a new polynomial invariant of knots and links. This invariant was rapidly connected with quantum groups, introduced by Drinfeld and Jimbo (1985), and methods in statistical mechanics. This was followed by Witten's introduction of methods of quantum field theory into the subject and the formulation by Witten and Atiyah (1988) of the concept of topological quantum field theories (TQFTs).

Two fundamental constructions of 3-dimensional TQFTs, which give rise in particular to scalar invariants of closed 3-manifolds, are due to Reshetikhin-Turaev [RT] and Turaev-Viro [TV]. The RT-construction is widely viewed as a mathematical realization of Witten's Chern-Simons TQFT, see [Wi]. The TV-construction is closely related to the Ponzano-Regge state sum model for 3-dimensional quantum gravity, see [Ca]. How these two constructions are related? Before addressing this question, let us briefly recall the definitions of the RT and TV invariants.

The Turaev-Viro approach uses as the main algebraic ingredient spherical fusion categories. A fusion category is a monoidal category with compatible left and right dualities such that all objects are direct sums of simple objects and the number of isomorphism classes of simple objects is finite. The condition of sphericity says that the left and right dimensions of all objects are equal. The form of the TV-construction widely viewed as the most general is due to Barrett and Westbury [BW1] who derived a topological invariant $|M|_{\mathcal{C}}$ of an arbitrary closed oriented 3-manifold M from a spherical fusion category $\mathcal{C}$ with invertible dimension. The definition of $|M|_{\mathcal{C}}$ goes by considering a certain state sum on a triangulation of M and proving that this sum depends only on M and not on the choice of triangulation. The key algebraic ingredients of the state sum are the so-called $6j$-symbols associated with $\mathcal{C}$.

The Reshetikhin-Turaev approach uses as the main algebraic ingredient modular categories, see [Tu1]. A modular category is a spherical fusion category endowed with braiding satisfying a non-degeneracy condition (invertibility of the S-matrix). The RT-construction associates with every closed oriented 3-manifold M a numerical invariant $\tau_{\mathcal{B}}(M)$ from a modular category $\mathcal{B}$. The definition of $\tau_{\mathcal{B}}(M)$ consists in presenting M by surgery along a framed link in the 3-sphere and then taking a certain linear combination of colorings of this link by simple objects of $\mathcal{B}$.

The first connections between the Reshetikhin-Turaev and Turaev-Viro constructions were established by Walker [Wa] and Turaev [Tu1]: if $\mathcal{B}$ is a modular category, then it is also a spherical category with invertible dimension and the Reshetikhin-Turaev and Turaev-Viro invariants are related by:

$$|M|_{\mathcal{B}} = \tau_{\mathcal{B}}(M)\, \tau_{\mathcal{B}}(-M) \tag{1}$$

for every oriented closed 3-manifold M, where $-M$ is the 3-manifold M with opposite orientation. If $\mathcal{B}$ is a unitary modular category, then $\tau_{\mathcal{B}}(-M) = \overline{\tau_{\mathcal{B}}(M)}$ and so $|M|_{\mathcal{B}} = |\tau_{\mathcal{B}}(M)|^2$.

But in general a spherical category need not to be braided and so cannot be used as input to define the Reshetikhin-Turaev invariant. However, for every monoidal category $\mathcal{C}$, Joyal and Street [JS] and Drinfeld (unpublished, see Majid [Ma1]) defined a braided monoidal category $Z(\mathcal{C})$ called the center of $\mathcal{C}$. A fundamental theorem of Müger [Mü2] says that the center of a non-zero dimensional spherical fusion category $\mathcal{C}$ over an algebraically closed field is modular. Combining with the results mentioned above, we observe that such a $\mathcal{C}$ gives rise to two 3-manifold invariants: $|M|_{\mathcal{C}}$ and $\tau_{Z(\mathcal{C})}(M)$. In 1995, Turaev conjectured that these invariants are equal, i.e., for any closed oriented 3-manifold M,

$$|M|_{\mathcal{C}} = \tau_{Z(\mathcal{C})}(M). \tag{2}$$

The conjecture (2) was previously known to be true in several special cases: when $\mathcal{C}$ is modular [Tu1, Wa], when $\mathcal{C}$ is the category of bimodules associated with a subfactor [KSW], and when $\mathcal{C}$ is the category of representations of a finite group. Note that for a modular category $\mathcal{B}$, Formula (1) can indeed be derived from Formula (2) since the category $\mathcal{Z}(\mathcal{B})$ is then braided equivalent to the Deligne tensor product $\mathcal{B} \boxtimes \overline{\mathcal{B}}$, where $\overline{\mathcal{B}}$ is the mirror of $\mathcal{B}$, and therefore Formula (2) can be rewritten as $|M|_\mathcal{B} = \tau_{\mathcal{B}\boxtimes\overline{\mathcal{B}}}(M) = \tau_\mathcal{B}(M)\,\tau_{\overline{\mathcal{B}}}(M) = \tau_\mathcal{B}(M)\,\tau_\mathcal{B}(-M)$.

The connecting thread of my research in recent years was the proof of the conjecture (2). In this survey, we present a proof of this conjecture and the tools we developed to this end, in particular the theory of Hopf monads. Most of the achievements were done in collaboration with Alain Bruguières [6, 8, 9, 10] or Vladimir Turaev [12].

Formula (2) relates two categorical approaches to 3-manifold invariants through the categorical center. This relationship sheds new light on both approaches and shows, in particular, that the Reshetikhin-Turaev construction is more general than the Turaev-Viro state sum construction.

My work on the subject, which is going to be described in more detail below, can be roughly summarized as follows:

1) Kirby elements, Hopf diagrams, and quantum invariants [6, 7]
Given a spherical fusion category $\mathcal{C}$, how can we compute $\tau_{\mathcal{Z}(\mathcal{C})}(M)$? Using the algorithm given by Reshetikhin and Turaev is not a practicable approach here, as that would require a description of the simple objects of $\mathcal{Z}(\mathcal{C})$ in terms of those of $\mathcal{C}$, and no such description is available in general. What we need is a different algorithm for computing $\tau_{\mathcal{Z}(\mathcal{C})}(M)$, which one should be able to perform inside $\mathcal{C}$, without reference to the simple objects of $\mathcal{Z}(\mathcal{C})$. Let $\mathcal{B}$ be a (non-necessarily semisimple) ribbon category $\mathcal{B}$ admitting a coend C, which is a Hopf algebra in $\mathcal{B}$ (see [Ma2]). We define in [7] the notion of Kirby elements of $\mathcal{B}$ by means of the structural morphisms of C. To each Kirby element α is associated a topological invariant $\tau_\mathcal{B}(M;\alpha)$ of closed oriented 3-manifolds M. This construction is made effective by encoding certain tangles by means of Hopf diagrams [6], from which the invariants $\tau_\mathcal{B}(M;\alpha)$ can be expressed in terms of certain structural morphisms of the coend C. If C admits a two-sided integral Λ, then Λ is a Kirby element and $\tau_\mathcal{B}(M;\Lambda)$ is equals to the Lyubashenko invariant [Lyu2], and to the Reshetikhin-Turaev invariant $\tau_\mathcal{B}(M)$ if $\mathcal{B}$ is moreover $\mathcal{B}$ is semisimple. Hence, when $\mathcal{C}$ is a spherical fusion category, we may compute $\tau_{\mathcal{Z}(\mathcal{C})}(M)$ provided we can describe explicitly the structural morphisms of the coend of $\mathcal{Z}(\mathcal{C})$. In other words, we need an algebraic interpretation of the center construction. This motivated the introduction of the notion of Hopf monads.

2) Hopf monads and categorical centers [8, 10, 11]
Recall that a monad T on a category $\mathcal{C}$ is a monoid in the monoidal category of endofunctors of $\mathcal{C}$. Then one defines a category $\mathcal{C}^T$ of T-modules in $\mathcal{C}$. Following Moerdijk [Mo], a bimonad on a monoidal category $\mathcal{C}$ is a monad T on $\mathcal{C}$ which is comonoidal: it comes with some coproduct and counit making $\mathcal{C}^T$ monoidal and the forgetful functor $U_T\colon \mathcal{C}^T \to \mathcal{C}$ strict monoidal. There is no straightforward generalization of the notion of antipode to the monoidal setting. When $\mathcal{C}$ is an autonomous category (that is, a monoidal category whose objects have duals), according to Tannaka theory, one expects that a bimonad T be Hopf if and only if $\mathcal{C}^T$ is autonomous. This turns out to be equivalent to the existence of a unary antipode. That is the definition of a Hopf monad we gave in [8]. For example, any comonoidal adjunction between autonomous categories gives rise to a Hopf monad. This definition of Hopf monad is satisfactory for applications to quantum topology,

as the categories involved are autonomous, but it has some drawbacks for other applications: for instance, it doesn't encompass infinite-dimensional Hopf algebras since the category of vector spaces of arbitrary dimension is not autonomous. That is why, in [11], we generalize the above definition: a *Hopf monad* on an arbitrary monoidal category $\mathcal{C}$ is a bimonad on $\mathcal{C}$ whose fusion operators are invertible. Hopf monads generalize Hopf algebras to arbitrary monoidal categories. On a monoidal category with internal Homs, a Hopf monad is a bimonad admitting a binary antipode. For example, Hopf algebroids are linear Hopf monads on a category of bimodules admitting a right adjoint. It turns out that certain classical results on Hopf algebras extend naturally to Hopf monads, such as Maschke's semisimplicity criterion and Sweedler's theorem on the structure of Hopf modules. Also the notion of Hopf monad is suitable for Tannaka reconstruction: for example, any finite tensor category is the category of finite-dimensional modules over a Hopf algebroid.

The whole point of introducing Hopf monads is that they provide an algebraic interpretation of the center construction, see [10]. The center $\mathcal{Z}(\mathcal{C})$ of an autonomous category $\mathcal{C}$ is monadic over $\mathcal{C}$ (if certain coends exist in $\mathcal{C}$). Its monad Z is a quasitriangular Hopf monad on $\mathcal{C}$ and $\mathcal{Z}(\mathcal{C})$ is isomorphic to the braided category $\mathcal{C}^Z$ of Z- modules. More generally, let T be a Hopf monad on an autonomous category $\mathcal{C}$. We construct a Hopf monad Z_T on $\mathcal{C}$, the *centralizer of* T, and a canonical distributive law $\Omega\colon TZ_T \to Z_TT$. By Beck's theory, this has two consequences. On one hand, the composition $D_T = Z_T \circ_\Omega T$ is a quasitriangular Hopf monad on $\mathcal{C}$, called the *double of* T, and $\mathcal{Z}(\mathcal{C}^T) \simeq \mathcal{C}^{D_T}$ as braided categories. As an illustration, this allows us to define the double of any Hopf algebra in a braided autonomous category, generalizing (but not straightforwardly) the Drinfeld double of finite dimensional Hopf algebras. On the other hand, the canonical distributive law Ω also lifts Z_T to a Hopf monad $\tilde{Z}_T^\Omega$ on $\mathcal{C}^T$, which describes the coend of $\mathcal{C}^T$. For $T = Z$, this gives an explicit description of the Hopf algebra structure of the coend of $\mathcal{Z}(\mathcal{C})$ in terms of the structural morphisms of $\mathcal{C}$. Such a description is useful in quantum topology, especially when $\mathcal{C}$ is a spherical fusion category, as $\mathcal{Z}(\mathcal{C})$ is then modular.

3) On two approaches to 3-dimensional TQFTs [12]
We show that the conjecture (2) is true: the Turaev-Viro and Reshetikhin-Turaev invariants are related via the categorical center, i.e., if $\mathcal{C}$ is non-zero dimensional spherical fusion category over an algebraically closed field, then $|M|_\mathcal{C} = \tau_{Z(\mathcal{C})}(M)$ for any closed oriented 3-manifold M. As a corollary, by the results above, we get that the state sum $|M|_\mathcal{C}$ can be efficiently computed in terms of Hopf diagrams and the structural morphisms of the coend of $\mathcal{Z}(\mathcal{C})$.

Our proof is based on topological quantum field theory (TQFT). For a modular category $\mathcal{B}$, the Reshetikhin-Turaev invariant $\tau_\mathcal{B}(M)$ extends to a 3-dimensional TQFT $\tau_\mathcal{B}$ derived from $\mathcal{B}$, see [RT, Tu1]. For a spherical fusion category $\mathcal{C}$ with invertible dimension, we extend the state sum invariant $|M|_\mathcal{C}$ to a 3-dimensional TQFT $|.|_\mathcal{C}$ which we define in terms of state sums on skeletons of 3-manifolds. It is crucial for the proof of Formula (2) that we allow non-generic skeletons, i.e., skeletons with edges incident to ≥ 4 regions. In particular, we give a new state sum on any triangulation t of a closed oriented 3-manifold M, different from the one in [TV, BW1]. In the latter, the labels are attributed to the edges and the Boltzmann weights are the $6j$-symbols computed in the tetrahedra; in the former, the labels are attributed to the faces and the Boltzmann weights are computed in the vertices by means of an invariant of $\mathcal{C}$- colored graphs in the sphere. (It is non-obvious but true that these two state sums are equal.) Our main result in [12] is that for any non-zero dimensional spherical fusion category $\mathcal{C}$ over an algebraically closed field, the TQFTs $|.|_\mathcal{C}$ and $\tau_{Z(\mathcal{C})}$ are isomorphic:

$$|\Sigma|_\mathcal{C} \simeq \tau_{Z(\mathcal{C})}(\Sigma) \quad \text{and} \quad |M|_\mathcal{C} \simeq \tau_{Z(\mathcal{C})}(M)$$

for any closed oriented surface Σ and any oriented 3-cobordism M. The proof involves a detailed study of transformations of skeletons of 3-manifolds and the computation of the coend of $Z(\mathcal{C})$ provided by the theory of Hopf monads.

This survey is organized as follows. Section 1 deals with preliminaries on categories. In Section 2, we define and study Kirby elements and Hopf diagrams. Section 3 is devoted to the theory of Hopf monads. In Section 4, we prove the conjecture (2). Section 5 deals with some of my other works and the perspectives.

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Throughout the survey, the symbol $\Bbbk$ denotes a commutative ring.

1. Preliminaries on categories

In this section, we recall some basic definitions on categories. Most of the material of this section is classical. We refer to [Mac, Kas, Tu1] for details.

1.1. **Autonomous categories.** Monoidal categories are assumed to be strict. Let $\mathcal{C}$ be a monoidal category, with monoidal product $\otimes$ and unit object $\mathbb{1}$. Recall that a *duality* in $\mathcal{C}$ is a quadruple (X, Y, e, d), where X, Y are objects of $\mathcal{C}$, $e \colon X \otimes Y \to \mathbb{1}$ (the *evaluation*) and $c \colon \mathbb{1} \to Y \otimes X$ (the *coevaluation*) are morphisms in $\mathcal{C}$, such that:

$$(e \otimes \mathrm{id}_X)(\mathrm{id}_X \otimes c) = \mathrm{id}_X \quad \text{and} \quad (\mathrm{id}_Y \otimes e)(c \otimes \mathrm{id}_Y) = \mathrm{id}_Y.$$

Then (X, e, c) is a *left dual* of Y, and (Y, e, c) is a *right dual* of X.

If $D = (X, Y, e, d)$ and $D' = (X', Y', e', d')$ are two dualities, two morphisms $f \colon X \to X'$ and $g \colon Y' \to Y$ are *in duality with respect to D and D'* if

$$e'(f \otimes \mathrm{id}_{Y'}) = e(\mathrm{id}_X \otimes g) \quad \big(\text{or, equivalently, } (\mathrm{id}_{Y'} \otimes f)d = (g \otimes \mathrm{id}_X)d'\big).$$

In that case we write $f = {}^\vee g_{D,D'}$ and $g = f^\vee_{D,D'}$, or simply $f = {}^\vee g$ and $g = f^\vee$. Note that this defines a bijection between $\mathrm{Hom}_\mathcal{C}(X, X')$ and $\mathrm{Hom}_\mathcal{C}(Y', Y)$.

Left and right duals, if they exist, are essentially unique: if (Y, e, d) and (Y', e', d') are right duals of some object X, then there exists a unique isomorphism $u \colon Y \to Y'$ such that $e' = e(\mathrm{id}_X \otimes u^{-1})$ and $d' = (u \otimes \mathrm{id}_X)d$.

A *left autonomous* category is a monoidal category for which every object X admits a left dual $({}^\vee X, \mathrm{ev}_X, \mathrm{coev}_X)$. Likewise, a *right autonomous* category is a monoidal category for which every object X admits a right dual $(X^\vee, \widetilde{\mathrm{ev}}_X, \widetilde{\mathrm{coev}}_X)$.

Assume $\mathcal{C}$ is a left autonomous category and, for each object X, pick a left dual $(X^*, \mathrm{ev}_X, \mathrm{coev}_X)$. This data defines a strong monoidal functor $?^* \colon \mathcal{C}^{\mathrm{op},\otimes\mathrm{op}} \to \mathcal{C}$, where $\mathcal{C}^{\mathrm{op},\otimes\mathrm{op}}$ is the opposite category to $\mathcal{C}$ with opposite monoidal structure. This monoidal functor is called the *left dual functor*. Notice that the actual choice of left duals is innocuous in the sense that different choices of left duals define canonically isomorphic left dual functors.

Likewise one defines the strong monoidal *right dual functor* $?^\vee : \mathcal{C}^{\mathrm{op},\otimes\mathrm{op}} \to \mathcal{C}$ associated with a right autonomous category $\mathcal{C}$.

An *autonomous* (or *rigid*) *category* is a monoidal category which is left and right autonomous.

Subsequently, when dealing with left or right autonomous categories, we shall always assume tacitly that left duals or right duals have been chosen. Moreover, in formulae, we will often abstain (by abuse) from writing down the following canonical isomorphisms:

$$\begin{aligned} &{}^\vee(X^\vee) \cong X, \qquad &&{}^\vee(X \otimes Y) \cong {}^\vee Y \otimes {}^\vee X, \qquad &&{}^\vee\mathbb{1} \cong \mathbb{1},\\ &({}^\vee X)^\vee \cong X, \qquad &&(X \otimes Y)^\vee \cong Y^\vee \otimes X^\vee, \qquad &&\mathbb{1}^\vee \cong \mathbb{1}. \end{aligned}$$

1.2. **Pivotal categories.** A *pivotal* (or *sovereign*) *category* is a left autonomous category endowed with *pivotal structure*, that is, a strong monoidal natural transformation $\phi_X : X \to {}^{\vee\vee}X$. Such a transformation is then an isomorphism. A sovereign category $\mathcal{C}$ is autonomous: for each object X of $\mathcal{C}$, set:

$$\begin{aligned} \widetilde{\mathrm{ev}}_X &= \mathrm{ev}_{{}^\vee X}(\phi_X \otimes \mathrm{id}_{{}^\vee X}) \colon X \otimes {}^\vee X \to \mathbb{1},\\ \widetilde{\mathrm{coev}}_X &= (\mathrm{id}_{{}^\vee X} \otimes \phi_X^{-1})\mathrm{coev}_{{}^\vee X} : \mathbb{1} \to {}^\vee X \otimes X. \end{aligned}$$

Then $({}^\vee X, \widetilde{\mathrm{ev}}_X, \widetilde{\mathrm{coev}}_X)$ is a right dual of X. Moreover the right dual functor $?^\vee$ defined by this choice of right duals coincides with the left dual functor ${}^\vee ?$ as a strong monoidal functor. We denote the functor ${}^\vee ? = ?^\vee$ by $?^*$ and call it the *dual functor*. In particular, in a pivotal category, $X^* = {}^\vee X = X^\vee$ and $f^* = {}^\vee f = f^\vee$ for any object X and any morphism f.

1.3. **Traces and dimensions.** For an endomorphism f of an object X of a pivotal category $\mathcal{C}$, one defines the *left* and *right traces* $\operatorname{tr}_l(f), \operatorname{tr}_r(f) \in \operatorname{End}_{\mathcal{C}}(\mathbb{1})$ by

$$\operatorname{tr}_l(f) = \operatorname{ev}_X(\operatorname{id}_{X^*} \otimes f)\widetilde{\operatorname{coev}}_X \quad \text{and} \quad \operatorname{tr}_r(f) = \widetilde{\operatorname{ev}}_X(f \otimes \operatorname{id}_{X^*})\operatorname{coev}_X.$$

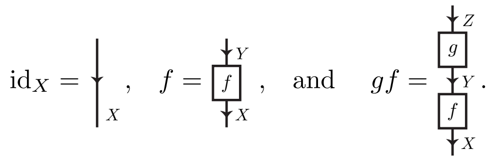
Both traces are symmetric: $\operatorname{tr}_l(gh) = \operatorname{tr}_l(hg)$ and $\operatorname{tr}_r(gh) = \operatorname{tr}_r(hg)$ for any morphisms $g\colon X \to Y$ and $h\colon Y \to X$ in $\mathcal{C}$. Also $\operatorname{tr}_l(f) = \operatorname{tr}_r(f^*) = \operatorname{tr}_l(f^{**})$ for any endomorphism f of an object (and similarly with l, r exchanged). If

$$\alpha \otimes \operatorname{id}_X = \operatorname{id}_X \otimes \alpha \quad \text{for all } \alpha \in \operatorname{End}_{\mathcal{C}}(\mathbb{1}) \text{ and } X \in \operatorname{Ob}(\mathcal{C}), \tag{3}$$

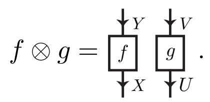
then the traces $\operatorname{tr}_l, \operatorname{tr}_r$ are $\otimes$-multiplicative: $\operatorname{tr}_l(f \otimes g) = \operatorname{tr}_l(f)\operatorname{tr}_l(g)$ and $\operatorname{tr}_r(f \otimes g) = \operatorname{tr}_r(f)\operatorname{tr}_r(g)$ for all endomorphisms f, g of objects of $\mathcal{C}$.

The *left* and *right dimensions* of $X \in \operatorname{Ob}(\mathcal{C})$ are defined by $\dim_l(X) = \operatorname{tr}_l(\operatorname{id}_X)$ and $\dim_r(X) = \operatorname{tr}_r(\operatorname{id}_X)$. Clearly, $\dim_l(X) = \dim_r(X^*) = \dim_l(X^{**})$ (and similarly with l, r exchanged). Note that isomorphic objects have the same dimensions and $\dim_l(\mathbb{1}) = \dim_r(\mathbb{1}) = \operatorname{id}_{\mathbb{1}}$. If $\mathcal{C}$ satisfies (3), then left and right dimensions are $\otimes$-multiplicative: $\dim_l(X \otimes Y) = \dim_l(X)\dim_l(Y)$ and $\dim_r(X \otimes Y) = \dim_r(X)\dim_r(Y)$ for any $X, Y \in \operatorname{Ob}(\mathcal{C})$.

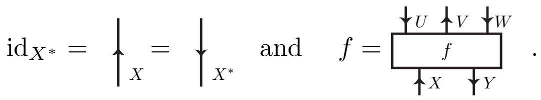
1.4. **Penrose graphical calculus.** We represent morphisms in a category $\mathcal{C}$ by plane diagrams to be read from the bottom to the top. The diagrams are made of oriented arcs colored by objects of $\mathcal{C}$ and of boxes colored by morphisms of $\mathcal{C}$. The arcs connect the boxes and have no mutual intersections or self-intersections. The identity id_X of $X \in \operatorname{Ob}(\mathcal{C})$, a morphism $f\colon X \to Y$, and the composition of two morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ are represented as follows:



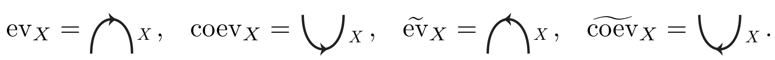
If $\mathcal{C}$ is monoidal, then the monoidal product of two morphisms $f\colon X \to Y$ and $g\colon U \to V$ is represented by juxtaposition:



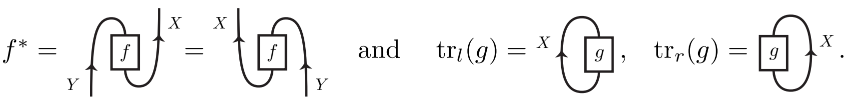
In a pivotal category, if an arc colored by X is oriented upwards, then the corresponding object in the source/target of morphisms is X^*. For example, id_{X^*} and a morphism $f\colon X^* \otimes Y \to U \otimes V^* \otimes W$ may be depicted as:



The duality morphisms are depicted as follows:



The dual of a morphism $f\colon X \to Y$ and the traces of a morphism $g\colon X \to X$ can be depicted as follows:



If $\mathcal{C}$ is pivotal, then the morphisms represented by the diagrams are invariant under isotopies of the diagrams in the plane keeping fixed the bottom and top endpoints.

1.5. **Linear categories.** A *monoidal* $\Bbbk$*-category* is a monoidal category $\mathcal{C}$ such that its hom-sets are (left) $\Bbbk$-modules, the composition and monoidal product of morphisms are $\Bbbk$-bilinear, and $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$ is a free $\Bbbk$-module of rank one. Then the map $\Bbbk \to \mathrm{End}_{\mathcal{C}}(\mathbb{1}), k \mapsto k\,\mathrm{id}_{\mathbb{1}}$ is a $\Bbbk$-algebra isomorphism. It is used to identify $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \Bbbk$.

A pivotal $\Bbbk$-category satisfies (3). Therefore the traces $\mathrm{tr}_l, \mathrm{tr}_r$ and the dimensions $\dim_l, \dim_r$ in such a category are $\otimes$-multiplicative. Clearly, $\mathrm{tr}_l, \mathrm{tr}_r$ are $\Bbbk$-linear.

1.6. **Spherical categories.** A *spherical category* is a pivotal category whose left and right traces are equal, i.e., $\mathrm{tr}_l(g) = \mathrm{tr}_r(g)$ for every endomorphism g of an object. Then $\mathrm{tr}_l(g)$ and $\mathrm{tr}_r(g)$ are denoted $\mathrm{tr}(g)$ and called the *trace of* g. In particular, the left and right dimensions of an object X are equal, denoted $\dim(X)$, and called the *dimension of* X.

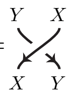
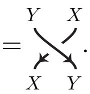
For spherical categories, the corresponding Penrose graphical calculus has the following property: the morphisms represented by diagrams are invariant under isotopies of diagrams in the 2-sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$, i.e., are preserved under isotopies pushing arcs of the diagrams across ∞. For example, the diagrams above representing $\mathrm{tr}_l(g)$ and $\mathrm{tr}_r(g)$ are related by such an isotopy. The condition $\mathrm{tr}_l(g) = \mathrm{tr}_r(g)$ for all g is therefore necessary (and in fact sufficient) to ensure this property.

1.7. **Braided categories.** A *braiding* in a monoidal category $\mathcal{B}$ is a natural isomorphism $\tau = \{\tau_{X,Y} \colon X \otimes Y \to Y \otimes X\}_{X,Y \in \mathrm{Ob}(\mathcal{B})}$ such that

$$\tau_{X,Y\otimes Z} = (\mathrm{id}_Y \otimes \tau_{X,Z})(\tau_{X,Y} \otimes \mathrm{id}_Z) \quad \text{and} \quad \tau_{X\otimes Y,Z} = (\tau_{X,Z} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes \tau_{Y,Z}).$$

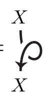
These conditions imply that $\tau_{X,\mathbb{1}} = \tau_{\mathbb{1},X} = \mathrm{id}_X$ for all $X \in \mathrm{Ob}(\mathcal{B})$.

A monoidal category endowed with a braiding is said to be *braided*. The braiding and its inverse are depicted as follows

$\tau_{X,Y} =$  and $\tau^{-1}_{Y,X} =$  .

A braided category satisfies (3) and so any braided pivotal category has $\otimes$-multiplicative left and right traces and dimensions.

For any object X of a braided pivotal category $\mathcal{B}$, one defines a morphism

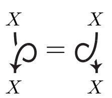
$\theta_X =$  $= (\mathrm{id}_X \otimes \widetilde{\mathrm{ev}}_X)(\tau_{X,X} \otimes \mathrm{id}_{X^*})(\mathrm{id}_X \otimes \mathrm{coev}_X) \colon X \to X.$

This morphism, called the *twist*, is invertible and

$\theta_X^{-1} =$  $= (\mathrm{ev}_X \otimes \mathrm{id}_X)(\mathrm{id}_{X^*} \otimes \tau_{X,X})(\widetilde{\mathrm{coev}}_X \otimes \mathrm{id}_X) \colon X \to X.$

Note that $\theta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$, $\theta_{X\otimes Y} = (\theta_X \otimes \theta_Y)\tau_{Y,X}\tau_{X,Y}$ for any $X, Y \in \mathrm{Ob}(\mathcal{B})$. The twist is natural: $\theta_Y f = f \theta_X$ for any morphism $f \colon X \to Y$ in $\mathcal{B}$.

1.8. **Ribbon categories.** A *ribbon category* is a braided pivotal category $\mathcal{B}$ whose twist is self-dual, i.e., $(\theta_X)^* = \theta_{X^*}$ for all $X \in \mathrm{Ob}(\mathcal{B})$. This is equivalent to the equality of morphisms



for any $X \in \mathrm{Ob}(\mathcal{B})$. In a ribbon category, for any $X \in \mathrm{Ob}(\mathcal{B})$,

$\theta_X^{-1} =$  $=$  .

A ribbon category $\mathcal{B}$ is spherical and gives rise to topological invariants of links in S^3. Namely, every $\mathcal{B}$-colored framed oriented link $L \subset S^3$ determines an endomorphism of the unit object $F_{\mathcal{B}}(L) \in \mathrm{End}(\mathbb{1})$ which turns out to be a topological invariant of L. Here L is $\mathcal{B}$-colored if every component of L is endowed with an object of $\mathcal{B}$ (called the color of this component). The definition of $F_{\mathcal{B}}(L)$ goes by an application of the Penrose calculus to a diagram of L; a new feature is that with the positive and negative crossings of the diagram one associates the braiding and its inverse, respectively. For more on this, see [Tu1].

1.9. **Fusion categories.** An object X of a monoidal $\Bbbk$-category $\mathcal{C}$ is *simple* if $\mathrm{End}_{\mathcal{C}}(X)$ is a free $\Bbbk$-module of rank 1. Equivalently, X is simple if the $\Bbbk$-homomorphism $\Bbbk \to \mathrm{End}_{\mathcal{C}}(X)$, $k \mapsto k\,\mathrm{id}_X$ is an isomorphism. By the definition of a monoidal $\Bbbk$-category, the unit object $\mathbb{1}$ is simple.

A *pre-fusion category* (over $\Bbbk$) is a pivotal $\Bbbk$-category $\mathcal{C}$ such that

(a) Any finite family of objects of $\mathcal{C}$ has a direct sum in $\mathcal{C}$;
(b) Each object of $\mathcal{C}$ is a finite direct sum of simple objects;
(c) For any non-isomorphic simple objects i, j of $\mathcal{C}$, we have $\mathrm{Hom}_{\mathcal{C}}(i,j) = 0$.

Conditions (b) and (c) imply that all the Hom spaces in $\mathcal{C}$ are free $\Bbbk$-modules of finite rank. The *multiplicity* of a simple object i in any $X \in \mathrm{Ob}(\mathcal{C})$ is the integer

$$N_X^i = \mathrm{rank}_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(X, i) = \mathrm{rank}_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(i, X) \geq 0.$$

This integer depends only on the isomorphism classes of i and X.

A set I of simple objects of a pre-fusion category $\mathcal{C}$ is *representative* if $\mathbb{1} \in I$ and every simple object of $\mathcal{C}$ is isomorphic to a unique element of I. Condition (b) above implies that for such I and any $X \in \mathrm{Ob}(\mathcal{C})$, there is a finite family of morphisms $(p_\alpha \colon X \to i_\alpha, q_\alpha \colon i_\alpha \to X)_{\alpha \in \Lambda}$ in $\mathcal{C}$ such that

$$\mathrm{id}_X = \sum_{\alpha \in \Lambda} q_\alpha p_\alpha, \quad i_\alpha \in I, \quad \text{and} \quad p_\alpha q_\beta = \delta_{\alpha,\beta}\, \mathrm{id}_{i_\alpha} \quad \text{for all} \quad \alpha, \beta \in \Lambda,$$

where $\delta_{\alpha,\beta}$ is the Kronecker symbol. Such a family $(p_\alpha, q_\alpha)_{\alpha \in \Lambda}$ is called an *I-partition* of X. For $i \in I$, set $\Lambda^i = \Lambda^i_X = \{\alpha \in \Lambda \,|\, i_\alpha = i\}$. Then $(p_\alpha \colon X \to i)_{\alpha \in \Lambda^i}$ is a basis of $\mathrm{Hom}_{\mathcal{C}}(X, i)$ and $(q_\alpha \colon i \to X)_{\alpha \in \Lambda^i}$ is a basis of $\mathrm{Hom}_{\mathcal{C}}(i, X)$. Therefore $\#\Lambda^i = N_X^i$, $\#\Lambda = \sum_{i \in I} N_X^i$, and $\dim(X) = \sum_{i \in I} \dim(i) N_X^i$.

In a pre-fusion category $\mathcal{C}$, the left and right dimensions of any simple object of $\mathcal{C}$ are invertible in $\Bbbk$. Furthermore $\mathcal{C}$ is spherical if and only if $\dim_l(i) = \dim_r(i)$ for any simple object i of $\mathcal{C}$.

By a *fusion category*, we mean a pre-fusion category $\mathcal{C}$ such that the set of isomorphism classes of simple objects of $\mathcal{C}$ is finite. A standard example of a fusion category is the category of finite rank representations (over $\Bbbk$) of a finite group whose order is relatively prime to the characteristic of $\Bbbk$. The category of representations of an involutory finite dimensional Hopf algebra over a field of characteristic zero is a fusion category. For more examples, see [ENO].

The dimension of a fusion category $\mathcal{C}$ is

$$\dim(\mathcal{C}) = \sum_{i \in I} \dim_l(i) \dim_r(i) \in \Bbbk.$$

By [ENO], if $\Bbbk$ is an algebraically closed field of characteristic zero, then $\dim(\mathcal{C}) \neq 0$. For spherical $\mathcal{C}$, we have $\dim(\mathcal{C}) = \sum_{i \in I} (\dim(i))^2$.

Let $\mathcal{B}$ be a ribbon fusion category. Note that for any simple object i of $\mathcal{B}$, the twist $\theta_i \colon i \to i$ is multiplication by an invertible scalar $v_i \in \Bbbk$. We set

$$\Delta_\pm = \sum_{i \in I} v_i^{\pm 1} (\dim(i))^2 \in \Bbbk, \tag{4}$$

where I is a representative set of simple objects of $\mathcal{B}$.

1.10. **Modular categories.** A *modular category* (over $\Bbbk$) is a ribbon fusion category $\mathcal{B}$ (over $\Bbbk$) such that the matrix $S = [\mathrm{tr}(\tau_{j,i}\tau_{i,j})]_{i,j\in I}$ is invertible, where I is a representative set of simple objects of $\mathcal{B}$ and τ is the braiding of $\mathcal{B}$. The matrix S is called the *S-matrix* of $\mathcal{B}$.

If $\mathcal{B}$ is a modular category, then its dimension $\dim(\mathcal{B})$ and the scalar $\Delta_\pm$ defined in (4) are invertible and satisfy $\Delta_+\Delta_- = \dim(\mathcal{B})$, see [Tu1].

We say that a modular category $\mathcal{B}$ is *anomaly free* if $\Delta_+ = \Delta_-$.

1.11. **The center of a monoidal category.** Let $\mathcal{C}$ be a monoidal category. A *half braiding* of $\mathcal{C}$ is a pair (A,σ), where $A \in \mathrm{Ob}(\mathcal{C})$ and

$$\sigma = \{\sigma_X \colon A\otimes X \to X\otimes A\}_{X\in \mathrm{Ob}(\mathcal{C})}$$

is a natural isomorphism such that

$$\sigma_{X\otimes Y} = (\mathrm{id}_X \otimes \sigma_Y)(\sigma_X \otimes \mathrm{id}_Y)$$

for all $X, Y \in \mathrm{Ob}(\mathcal{C})$. This condition implies that $\sigma_{\mathbb{1}} = \mathrm{id}_A$.

The *center of* $\mathcal{C}$ is the braided category $\mathcal{Z}(\mathcal{C})$ defined as follows. The objects of $\mathcal{Z}(\mathcal{C})$ are half braidings of $\mathcal{C}$. A morphism $(A,\sigma)\to(A',\sigma')$ in $\mathcal{Z}(\mathcal{C})$ is a morphism $f\colon A\to A'$ in $\mathcal{C}$ such that $(\mathrm{id}_X\otimes f)\sigma_X = \sigma'_X(f\otimes \mathrm{id}_X)$ for all $X\in\mathrm{Ob}(\mathcal{C})$. The unit object of $\mathcal{Z}(\mathcal{C})$ is $\mathbb{1}_{\mathcal{Z}(\mathcal{C})} = (\mathbb{1}, \{\mathrm{id}_X\}_{X\in\mathrm{Ob}(\mathcal{C})})$ and the monoidal product is

$$(A,\sigma)\otimes(B,\rho) = \big(A\otimes B, (\sigma\otimes \mathrm{id}_B)(\mathrm{id}_A\otimes\rho)\big).$$

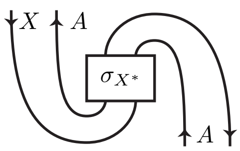
The braiding τ in $\mathcal{Z}(\mathcal{C})$ is defined by

$$\tau_{(A,\sigma),(B,\rho)} = \sigma_B \colon (A,\sigma)\otimes(B,\rho)\to(B,\rho)\otimes(A,\sigma).$$

There is a *forgetful functor* $\mathcal{Z}(\mathcal{C})\to\mathcal{C}$ assigning to every half braiding (A,σ) the underlying object A and acting in the obvious way on the morphisms. This is a strict monoidal functor.

If $\mathcal{C}$ is a monoidal $\Bbbk$-category, then so $\mathcal{Z}(\mathcal{C})$ and the forgetful functor is $\Bbbk$-linear. Observe that $\mathrm{End}_{\mathcal{Z}(\mathcal{C})}(\mathbb{1}_{\mathcal{Z}(\mathcal{C})}) = \mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \Bbbk$.

If $\mathcal{C}$ is pivotal, then so is $\mathcal{Z}(\mathcal{C})$ with $(A,\sigma)^* = (A^*,\sigma^\dagger)$, where

$\sigma^\dagger_X =$  $: A^*\otimes X\to X\otimes A^*,$

and $\mathrm{ev}_{(A,\sigma)} = \mathrm{ev}_A$, $\mathrm{coev}_{(A,\sigma)} = \mathrm{coev}_A$, $\widetilde{\mathrm{ev}}_{(A,\sigma)} = \widetilde{\mathrm{ev}}_A$, $\widetilde{\mathrm{coev}}_{(A,\sigma)} = \widetilde{\mathrm{coev}}_A$. The (left and right) traces of morphisms and dimensions of objects in $\mathcal{Z}(\mathcal{C})$ are the same as in $\mathcal{C}$. If $\mathcal{C}$ is spherical, then so is $\mathcal{Z}(\mathcal{C})$.

1.12. **The center of a fusion category.** The center $\mathcal{Z}(\mathcal{C})$ of a spherical fusion category $\mathcal{C}$ over $\Bbbk$ is a ribbon $\Bbbk$-category.

Theorem 1.1 ([Mü2, Theorem 1.2, Proposition 5.18])**.** *Let $\mathcal{C}$ be a spherical fusion category over an algebraically closed field such that $\dim\mathcal{C}\neq 0$. Then $\mathcal{Z}(\mathcal{C})$ is an anomaly free modular category with $\Delta_+ = \Delta_- = \dim(\mathcal{C})$.*

Note that $\dim\big(\mathcal{Z}(\mathcal{C})\big) = \Delta_+\Delta_- = (\dim(\mathcal{C}))^2$.

2. Kirby elements, Hopf diagrams, and quantum invariants

In this section, we review a general construction of quantum 3-manifolds invariants (based on surgical presentation of 3-manifolds) and a method for computing them via Hopf diagrams. We refer to [6, 7] for details.

2.1. **Coends.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Denote by $\mathcal{C}^{\mathrm{op}}$ the category opposite to $\mathcal{C}$ (obtained from $\mathcal{C}$ by reversing all the arrows). A *dinatural transformation* from a functor $F\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ to an object D of $\mathcal{D}$ is a family

$$d = \{d_X : F(X, X) \to D\}_{X \in \mathrm{Ob}(\mathcal{C})}$$

of morphisms in $\mathcal{D}$ such that $d_X F(f, \mathrm{id}_X) = d_Y F(\mathrm{id}_Y, f)$ for every morphism $f\colon X \to Y$ in $\mathcal{C}$. The *composition* of such a d with a morphism $\varphi\colon D \to D'$ in $\mathcal{D}$ is the dinatural transformation $\varphi \circ d = \{\varphi \circ d_X : F(X, X) \to D'\}_{X \in \mathrm{Ob}(\mathcal{C})}$ from F to D'. A *coend* of F is a pair (C, ρ) consisting in an object C of $\mathcal{D}$ and a dinatural transformation ρ from F to C satisfying the following universality condition: every dinatural transformation d from F to an object of $\mathcal{D}$ is the composition of ρ with a morphism in $\mathcal{D}$ and the latter morphism is uniquely determined by d. If F has a coend (C, ρ), then it is unique (up to unique isomorphism). One writes $C = \int^{X \in \mathcal{C}} F(X, X)$. For more on coends, see [Mac].

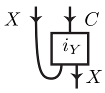
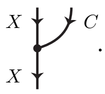
For example, let $\mathcal{C}$ be a fusion category and I be a (finite) representative set of simple objects of $\mathcal{C}$. If $\mathcal{D}$ is a $\Bbbk$-category which admits finite direct sums, then any $\Bbbk$-linear functor $F\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ has a coend (C, ρ). Here $C = \oplus_{i \in I} F(i, i)$ and $\rho = \{\rho_X : F(X, X) \to C\}_{X \in \mathrm{Ob}(\mathcal{C})}$ is computed by $\rho_X = \sum_\alpha F(q_X^\alpha, p_X^\alpha)$, where $(p_X^\alpha, q_X^\alpha)_\alpha$ is any I-partition of X. An arbitrary dinatural transformation d from F to an object D of $\mathcal{D}$ is the composition of ρ with $\sum_{i \in I} d_i : C \to D$.

2.2. **Coends of autonomous categories.** Let $\mathcal{C}$ be an autonomous category. If it exists, the coend $C = \int^{X \in \mathcal{C}} {}^\vee X \otimes X$ of the functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ defined by the formula $(X, Y) \mapsto {}^\vee X \otimes Y$ is called the *coend of* $\mathcal{C}$.

Let $\mathcal{C}$ be a pivotal category which admits a coend C. The left and right dimensions of C do not depend on the choice of the pivotal structure on $\mathcal{C}$. If they coincide (for instance when $\mathcal{C}$ is a spherical category or $\mathcal{C}$ is a fusion category), they are called the *dimension of* $\mathcal{C}$ and denoted $\dim(\mathcal{C})$.

For example, a fusion category $\mathcal{C}$ has a coend $C = \bigoplus_{i \in I} i^* \otimes i$, where I is a representative set of simple objects of $\mathcal{C}$. We have $\dim_l(C) = \dim_r(C)$. Therefore $\dim(\mathcal{C}) = \sum_{i \in I} \dim_l(i) \dim_r(i)$, recovering the definition given in Section 1.9.

2.3. **Forms associated with ribbon string links.** Throughout the rest of the section, the symbol $\mathcal{B}$ denotes a ribbon category admitting a coend C. Denote the universal dinatural transformation of C by $\rho = \{\rho_X : X^* \otimes X \to C\}_{X \in \mathrm{Ob}(\mathcal{B})}$. For any $X \in \mathrm{Ob}(\mathcal{B})$, set

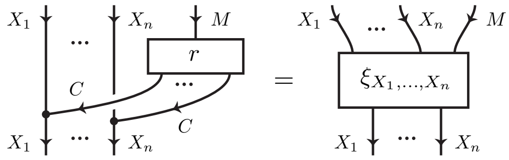
$\delta_X =$  $= (\mathrm{id}_X \otimes \rho_X)(\mathrm{coev}_X \otimes \mathrm{id}_X)\colon X \to X \otimes C$, depicted as  .

Using the general theory of coends (see [Mac]), we have the following universal property: for any natural transformation

$$\xi = \{\xi_{X_1, \dots, X_n} : X_1 \otimes \cdots \otimes X_n \to X_1 \otimes \cdots \otimes X_n \otimes M\}_{X_1, \dots, X_n \in \mathrm{Ob}(\mathcal{B})}$$

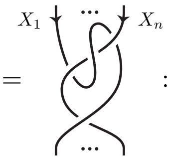
where M is an object of $\mathcal{B}$, there exists a unique morphism $r\colon C^{\otimes n} \to M$ such that:

(5)

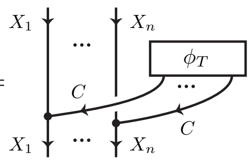


for all objects $X_1, \dots, X_n$ of $\mathcal{B}$.

Now let T be a ribbon n-string link with n a non-negative integer. Recall T is a ribbon (n,n)-tangle consisting of n arc components, without any closed component, such that the k-th arc $(1 \le k \le n)$ joins the k-th bottom endpoint to the k-th top endpoint. We orient T from top to bottom. By Penrose calculus, coloring the n components of T with objects $X_1, \dots, X_n$ of $\mathcal{B}$ yields a morphism

$T_{X_1,\cdots,X_n} =$  $: X_1 \otimes \cdots \otimes X_n \to X_1 \otimes \cdots \otimes X_n.$

Following Lyubashenko [Lyu1], since $T_{X_1,\cdots,X_n}$ is natural in each variable X_k, there exists a unique morphism: $\phi_T\colon C^{\otimes n} \to \mathbb{1}$ such that, for any $X_1, \dots, X_n \in \mathrm{Ob}(\mathcal{B})$,

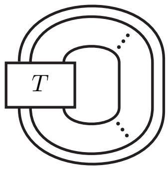
(6) $T_{X_1,\cdots,X_n} =$  .

Two natural questions arise in this context:

- How to evaluate the forms ϕ_T to get invariants of framed links and, further, of 3-manifolds?
- How to compute the forms ϕ_T which are defined by universal property?

We address the first question by introducing Kirby elements (see Sections 2.4 and 2.5) and the second one by introducing Hopf diagrams (see Section 2.8).

2.4. **Kirby elements and quantum invariants.** Let L be a framed link in S^3 with n components. There always exists a (non-unique) ribbon n-string link T such that L is isotopic to the closure of T, that is,

$L \sim$  .

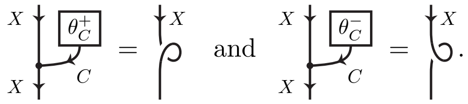
For $\alpha \in \mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, C)$, set

$$\tau_{\mathcal{B}}(L;\alpha) = \phi_T \circ \alpha^{\otimes n} \in \mathrm{End}_{\mathcal{B}}(\mathbb{1}),$$

where $\phi_T\colon C^{\otimes n} \to \mathbb{1}$ is defined as in (6).

Let us define a *Kirby element* of $\mathcal{B}$ to be a morphism $\alpha \in \mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, C)$ such that, for any framed link L, $\tau_{\mathcal{B}}(L;\alpha)$ is well-defined and invariant under isotopies and 2-handle slides of L.

A Kirby element α of $\mathcal{B}$ is said to be *normalizable* if $\tau_{\mathcal{B}}(\bigcirc^{+1};\alpha)$ and $\tau_{\mathcal{B}}(\bigcirc^{-1};\alpha)$ are invertible in the commutative monoid $\mathrm{End}_{\mathcal{B}}(\mathbb{1})$, where $\bigcirc^{\pm 1}$ denotes the unknot with framing ± 1. Using the universal property (5), define the morphisms $\theta_C^+ \colon C \to \mathbb{1}$ and $\theta_C^- \colon C \to \mathbb{1}$ by:



If α is a Kirby element of $\mathcal{B}$, then $\tau_{\mathcal{B}}(\bigcirc^{\pm 1}; \alpha) = \theta_C^{\pm}\alpha$ and so α is normalizable if and only if $\theta_C^{\pm}\alpha$ are invertible in $\mathrm{End}_{\mathcal{B}}(\mathbb{1})$.

Let M be a closed oriented 3-manifold. Present M by surgery on S^3 along a framed link L (see [Li]). An immediate consequence of the Kirby theorem [Ki] is that if α is a normalizable Kirby element of $\mathcal{B}$, then

$$(7) \qquad \tau_{\mathcal{B}}(M; \alpha) = (\theta_C^+\alpha)^{b_-(L) - n_L}\, (\theta_C^-\alpha)^{-b_-(L)}\, \tau_{\mathcal{B}}(L; \alpha) \in \mathrm{End}_{\mathcal{B}}(\mathbb{1})$$

is topological invariant of M, where n_L is the number of components of L and $b_-(L)$ is the number of negative eigenvalues of the linking matrix of L. This invariant is multiplicative with respect to the connected sum. We have: $\tau_{\mathcal{B}}(S^3; \alpha) = 1$.

Note that if α is a normalizable Kirby element and k is an automorphism of $\mathbb{1}$, then αk is also a normalizable Kirby element. The normalization of the invariant $\tau_{\mathcal{B}}(M; \alpha)$ has been chosen so that $\tau_{\mathcal{B}}(M; \alpha k) = \tau_{\mathcal{B}}(M; \alpha)$.

Determining all Kirby elements of $\mathcal{B}$ seems hopeless. In the next section, we define a class of Kirby elements of $\mathcal{B}$, called algebraic Kirby elements, characterized in purely algebraic terms (using the Hopf algebra structure of the coend C). This class is sufficiently large to contain the Kirby elements corresponding to the Reshetikhin-Turaev invariants [RT, Tu1], Lyubashenko invariant [Lyu2], and the Hennings-Kauffman-Radford invariant [He, KR] when these are well-defined. Note that there exist normalizable algebraic Kirby elements which give rise to new 3-manifolds invariants, i.e., to invariants different from those above (see Section 2.7 below).

2.5. **Algebraic Kirby elements.** Recall that a Hopf algebra in a braided category with braiding τ is an objet A of the category endowed with a product $m \colon A \otimes A \to A$, a unit $u \colon \mathbb{1} \to A$, a coproduct $\Delta \colon A \to A \otimes A$, a counit $\varepsilon \colon A \to \mathbb{1}$, and an antipode $S \colon A \to A$, which are morphism in the category such that (A, m, u) is an algebra, (A, Δ, ε) is a coalgebra, and

$$\Delta m = (m \otimes m)(\mathrm{id}_A \otimes \tau_{A,A} \otimes \mathrm{id}_A)(\Delta \otimes \Delta), \qquad \Delta u = u \otimes u,$$
$$\varepsilon m = \varepsilon \otimes \varepsilon, \qquad \varepsilon u = \mathrm{id}_{\mathbb{1}},$$
$$m(S \otimes \mathrm{id}_A)\Delta = u\varepsilon = m(\mathrm{id}_A \otimes S)\Delta.$$

We assume also that Hopf algebras have invertible antipode.

The coend C of the category $\mathcal{B}$ is a Hopf algebra in $\mathcal{B}$, see [Ma2, Lyu1]. Its product $m_C \colon C \otimes C \to C$, unit $u_C \colon \mathbb{1} \to C$, coproduct $\Delta_C \colon C \to C \otimes C$, counit $\varepsilon_C \colon C \to \mathbb{1}$, antipode $S_C \colon C \to C$, and inverse of the antipode $S_C^{-1} \colon C \to C$ are defined in Figure 1 by using the universal property (5).

Theorem 2.1 ([7, Theorem 2.5])**.** *Any morphism $\alpha \colon \mathbb{1} \to C$ in $\mathcal{B}$ such that:*

$$S_C\alpha = \alpha \quad \textit{and} \quad (m_C \otimes \mathrm{id}_C)(\mathrm{id}_C \otimes \Delta_C)(\alpha \otimes \alpha) = \alpha \otimes \alpha$$

is a Kirby element of $\mathcal{B}$.

Morphisms $\alpha \colon \mathbb{1} \to C$ satisfying the conditions of Theorem 2.1 are called *algebraic Kirby elements* of $\mathcal{B}$. For instance, the unit u_C of C is a normalizable algebraic Kirby element (its associated 3-manifolds invariant $\tau_{\mathcal{B}}(M; u_C)$ is the trivial one).

A more interesting example of an algebraic Kirby element is an S_C-invariant integral Λ of the Hopf algebra C, i.e., a morphism $\Lambda \colon \mathbb{1} \to C$ such that $S_C(\Lambda) = \Lambda$ and $m_C(\Lambda \otimes \mathrm{id}_C) = \Lambda\, \varepsilon_C = m_C(\mathrm{id}_C \otimes \Lambda)$. For the existence of such integrals,

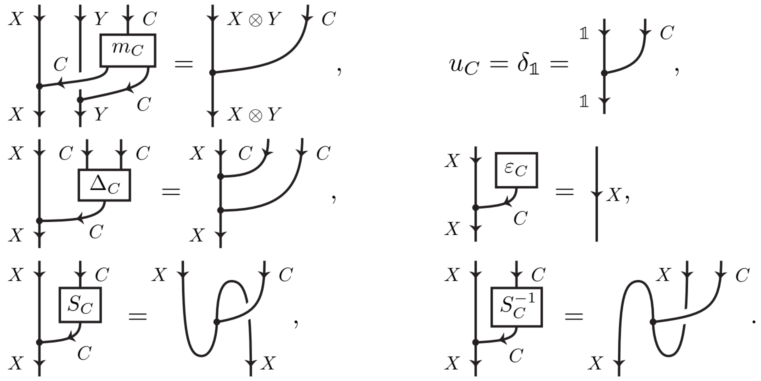


FIGURE 1. Structural morphisms of the coend

we refer to [BKLT]. If Λ is normalizable, then the associated invariant is the Lyubashenko's one [Lyu2], up to a different normalization.

Remark 2.2. When $\mathcal{B}$ is a ribbon $\Bbbk$-category admitting a coend C, we can relax the definition of an algebraic Kirby element by considering morphisms $\alpha\colon \mathbb{1} \to C$ in $\mathcal{B}$ such that $S_C\alpha - \alpha \in \mathrm{Negl}_{\mathcal{B}}(\mathbb{1}, C)$ and

$$(m_C \otimes \mathrm{id}_C)(\mathrm{id}_C \otimes \Delta_C)(\alpha \otimes \alpha) - \alpha \otimes \alpha \in \mathrm{Negl}_{\mathcal{B}}(\mathbb{1}, C \otimes C),$$

where $\mathrm{Negl}_{\mathcal{B}}(X, Y)$ denotes the $\Bbbk$-subspace of $\mathrm{Hom}_{\mathcal{B}}(X, Y)$ formed by morphisms $f\colon X \to Y$ which are negligible, that is, such that $\mathrm{tr}(gf) = 0$ for all $g \in \mathrm{Hom}_{\mathcal{B}}(Y, X)$. Such morphisms $\alpha\colon \mathbb{1} \to C$ are indeed Kirby elements (see [7, Theorem 2.5]). For more on this and the related semisimplification process, we refer to [7].

In the next two sections, we focus on the case of ribbon fusion categories and of categories of representations of finite dimensional ribbon Hopf algebras.

2.6. **The case of a fusion category.** In this section, we assume that $\mathcal{B}$ is a ribbon fusion category over $\Bbbk$. Recall that $\mathcal{B}$ admits a coend $C = \bigoplus_{i \in I} i^* \otimes i$, see Section 1.9. Let I be a (finite) representative set of simple objects of $\mathcal{B}$ and set

$$\Lambda_{\mathcal{B}} = \sum_{i \in I} \dim(i)\, \widetilde{\mathrm{coev}}_i \colon \mathbb{1} \to \bigoplus_{i \in I} i^* \otimes i = C.$$

If $\mathcal{B}$ is modular, then $\Lambda_{\mathcal{B}}$ is a is an S_C-invariant integral of C and so is an algebraic Kirby element (giving thus the Lyubashenko and Reshetikhin-Turaev invariants defined with $\mathcal{B}$). In general $\Lambda_{\mathcal{B}}$ is not a two-sided integral of B. Nevertheless:

Theorem 2.3 ([7, Theorem 3.4])**.** *$\Lambda_{\mathcal{B}}$ is an algebraic Kirby element of $\mathcal{B}$.*

In some particular cases, all algebraic Kirby elements of $\mathcal{B}$ are of this form:

Corollary 2.4 ([7, Corollary 3.8])**.** *If either*

(i) *the category $\mathcal{B}$ is Picard ($X^* \otimes X \cong \mathbb{1}$ for every simple object X of $\mathcal{B}$);*
(ii) *$\Bbbk = \mathbb{R}$ or $\mathbb{C}$ and the quantum dimensions of the simple objects are positive;*

then every algebraic Kirby elements of $\mathcal{B}$ is of the form $\Lambda_{\mathcal{D}}$ for some full ribbon fusion subcategory $\mathcal{D}$ of $\mathcal{B}$.

The algebraic Kirby element $\Lambda_{\mathcal{B}}$ is normalizable if and only if the scalars $\Delta_{\pm}$ defined in (4) are invertible. In that case, by Section 2.4, we get a 3-manifold invariant $\tau_{\mathcal{B}}(M; \Lambda_{\mathcal{B}})$.

If $\mathcal{B}$ is modular and $\dim(\mathcal{B})$ has a square root D in $\Bbbk$, then $\Lambda_\mathcal{B}$ is normalizable and $\tau_\mathcal{B}(M;\Lambda_\mathcal{B})$ corresponds to the Reshetikhin-Turaev invariant $\tau_\mathcal{B}(M)$. More precisely, for any closed oriented 3-manifold M,

$$\tau_\mathcal{B}(M) = D^{-1}\Big(\frac{D}{\Delta_-}\Big)^{b_1(M)}\tau_\mathcal{B}(M;\Lambda_\mathcal{B}), \tag{8}$$

where $b_1(M)$ is the first Betti number of M.

When $\mathcal{B}$ is not modular, and even when $\dim(\mathcal{B})=0$, the invariant $\tau_\mathcal{B}(M;\Lambda_\mathcal{B})$ may be still defined (see Section 3.18 below for an example).

2.7. **The case of a category of representations.** In this section $\Bbbk$ is a field. Let H be a finite dimensional Hopf $\Bbbk$-algebra, with coproduct Δ, counit ε, and antipode S. Recall that a left integral for H is an element $\Lambda \in H$ such that $x\Lambda = \varepsilon(x)\Lambda$ for every $x \in H$. A right integral for H^* is an element $\lambda \in H^*$ such that $\lambda(x_{(1)})x_{(2)} = \lambda(x)\,1$ for all $x \in H$, where $\Delta(x) = x_{(1)} \otimes x_{(2)}$. By the uniqueness of integrals (since H is finite dimensional), there exists a unique algebra map $\nu\colon H \to \Bbbk$ such that $\Lambda x = \nu(x)\Lambda$ for any $x \in H$. The form ν is called the distinguished grouplike element of H^*. The Hopf algebra H is said to be unimodular if its integrals are two sided, that is, if $\nu = \varepsilon$.

Denote by mod_H the ribbon $\Bbbk$-category of finite dimensional left H-modules. The coend of mod_H is $C = H^* = \mathrm{Hom}_\Bbbk(H,\Bbbk)$ endowed with the coadjoint action $\triangleright$ defined by $(h \triangleright f)(x) = f(S(h_{(1)})xh_{(2)})$ for $f \in H^*$ and $h,x \in H$.

Let H be a finite-dimensional ribbon Hopf $\Bbbk$-algebra with R-matrix $R \in H \otimes H$ and twist $\theta \in H$. Let $\lambda \in H^*$ be a non-zero right integral for H^* and ν be the distinguished grouplike element of H^*. Denote by $\leftharpoonup$ the right H^*-action on H defined by $x \leftharpoonup f = f(x_{(1)})x_{(2)}$ for $f \in H^*$. The element $h_\nu = (\mathrm{id}_H \otimes \nu)(R) \in H$ is a grouplike element of H. Consider the set $\mathcal{AK}(H)$ made of elements $z \in H$ satisfying:

(i) $(x \leftharpoonup \nu)z = zx$ for every $x \in H$;
(ii) $(S(z) \leftharpoonup \nu)h_\nu = z$;
(iii) $\lambda(zx_{(1)})zx_{(2)} = \lambda(zx)z$ for every $x \in H$.

The $\Bbbk$-linear map $\phi\colon H \to \mathrm{Hom}_\Bbbk(\Bbbk, H^*)$, defined by $\phi_z(1_\Bbbk)(x) = \lambda(zx)$ for $z,x \in H$, is a $\Bbbk$-linear isomorphism. It induces a $\Bbbk$-linear isomorphism between the vector spaces $L(H) = \{z \in H \mid z \text{ satisfies (i)}\}$ and $\mathrm{Hom}_{\mathrm{mod}_H}(\Bbbk, C)$.

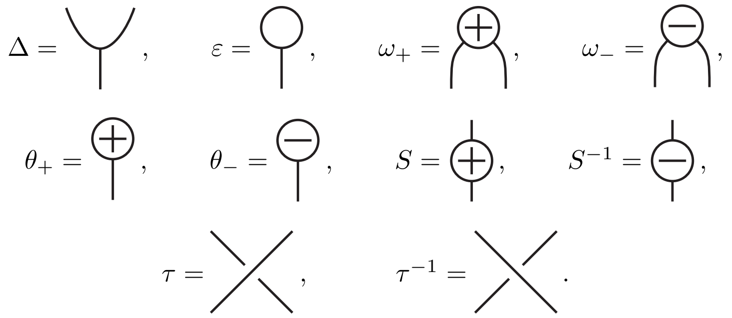
Theorem 2.5 ([7, Theorem 4.7])**.** *The set of algebraic Kirby elements of* mod_H *is* $\{\phi_z \mid z \in \mathcal{AK}(H)\}$. *Furthermore, for* $z \in \mathcal{AK}(H)$, *the Kirby element* ϕ_z *of* mod_H *is normalizable if and only if* $\lambda(z\theta) \neq 0 \neq \lambda(z\theta^{-1})$.

The Hopf algebra H is unimodular if and only if $1 \in \mathcal{AK}(H)$, see [7, Corollary 4.8]. Furthermore, if H is unimodular and $\lambda(\theta) \neq 0 \neq \lambda(\theta^{-1})$, then the Kirby element $\phi_1 = \lambda$ is normalizable and the corresponding 3-manifolds invariant is the Hennings-Kauffman-Radford invariant defined with the opposite ribbon Hopf algebra H^{op} to H (see [7, Corollary 4.16]).

In [7, Section 5], by studying in detail an example of a finite dimension ribbon Hopf algebra H, we show that there exist Kirby elements of mod_H leading to 3-manifolds invariants which do not correspond to the Lyubashenko invariant nor the Hennings-Kauffman-Radford. These invariants also do not correspond to the Reshetikhin-Turaev invariants defined with the full ribbon fusion subcategories of the semisimple quotient of mod_H. This means that this method allows to define new 3-manifolds invariants (which are 'non-semisimple').

2.8. **Hopf diagrams.** For a precise treatment of the theory of Hopf diagrams, we refer to [6]. Note that Habiro, shortly after us, had similar results in [Hab].

A *Hopf diagram* is a planar diagram, with inputs but no output (diagrams are read from bottom to top), obtained by stacking the following generators:



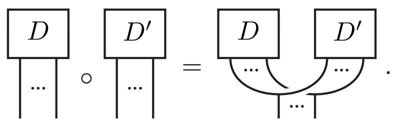
For example, the following diagrams are Hopf diagrams:

A Hopf diagram with 1 input

A Hopf diagram with 2 inputs

Hopf diagrams are submitted to the relations of Figure 2, plus relations expressing that τ is an invertible QYBE solution which is natural with respect to the other generators. In particular, the relations of Figure 2 say that Δ behaves as a coproduct with counit ε, S behaves as an antipode, $\omega_\pm$ behaves as a Hopf pairing, and $\theta_\pm$ behaves as a twist form. The last two relations of Figure 2 are nothing but the Markov relations for pure braids.

Hopf diagrams form a category $\mathcal{D}iag$. Objects of $\mathcal{D}iag$ are the non-negative integers. For two non-negative integers m and n, the set $\mathrm{Hom}_{\mathcal{D}iag}(m,n)$ of morphisms from m to n in $\mathcal{D}iag$ is the empty set if $m \neq n$ and is the set of Hopf diagrams with m inputs (up to their relations) if $m = n$. The composition of two Hopf diagrams D and D' (with the same number of inputs) is defined as:



The identity of n is the Hopf diagram obtained by juxtaposing n copies of ε. The category $\mathcal{D}iag$ is a monoidal category: $m \otimes n = m + n$ on objects and the monoidal product $D \otimes D'$ of two Hopf diagrams D and D' is the Hopf diagram obtained by juxtaposing D on the left of D'.

Let us denote by RSL the category of ribbon string links. The objects of RSL are the non-negative integers. For two non-negative integers m and n, the set $\mathrm{Hom}_{\mathrm{RSL}}(m,n)$ of morphisms from m to n in RSL is the empty set if $m \neq n$ and is the set of (isotopy classes) of ribbon n-string links (see Section 2.3) if $m = n$. The composition $T' \circ T$ of two ribbon n-string links is given by stacking T' on the top of T. Identities are the trivial string links. Note that the category RSL is a monoidal category: $m \otimes n = m + n$ on objects and the monoidal product $T \otimes T'$ of two ribbon string links T and T' is the ribbon string link obtained by juxtaposing T on the left of T'.

Hopf diagrams give a 'Hopf algebraic' description of ribbon string links. Indeed, any Hopf diagram D with n inputs gives rise to a ribbon n-string link $\Phi(D)$ in the following way: using the rules of Figure 3, we obtain a ribbon n-handle h_D, that is, a ribbon $(2n, 0)$-tangle consisting of n arc components, without any closed

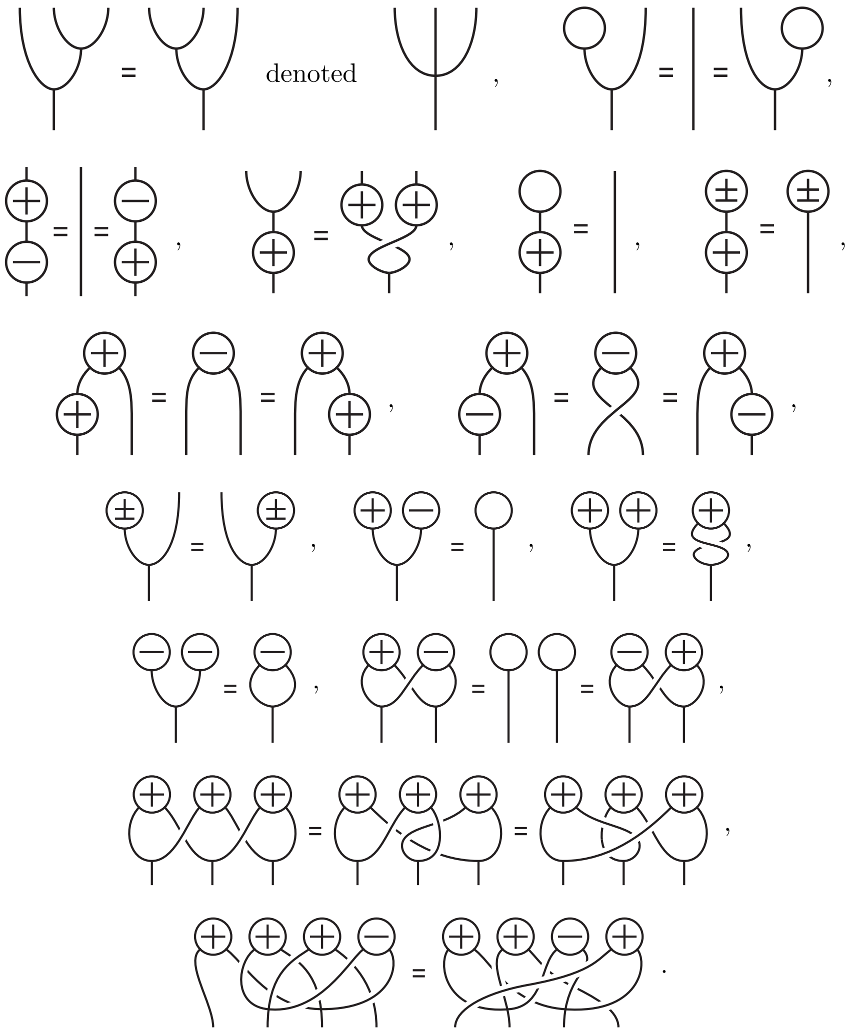
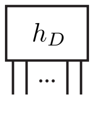
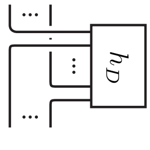


FIGURE 2. Relations on Hopf diagrams

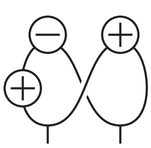
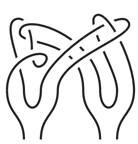
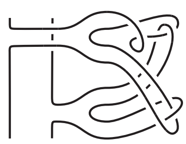
component, such that the k-th arc joins the $(2k-1)$-th and $2k$-th bottom endpoints. Then, by rotating h_D, we get a ribbon n-string link $\Phi(D)$:

D Hopf diagram $\rightsquigarrow$ 

$\rightsquigarrow$ $\Phi(D) =$ 

.

For example:

$D =$  $\rightsquigarrow h_D =$  $\rightsquigarrow \Phi(D) =$  $\sim$  .

This leads to a functor $\Phi\colon \mathcal{D}iag \to \mathrm{RSL}$, defined on objects by $n \mapsto \Phi(n) = n$ and on morphisms by $D \mapsto \Phi(D)$.

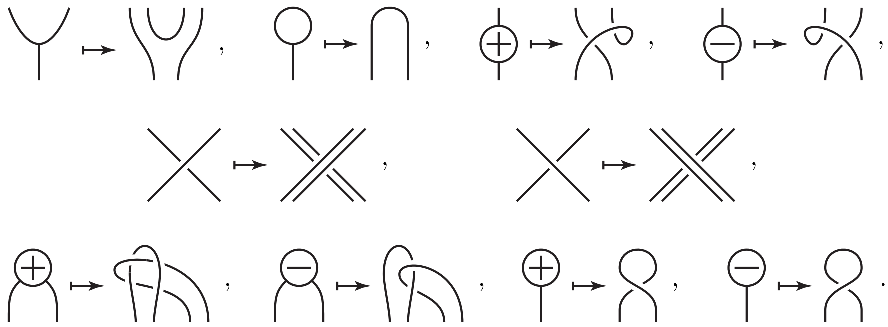


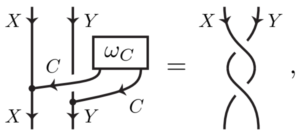
FIGURE 3. Rules for transforming Hopf diagrams to tangles

Theorem 2.6 ([6, Theorem 4.5])**.** $\Phi\colon \mathcal{D}iag \to \mathrm{RSL}$ *is a well-defined monoidal functor and there exists (constructive proof) a monoidal functor* $\Psi\colon \mathrm{RSL} \to \mathcal{D}iag$ *which satisfies* $\Phi \circ \Psi = 1_{\mathrm{RSL}}$.

Note that by 'constructing proof' we mean there is an explicit algorithm that associates to a ribbon string T a Hopf diagram $\Psi(T)$ such that $\Phi\big(\Psi(T)\big) = T$, see [6]. The key point is that such a functor Ψ exists thanks to the relations we put on Hopf diagrams.

2.9. **Computing quantum invariants from Hopf diagrams.** Let now $\mathcal{B}$ be a ribbon category which admits a coend C. Let us answer to the second question of Section 2.3: given a ribbon n-string link T, how to compute the morphism $\phi_T\colon C^{\otimes n} \to \mathbb{1}$ which is defined in (6) by universal property?

Recall C is a Hopf algebra in $\mathcal{B}$ (see Section 2.5) and denote its coproduct, counit, and antipode by Δ_C, ε_C, and S_C respectively. The twist of $\mathcal{B}$ and its inverse are encoded by morphisms $\theta_C^{\pm}\colon C \to \mathbb{1}$, see Section 2.4. Furthermore, the morphism $\omega_C\colon C \otimes C \to \mathbb{1}$, defined by

(9) 

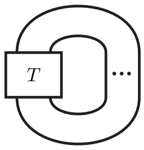
is a Hopf pairing for C. Finally, we set $\omega_C^{+} = \omega_C(S_C^{-1} \otimes \mathrm{id}_C)$ and $\omega_C^{-} = \omega_C$. Denote by $\{\tau_{X,Y}\colon X \otimes Y \to Y \otimes X\}_{X,Y \in \mathrm{Ob}(\mathcal{C})}$ the braiding of $\mathcal{B}$.

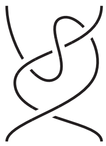
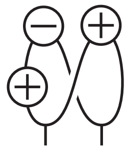
Theorem 2.7 ([6, Theorem 5.1])**.** *Let T be a ribbon n-string link. Let D be any Hopf diagram (with n entries) which encodes T, that is, such that $\Phi(D) = T$ (recall there is an algorithm producing such a Hopf diagram). Then the morphism $\phi_T\colon C^{\otimes n} \to \mathbb{1}$ defined by T is given by replacing in D the Hopf diagrams generators Δ, ε, $\omega_{\pm}$, $\theta_{\pm}$, $S^{\pm 1}$, and $\tau^{\pm 1}$ by the morphisms Δ_C, ε_C, $\omega_C^{\pm}$, $\theta_C^{\pm}$, $S_C^{\pm 1}$, and $\tau_{C,C}^{\pm 1}$ respectively.*

Remark that the product and unit of the Hopf algebra C are not needed to represent Hopf diagrams.

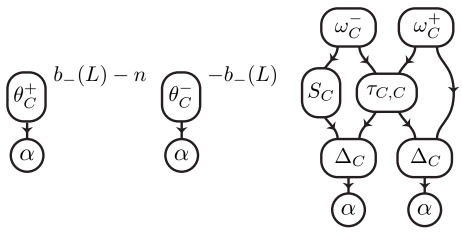
A consequence of Theorem 2.7 is that, given a normalizable Kirby element α of $\mathcal{B}$, the 3-manifold invariant $\tau_{\mathcal{B}}(M;\alpha)$ is computed using only the Kirby element α and some structural morphisms of the Hopf algebra C.

2.10. **Summary.** Let us summarize the above universal construction of quantum invariants, starting from a ribbon category $\mathcal{B}$ which admits a coend C. Pick a normalizable Kirby element α of $\mathcal{B}$, for example an algebraic Kirby element (recall that algebraic Kirby elements can be found by solving some purely algebraic system, see Section 2.5). It gives rise to the invariant $\tau_{\mathcal{B}}(M;\alpha)$ of 3-manifolds (see Section 2.4). Let M be a closed oriented 3-manifold. Present M by surgery along a framed link L in S^3, which can be viewed as the closure of a ribbon n-string link T where n is the number of components of L. Encode the string link T by a Hopf diagram D (there is an algorithm producing such a Hopf diagram):

$M \simeq S^3_L,\quad L \sim$ 

with $T =$  $\leftarrow\!\!\shortmid\ D =$  .

The morphism $\phi_T\colon C^{\otimes n} \to \mathbb{1}$ associated to T can be computed by replacing the generators of D by the corresponding structural morphisms of the coend C. Then evaluate ϕ_T with the Kirby element α and normalize to get the invariant:

$\tau_{\mathcal{B}}(M;\alpha) =$ 

,

that is,

$$\tau_{\mathcal{B}}(M;\alpha) = (\theta^+_C\alpha)^{b_-(L)-n}\,(\theta^+_C\alpha)^{-b_-(L)}\,(\omega^-_C\otimes\omega^+_C)(S_C\otimes\tau_{C,C}\otimes \mathrm{id}_C)(\Delta_C\alpha\otimes\Delta_C\alpha).$$

For example, we get

$$\tau_{\mathcal{B}}(S^1\times S^2;\alpha) = (\theta^+_C\alpha)^{-1}\,\varepsilon_C\alpha \quad\text{and}\quad \tau_{\mathcal{B}}(\mathbb{P};\alpha) = (\theta^+_C\alpha)^{-1}\,(\omega^+_C\Delta_C\otimes\theta^-_C)\Delta_C\alpha,$$

where $\mathbb{P}$ is the Poincaré sphere (which is obtained by surgery along the right-handed trefoil with framing $+1$).

In particular, to compute such quantum invariants defined from the center $\mathcal{Z}(\mathcal{C})$ of a spherical fusion category $\mathcal{C}$, one needs to give an explicit description of the structural morphism of the coend of $\mathcal{Z}(\mathcal{C})$ in terms of the category $\mathcal{C}$ (see Section 3.18 below). Providing such a description was our original motivation for introducing Hopf monads.

3. Hopf monads and categorical centers

In this section, we review the theory of Hopf monads we introduced in [8, 10, 11].

3.1. **Monads an their modules.** Any category $\mathcal{C}$ gives rise to a category $\mathrm{End}\,(\mathcal{C})$ whose objects are functors $\mathcal{C} \to \mathcal{C}$ and whose morphisms are natural transformations of such functors. The category $\mathrm{End}\,(\mathcal{C})$ is a (strict) monoidal category with tensor product being composition of functors and unit object being the identity functor $1_\mathcal{C} \colon \mathcal{C} \to \mathcal{C}$. A *monad* on $\mathcal{C}$ is a monoid in the category $\mathrm{End}\,(\mathcal{C})$, that is, a triple (T, μ, η) consisting of a functor $T \colon \mathcal{C} \to \mathcal{C}$ and two natural transformations

$$\mu = \{\mu_X : T^2(X) \to T(X)\}_{X \in \mathrm{Ob}(\mathcal{C})} \quad \text{and} \quad \eta = \{\eta_X : X \to T(X)\}_{X \in \mathrm{Ob}(\mathcal{C})},$$

called the *product* and the *unit* of T, such that for all $X \in \mathrm{Ob}(\mathcal{C})$,

$$\mu_X T(\mu_X) = \mu_X \mu_{T(X)} \quad \text{and} \quad \mu_X \eta_{T(X)} = \mathrm{id}_{T(X)} = \mu_X T(\eta_X).$$

For example, the identity functor $1_\mathcal{C} \colon \mathcal{C} \to \mathcal{C}$ is a monad on $\mathcal{C}$ (with identity as product and unit), called the *trivial monad*.

Given a monad T on $\mathcal{C}$, a *T-module in* $\mathcal{C}$ is a pair (M, r) where $M \in \mathrm{Ob}(\mathcal{C})$ and $r \colon T(M) \to M$ is a morphism in $\mathcal{C}$ such that $rT(r) = r\mu_M$ and $r\eta_M = \mathrm{id}_M$. A morphism from a T-module (M, r) to a T-module (N, s) is a morphism $f \colon M \to N$ in $\mathcal{C}$ such that $fr = sT(f)$. This defines the *category* $\mathcal{C}^T$ *of T-modules in* $\mathcal{C}$, with composition induced by that in $\mathcal{C}$. We denote by U_T the forgetful functor $\mathcal{C}^T \to \mathcal{C}$, defined by $U_T(M, r) = M$ and $U_T(f) = f$.

3.2. **Bimonads.** To define bimonads (introduced by Moerdijk [Mo]), we recall the notion of a comonoidal functor. A functor $F \colon C \to D$ between monoidal categories is *comonoidal* if it is endowed with a morphism $F_0 \colon F(\mathbb{1}) \to \mathbb{1}$ and a natural transformation

$$F_2 = \{F_2(X, Y) \colon F(X \otimes Y) \to F(X) \otimes F(Y)\}_{X,Y \in \mathrm{Ob}(\mathcal{C})}$$

which are coassociative and counitary, i.e., for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$,

$$(\mathrm{id}_{F(X)} \otimes F_2(Y, Z))F_2(X, Y \otimes Z) = (F_2(X, Y) \otimes \mathrm{id}_{F(Z)})F_2(X \otimes Y, Z)$$

and

$$(\mathrm{id}_{F(X)} \otimes F_0)F_2(X, \mathbb{1}) = \mathrm{id}_{F(X)} = (F_0 \otimes \mathrm{id}_{F(X)})F_2(\mathbb{1}, X).$$

A natural transformation $\varphi = \{\varphi_X \colon F(X) \to G(X)\}_{X \in \mathrm{Ob}(\mathcal{C})}$ between comonoidal functors is *comonoidal* if $G_0 \varphi_\mathbb{1} = F_0$ and $G_2(X, Y)\, \varphi_{X \otimes Y} = (\varphi_X \otimes \varphi_Y)F_2(X, Y)$ for all $X, Y \in \mathrm{Ob}(\mathcal{C})$.

Let $\mathcal{C}$ be a monoidal category. A *bimonad* on $\mathcal{C}$ is a monad (T, μ, η) on $\mathcal{C}$ such that the underlying functor $T \colon \mathcal{C} \to \mathcal{C}$ and the natural transformations μ and η are comonoidal. For a bimonad T on $\mathcal{C}$, the category $\mathcal{C}^T$ of T-modules has a monoidal structure with unit object $(\mathbb{1}, T_0)$ and monoidal product

$$(M, r) \otimes (N, s) = \big(M \otimes N, (r \otimes s)\, T_2(M, N)\big).$$

Note that the forgetful functor $U_T \colon \mathcal{C}^T \to \mathcal{C}$ is strict monoidal.

3.3. **Hopf monads.** Let $\mathcal{C}$ be a monoidal category. The *left fusion operator* and the *right fusion operator* of a bimonad T on $\mathcal{C}$ are the natural transformations

$$H^l = \{H^l \colon T(X \otimes T(Y)) \to T(X) \otimes T(Y)\}_{X,Y \in \mathrm{Ob}(\mathcal{C})}$$

and

$$H^r = \{H^r \colon T(T(X) \otimes Y) \to T(X) \otimes T(Y)\}_{X,Y \in \mathrm{Ob}(\mathcal{C})}$$

defined by

$$H^l_{X,Y} = (\mathrm{id}_{T(X)} \otimes \mu_Y)T_2(X, T(Y)) \quad \text{and} \quad H^r_{X,Y} = (\mu_X \otimes \mathrm{id}_{T(Y)})T_2(T(X), Y).$$

We define a *Hopf monad on* $\mathcal{C}$ to be a bimonad on $\mathcal{C}$ whose both left and right fusion operators are isomorphisms (see [11, Section 2]). Hopf monads on $\mathcal{C}$ form a category $\mathrm{HopfMon}(\mathcal{C})$, morphisms of Hopf monads being comonoidal morphisms of monads. The trivial Hopf monad $1_\mathcal{C}$ is an initial object of $\mathrm{HopfMon}(\mathcal{C})$.

3.4. **Hopf monads and Hopf algebras.** In this section, we characterize Hopf monads which are representable by Hopf algebras.

Let $\mathcal{C}$ be a monoidal category. Any Hopf algebra (A, σ) in the center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ gives rise to a Hopf monad $A \otimes_\sigma ?$, defined as follows. As an endofunctor of $\mathcal{C}$, $(A \otimes_\sigma ?)(X) = A \otimes X$ and $(A \otimes_\sigma ?)(f) = \mathrm{id}_A \otimes f$ for any object X and morphism f in $\mathcal{C}$. The product μ, unit η, and comonoidal structure of $A \otimes_\sigma ?$ are defined by

$$\mu_X = m \otimes \mathrm{id}_X, \qquad \eta_X = u \otimes \mathrm{id}_X,$$
$$(A \otimes_\sigma ?)_2(X, Y) = (\mathrm{id}_A \otimes \sigma_X \otimes \mathrm{id}_Y)(\Delta \otimes \mathrm{id}_{X \otimes Y}), \qquad (A \otimes_\sigma ?)_0 = \varepsilon,$$

for $X, Y \in \mathrm{Ob}(\mathcal{C})$, where m, u, Δ, and ε are the product, unit, coproduct, and counit of (A, σ), respectively. The axioms of a Hopf algebra (see Section 2.5) ensures that $A \otimes_\sigma ?$ is a Hopf monad (invertibility of the fusion operators comes from the axioms of the antipode). We say that a Hopf monad is *representable* if it is isomorphic to $A \otimes_\sigma ?$ for some Hopf algebra (A, σ) in $\mathcal{Z}(\mathcal{C})$.

A Hopf monad T on $\mathcal{C}$ is *augmented* if it is endowed with an *augmentation*, that is, a Hopf monad morphism $e\colon T \to 1_\mathcal{C}$. Augmented Hopf monads on $\mathcal{C}$ form a category $\mathrm{HopfMon}(\mathcal{C})/1_\mathcal{C}$, whose objects are augmented Hopf monads on $\mathcal{C}$, and morphisms between two augmented Hopf monads (T, e) and (T', e') are morphisms of Hopf monads $f\colon T \to T'$ such that $e'f = e$. For example, the Hopf monad $A \otimes_\sigma ?$ associated with a Hopf algebra (A, σ) in $\mathcal{Z}(\mathcal{C})$ is augmented with augmentation $\varepsilon \otimes ?$ defined by $(\varepsilon \otimes ?)_X = \varepsilon \otimes \mathrm{id}_X$ for $X \in \mathrm{Ob}(\mathcal{C})$.

Denote by $\mathrm{HopfAlg}(\mathcal{Z}(\mathcal{C}))$ the category of Hopf algebra in $\mathcal{Z}(\mathcal{C})$. The above construction defines a functor

$$\mathfrak{R}\colon \mathrm{HopfAlg}(\mathcal{Z}(\mathcal{C})) \to \mathrm{HopfMon}(\mathcal{C})/1_\mathcal{C}$$

which associates to each Hopf algebra (A, σ) in $\mathcal{Z}(\mathcal{C})$ the augmented Hopf monad $(A \otimes_\sigma ?, \varepsilon \otimes ?)$ and to each morphism of Hopf algebra f the Hopf monad morphism $\{f \otimes \mathrm{id}_X\}_{X \in \mathrm{Ob}(\mathcal{C})}$.

Theorem 3.1 ([11, Theorem 5.7])**.** *The functor* $\mathfrak{R}$ *is an equivalence of categories.*

In other words, representable Hopf monads are nothing but augmented Hopf monads. In [9, Remark 9.2] we give an example of a Hopf monad which is not representable.

Let $\mathcal{B}$ be a braided category with braiding τ. A Hopf algebra A in $\mathcal{B}$ gives rise to a Hopf algebra $(A, \tau_{A,-})$ in $\mathcal{Z}(\mathcal{B})$ and so to a Hopf monad $A \otimes_{\tau_{A,-}} ?$ on $\mathcal{B}$, denoted by $A \otimes ?$. Hopf monads on $\mathcal{B}$ which are representable by Hopf algebras in $\mathcal{B}$ are characterized as follows:

Corollary 3.2 ([11, Theorem 5.7])**.** *Let T be a Hopf monad on a braided category $\mathcal{B}$. Then T is isomorphic to the Hopf monad $A \otimes ?$ for some Hopf algebra A in $\mathcal{B}$ if and only if it is endowed with an augmentation $e\colon T \to 1_\mathcal{C}$ compatible with the braiding τ of $\mathcal{B}$ in the following sense:*

$$(e_X \otimes \mathrm{id}_{T(\mathbb{1})}) T_2(X, \mathbb{1}) = (e_X \otimes \mathrm{id}_{T(\mathbb{1})}) \tau_{T(\mathbb{1}), T(X)} T_2(\mathbb{1}, X)$$

for any $X \in \mathrm{Ob}(\mathcal{B})$.

3.5. **Hopf monads on closed monoidal categories.** In this section, we characterize Hopf monads on closed monoidal categories in terms of the existence of binary antipodes.

Let $\mathcal{C}$ be a monoidal category. For $X, Y \in \mathrm{Ob}(\mathcal{C})$, a *left internal Hom* from X to Y is an object $[X,Y]^l \in \mathrm{Ob}(\mathcal{C})$ endowed with a morphism $\mathrm{ev}_Y^X \colon [X,Y]^l \otimes X \to Y$ such that, for each $Z \in \mathrm{Ob}(\mathcal{C})$, the mapping

$$\begin{cases} \mathrm{Hom}_{\mathcal{C}}(Z,[X,Y]^l) & \to & \mathrm{Hom}_{\mathcal{C}}(Z \otimes X, Y) \\ f & \mapsto & \mathrm{ev}_Y^X(f \otimes \mathrm{id}_X) \end{cases}$$

is a bijection. If a left internal Hom from X to Y exists, it is unique up to unique isomorphism. A monoidal category $\mathcal{C}$ is *left closed* if left internal Homs exist in $\mathcal{C}$. This is equivalent to saying that, for every $X \in \mathrm{Ob}(\mathcal{C})$, the endofunctor $? \otimes X$ admits a right adjoint $[X, ?]^l$, with adjunction morphisms:

$$\mathrm{ev}_Y^X \colon [X,Y]^l \otimes X \to Y \quad \text{and} \quad \mathrm{coev}_Y^X \colon Y \to [X, Y \otimes X]^l,$$

called respectively the *left evaluation* and the *left coevaluation*.

One defines similarly *right internal Homs* and *right closed* monoidal categories. A monoidal category $\mathcal{C}$ is right closed if and only if, for every $X \in \mathrm{Ob}(\mathcal{C})$, the endofunctor $X \otimes ?$ has a right adjoint $[X, ?]^r$, with adjunction morphisms:

$$\widetilde{\mathrm{ev}}_Y^X \colon X \otimes [X,Y]^r \to Y \quad \text{and} \quad \widetilde{\mathrm{coev}}_Y^X \colon Y \to [X, X \otimes Y]^l,$$

called respectively the *right evaluation* and the *right coevaluation*.

A *closed monoidal category* is a monoidal category which is both left and right closed.

Let T be a bimonad on $\mathcal{C}$. If $\mathcal{C}$ is left closed, a *binary left antipode* for T is a natural transformation $\mathfrak{s}^l = \{\mathfrak{s}^l_{X,Y} \colon T[T(X),Y]^l \to [X, T(Y)]^l\}_{X,Y \in \mathrm{Ob}(\mathcal{C})}$ satisfying the following two axioms:

$$T\big(\mathrm{ev}_Y^X([\eta_X, Y]^l \otimes \mathrm{id}_X)\big) = \mathrm{ev}_{TY}^{TX}(s^l_{TX,Y} T[\mu_X, Y]^l \otimes \mathrm{id}_{TX}) T_2([TX,Y]^l, X),$$

$$[X, \mathrm{id}_{TY} \otimes \eta_X]^l \mathrm{coev}_{TY}^X = [X, (\mathrm{id}_{TY} \otimes \mu_X) T_2(Y, TX)]^l s^l_{X, Y \otimes TX} T(\mathrm{coev}_Y^{TX}),$$

for all $X, Y \in \mathrm{Ob}(\mathcal{C})$.

Similarly if $\mathcal{C}$ is right closed, a *binary right antipode* for T is a natural transformation $\mathfrak{s}^r = \{\mathfrak{s}^r_{X,Y} \colon T[T(X),Y]^r \to [X, T(Y)]^r\}_{X,Y \in \mathrm{Ob}(\mathcal{C})}$ satisfying:

$$T\big(\widetilde{\mathrm{ev}}_Y^X(\mathrm{id}_X \otimes [\eta_X, Y]^r)\big) = \widetilde{\mathrm{ev}}_{TY}^{TX}(\mathrm{id}_{TX} \otimes s^r_{TX,Y} T[\mu_X, Y]^r) T_2(X, [TX,Y]^r),$$

$$[X, \eta_X \otimes \mathrm{id}_{TY}]^r \widetilde{\mathrm{coev}}_{TY}^X = [X, (\mu_X \otimes \mathrm{id}_{TY}) T_2(TX, Y)]^r s^r_{X, TX \otimes Y} T(\widetilde{\mathrm{coev}}_Y^{TX}),$$

for all $X, Y \in \mathrm{Ob}(\mathcal{C})$.

Theorem 3.3 ([11, Theorem 3.6])**.** *Let T be a bimonad on a closed monoidal category $\mathcal{C}$. The following assertions are equivalent:*

(i) *The bimonad T is a Hopf monad on $\mathcal{C}$;*
(ii) *The bimonad T admits left and right binary antipodes;*
(iii) *The monoidal category $\mathcal{C}^T$ is closed and the forgetful functor U_T preserves left and right internal Homs.*

If the equivalent conditions of Theorem 3.3 are satisfied, then left and right internal Homs for any two T-modules (M,r) and (N,t) are given by

$$[(M,r),(N,t)]^l = \big([M,N]^l, [M,t]^l \mathfrak{s}^l_{M,N} T[r,N]^l\big),$$

$$[(M,r),(N,t)]^r = \big([M,N]^r, [M,t]^r \mathfrak{s}^r_{M,N} T[r,N]^r\big).$$

In addition to characterizing Hopf monads on closed monoidal categories, the left and right antipodes, when they exist, are unique and well-behaved with respect to the bimonad structure (see [11, Proposition 3.8]).

3.6. **Hopf monads on autonomous categories.** In this section, we characterize Hopf monads on autonomous categories in terms of the existence of unary antipodes, recovering the first definition of Hopf monad we gave in [8].

If T is a bimonad on a left autonomous category $\mathcal{C}$, a *left (unary) antipode* for T is a natural transformation $s^l = \{s^l_X : T({}^\vee T(X)) \to {}^\vee X\}_{X \in \mathrm{Ob}(\mathcal{C})}$ satisfying:

$$T_0 T(\mathrm{ev}_X) T({}^\vee\eta_X \otimes \mathrm{id}_X) = \mathrm{ev}_{T(X)}\big(s^l_{T(X)} T({}^\vee\mu_X) \otimes \mathrm{id}_{T(X)}\big) T_2({}^\vee T(X), X),$$
$$(\eta_X \otimes \mathrm{id}_{{}^\vee X})\mathrm{coev}_X T_0 = (\mu_X \otimes s^l_X) T_2(T(X), {}^\vee T(X)) T(\mathrm{coev}_{T(X)}),$$

for all $X \in \mathrm{Ob}(\mathcal{C})$.

Similarly if T is a bimonad on a right autonomous category $\mathcal{C}$, a *right (unary) antipode* for T is a natural transformation $s^r = \{s^r_X : T((TX)^\vee) \to X^\vee\}_{X \in \mathrm{Ob}(\mathcal{C})}$ satisfying:

$$T_0 T(\widetilde{\mathrm{ev}}_X) T(\mathrm{id}_X \otimes \eta^\vee_X) = \widetilde{\mathrm{ev}}_{T(X)}\big(\mathrm{id}_{T(X)} \otimes s^r_{T(X)} T(\mu^\vee_X)\big) T_2(X, T(X)^\vee),$$
$$(\mathrm{id}_{X^\vee} \otimes \eta_X)\widetilde{\mathrm{coev}}_X T_0 = (s^r_X \otimes \mu_X) T_2(T(X)^\vee, T(X)) T(\widetilde{\mathrm{coev}}_{T(X)}),$$

for all $X \in \mathrm{Ob}(\mathcal{C})$.

For example, if (A, σ) is a Hopf algebra in the center $\mathcal{Z}(\mathcal{C})$ of an autonomous category $\mathcal{C}$, then the Hopf monad $A \otimes_\sigma ?$ (see Section 3.4) admits left and right antipodes given by

$$s^l_X = (\mathrm{id}_{{}^\vee X} \otimes \mathrm{ev}_A)\tau_{A, {}^\vee X \otimes {}^\vee A}(S \otimes \mathrm{id}_{{}^\vee X \otimes {}^\vee A}),$$
$$s^r_X = (\mathrm{id}_{X^\vee} \otimes \widetilde{\mathrm{ev}}_A)(\tau_{A, X^\vee} \otimes \mathrm{id}_{{}^\vee A})(S^{-1} \otimes \mathrm{id}_{X^\vee \otimes A^\vee}),$$

for all $X \in \mathrm{Ob}(\mathcal{C})$.

An autonomous category $\mathcal{C}$ is closed: for $X, Y \in \mathrm{Ob}(\mathcal{C})$, $[X, Y]^l = Y \otimes {}^\vee X$ is a left internal Hom from X to Y, with left evaluation $\mathrm{ev}^X_Y = \mathrm{id}_Y \otimes \mathrm{ev}_X$, and $[X, Y]^r = X^\vee \otimes Y$ is a right internal Hom from X to Y, with right evaluation $\widetilde{\mathrm{ev}}^X_Y = \widetilde{\mathrm{ev}}_X \otimes \mathrm{id}_Y$. By [11, Theorem 3.10], the existence of a left (resp. right) unary antipode for a bimonad T on an autonomous category $\mathcal{C}$ is equivalent to the existence of a left (resp. right) binary antipode for T. If such is the case, antipodes are related by

$$\mathfrak{s}^l_{X,Y} = (\mathrm{id}_{T(X)} \otimes s^l_Y) T_2(X, {}^\vee T(Y)), \qquad s^l_X = (T_0 \otimes \mathrm{id}_{{}^\vee X})\mathfrak{s}^l_{X,\mathbb{1}},$$
$$\mathfrak{s}^r_{X,Y} = (s^r_Y \otimes \mathrm{id}_{T(X)}) T_2({}^\vee T(Y), X), \qquad s^r_X = (\mathrm{id}_{{}^\vee X} \otimes T_0)\mathfrak{s}^r_{X,\mathbb{1}},$$

for any $X, Y \in \mathrm{Ob}(\mathcal{C})$.

The following theorem characterizes Hopf monads on autonomous categories.

Theorem 3.4 ([8, Theorem 3.8])**.** *Let $\mathcal{C}$ be an autonomous category and T be a bimonad on $\mathcal{C}$. Then the following assertions are equivalent:*

(i) *The bimonad T is a Hopf monad;*
(ii) *The bimonad T admits left and right antipodes;*
(iii) *The monoidal category $\mathcal{C}^T$ is autonomous.*

If the equivalent conditions of Theorem 3.4 are satisfied, the left and right duals of any T-module (M, r) are given by ${}^\vee(M, r) = ({}^\vee M, s^l_M T({}^\vee r))$ and $(M, r)^\vee = (M^\vee, s^r_M T(r^\vee))$.

3.7. **Hopf monads and adjunctions.** Let $(F : \mathcal{C} \to \mathcal{D}, U : \mathcal{D} \to \mathcal{C})$ be an adjunction, with unit $\eta : 1_\mathcal{C} \to UF$ and counit $\varepsilon : FU \to 1_\mathcal{D}$. Then $T = UF$ is a monad with product $\mu = U(\varepsilon_F)$ and unit η. For example, if (T, μ, η) is a monad on a category $\mathcal{C}$, then the forgetful functor $U_T : \mathcal{C}^T \to \mathcal{C}$ admits a left adjoint $F_T : \mathcal{C} \to \mathcal{C}^T$, defined by $F_T(X) = (T(X), \mu_X)$ for any object X of $\mathcal{C}$ and $F_T(f) = T(f)$ for any morphism f of $\mathcal{C}$, and T is the monad of the adjunction (F_T, U_T). See [Mac] for details.

Let $(F: \mathcal{C} \to \mathcal{D}, U: \mathcal{D} \to \mathcal{C})$ be an adjunction between monoidal categories. Denote its unit by $\eta: 1_{\mathcal{C}} \to UF$ and its counit by $\varepsilon: FU \to 1_{\mathcal{D}}$. We say that (F, U) is *comonoidal* if F, U are comonoidal functors and η, ε are comonoidal natural transformations. In fact, (F, U) is a comonoidal if and only if U is strong monoidal. The monad $T = UF$ of a comonoidal adjunction (F, U) is a bimonad. For example, the adjunction (F_T, U_T) of a bimonad T is comonoidal (because U_T is strong monoidal) and its associated bimonad is T.

A comonoidal adjunction $(F: \mathcal{C} \to \mathcal{D}, U: \mathcal{D} \to \mathcal{C})$ is said to be a *Hopf adjunction* if the natural transformations

$$\mathbb{H}^l = \{\mathbb{H}^l_{c,d} = (F(c) \otimes \varepsilon_d) F_2(c, U(d)) : F(c \otimes U(d)) \to F(c) \otimes d\}_{c \in \mathrm{Ob}(\mathcal{C}), d \in \mathrm{Ob}(\mathcal{D})},$$
$$\mathbb{H}^r = \{\mathbb{H}^r_{d,c} = (\varepsilon_d \otimes F(c)) F_2(U(d), c) : F(U(d) \otimes c) \to d \otimes F(c)\}_{c \in \mathrm{Ob}(\mathcal{C}), d \in \mathrm{Ob}(\mathcal{D})},$$

are invertible, where η and ε are the unit and counit of the adjunction. The monad $T = UF$ of a Hopf adjunction (F, U) is Hopf monad. On the other hand:

Theorem 3.5 ([11, Theorem 2.15]). *Let T be a bimonad on a monoidal category $\mathcal{C}$. Then T is a Hopf monad if and only if the comonoidal adjunction (F_T, U_T) is a Hopf adjunction.*

Comonoidal adjunctions between autonomous categories give examples of Hopf monads:

Theorem 3.6 ([8, Theorem 3.8]). *Let (F, U) be a comonoidal adjunction between autonomous categories. Then $T = UF$ is a Hopf monad on $\mathcal{C}$.*

3.8. **Properties of Hopf monads.** Many fundamental results of the theory of Hopf algebras remain true for Hopf monads. For example, we extend the decomposition theorem of Hopf modules (see [8, Theorem 4.5] and [11, Theorem 6.11]), the Maschke criterium of semisimplicity (see [8, Theorem 6.5]), the existence of integral (see [8, Theorem 5.3]). In Section 3.14, we generalize the Drinfeld double of Hopf algebras to Hopf monads.

3.9. **Hopf algebroids and abelian tensor categories.** Let $\mathbb{k}$ be a commutative ring and R be a $\mathbb{k}$-algebra. Denote by ${}_R\mathrm{Mod}_R$ the category of R- bimodules. In [Sz], Szlachányi shows that left bialgebroids with base R (also called Takeuchi $\times_R$-bialgebras) are in 1-1 correspondence with $\mathbb{k}$- linear bimonads on ${}_R\mathrm{Mod}_R$ admitting a right adjoint. Let us define a *Hopf bialgebroid* to be a left bialgebroid whose associated bimonad on ${}_R\mathrm{Mod}_R$ is a Hopf monad. (This definition turns out to be equivalent to that given by Schauenburg in [Sc].) Since the monoidal category ${}_R\mathrm{Mod}_R$ is closed, we obtain from Section 3.5 a notion a left and right antipode for Hopf bialgebroids, see [11, Section 7].

The notion of Hopf algebroid is suitable for Tannaka reconstruction theory. Recall that a *tensor category over* $\mathbb{k}$ is an abelian autonomous $\mathbb{k}$-category. We say that a monoidal $\mathbb{k}$-category is *finite* if it is $\mathbb{k}$-linearly equivalent to the category of finite-dimensional left modules over some finite-dimensional $\mathbb{k}$-algebra.

Theorem 3.7 ([11, Theorem 7.6]). *Let $\mathcal{C}$ be a finite tensor category over a field $\mathbb{k}$. Then $\mathcal{C}$ is equivalent, as a tensor category, to the category of modules over a finite-dimensional left Hopf algebroid over $\mathbb{k}$.*

3.10. **Quasitriangular Hopf monads.** Let T be a bimonad on an monoidal category $\mathcal{C}$. An R-*matrix* for T is a natural transformation

$$R = \{R_{X,Y} : X \otimes Y \to T(Y) \otimes T(X)\}_{X,Y \in \mathrm{Ob}(\mathcal{C})}$$

satisfying:

$$(\mu_Y \otimes \mu_X) R_{T(X),T(Y)} T_2(X,Y) = (\mu_Y \otimes \mu_X) T_2(T(Y),T(X)) T(R_{X,Y}),$$
$$(\mathrm{id}_{T(Z)} \otimes T_2(X,Y)) R_{X\otimes Y,Z}$$
$$= (\mu_Z \otimes \mathrm{id}_{T(X)\otimes T(Y)})(R_{X,T(Z)} \otimes \mathrm{id}_{T(Y)})(\mathrm{id}_X \otimes R_{Y,Z}),$$
$$(T_2(Y,Z) \otimes \mathrm{id}_{T(X)}) R_{X,Y\otimes Z}$$
$$= (\mathrm{id}_{T(Y)\otimes T(Z)} \otimes \mu_X)(\mathrm{id}_{T(Y)} \otimes R_{T(X),Z})(R_{X,Y} \otimes \mathrm{id}_Z),$$

for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$. An R-matrix for T satisfies some QYB equation, see [8].

If T is a Hopf monad on an autonomous category $\mathcal{C}$, then an R-matrix R for T is invertible with respect to some convolution product and yields a braiding τ on $\mathcal{C}^T$ as follows:

$$\tau_{(M,r),(N,s)} = (s \otimes t) R_{M,N} \colon (M,r) \otimes (N,s) \to (N,s) \otimes (M,r)$$

for any T-modules (M,r) and (N,s). This assignment gives a 1-1 correspondence between R-matrices for T and braidings on $\mathcal{C}^T$. See [8, Section 8.2] for details.

3.11. **Distributive laws.** Let (P,m,u) and (T,μ,η) be monads on a category $\mathcal{C}$. Following Beck [Be], a *distributive law of T over P* is a natural transformation $\Omega = \{\Omega_X \colon TP(X) \to PT(X)\}_{X\in\mathrm{Ob}(\mathcal{C})}$ satisfying

$$\Omega_X T(m_X) = m_{T(X)} P(\Omega_X) \Omega_{P(X)}; \qquad \Omega_X T(u_X) = u_{T(X)};$$
$$\Omega_X \mu_{P(X)} = P(\mu_X) \Omega_{T(X)} T(\Omega_X); \qquad \Omega_X \eta_{P(X)} = P(\eta_X);$$

for all $X \in \mathrm{Ob}(\mathcal{C})$. These axioms ensure that the functor $PT \colon \mathcal{C} \to \mathcal{C}$ is a monad on $\mathcal{C}$ with product p and unit e given by

$$p_X = m_{T(X)} P^2(\mu_X) P(\Omega_{T(X)}) \quad \text{and} \quad e_X = u_{T(X)} \eta_X \quad \text{for any} \quad X \in \mathrm{Ob}(\mathcal{C}).$$

The monad (PT,p,e) is denoted by $P \circ_\Omega T$. A distributive law Ω of T over P also defines a lift of P to a monad $(\tilde{P},\tilde{m},\tilde{u})$ on the category $\mathcal{C}^T$ by

$$\tilde{P}(M,r) = \big(P(M), P(r)\Omega_M\big), \quad \tilde{m}_{(M,r)} = m_M, \quad \tilde{u}_{(M,r)} = u_M,$$

and the categories $(\mathcal{C}^T)^{\tilde{P}}$ and $\mathcal{C}^{P\circ_\Omega T}$ are isomorphic.

If P and T are Hopf monads on a monoidal category $\mathcal{C}$ and Ω is comonoidal, then $P \circ_\Omega T$ is a Hopf monad on $\mathcal{C}$, $\tilde{P}$ is a Hopf monad on $\mathcal{C}^T$, and $(\mathcal{C}^T)^{\tilde{P}} \simeq \mathcal{C}^{P\circ_\Omega T}$ as monoidal categories (see [11, Corollary 4.7]). If $\mathcal{C}$ is furthermore autonomous, then Ω is invertible (see [10, Proposition 4.12]).

3.12. **The centralizer of a Hopf monad.** Let $\mathcal{C}$ be an autonomous category. A functor $T \colon \mathcal{C} \to \mathcal{C}$ is *centralizable* if for every object X of $\mathcal{C}$, the functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ carrying any pair (Y_1,Y_2) to ${}^\vee T(Y_1) \otimes X \otimes Y_2$ has a coend

$$Z_T(X) = \int^{Y\in\mathcal{C}} {}^\vee T(Y) \otimes X \otimes Y.$$

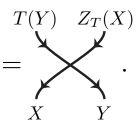
The correspondence $X \mapsto Z_T(X)$ extends to a functor $Z_T \colon \mathcal{C} \to \mathcal{C}$, called the *centralizer* of T, so that the associated universal dinatural transformation

$$i_{X,Y} \colon {}^\vee T(Y) \otimes X \otimes Y \to Z_T(X) \tag{10}$$

is natural in X and dinatural in Y. For $X, Y \in \mathrm{Ob}(\mathcal{C})$, set

$$\partial_{X,Y} = (\mathrm{id}_{T(Y)} \otimes i_{X,Y})(\mathrm{coev}_{T(X)} \otimes \mathrm{id}_{X\otimes Y}) \colon X \otimes Y \to T(Y) \otimes Z_T(X). \tag{11}$$

We depict $\partial_{X,Y}$ as:

$\partial_{X,Y} =$  .

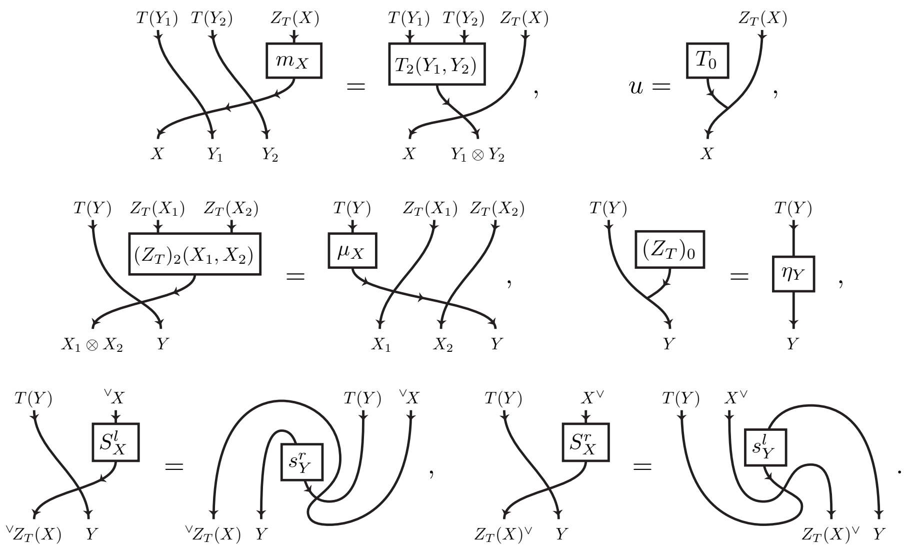


FIGURE 4. Structural morphisms of Z_T

Assume now that T is a Hopf monad on $\mathcal{C}$, with product μ, unit η, left antipode s^l, and right antipode s^r. By the factorization properties of coends, there exist unique natural transformations

$$
\begin{aligned}
&m\colon Z_T^2 \to Z_T, && (Z_T)_2\colon Z_T\otimes \to Z_T \otimes Z_T, && u\colon 1_{\mathcal{C}} \to Z_T,\\
&S^l\colon Z_T({}^\vee Z_T) \to {}^\vee 1_{\mathcal{C}}, && S^r\colon Z_T({Z_T}^\vee) \to 1_{\mathcal{C}}^\vee
\end{aligned}
$$

and a unique morphism $(Z_T)_0\colon Z_T(\mathbb{1}) \to \mathbb{1}$ such that the equalities of Figure 4 hold for all $X, Y, X_1, X_2, Y_1, Y_2 \in \mathrm{Ob}(\mathcal{C})$.

Theorem 3.8 ([10, Theorem 5.6])**.** *The centralizer Z_T of T is a Hopf monad on $\mathcal{C}$, with product m, unit u, comonoidal structure $\big((Z_T)_2, (Z_T)_0\big)$, left antipode S^l, and right antipode S^r.*

3.13. **The canonical distributive law.** Let T be a centralizable Hopf monad on an autonomous category $\mathcal{C}$ and Z_T be its centralizer with associated universal dinatural transformation i as in (10). Since a Hopf monad preserves colimits and so coends (see [8, Remark 3.13]), for any $X \in \mathrm{Ob}(\mathcal{C})$, the dinatural transformation

$$\{T(i_{X,Y})\colon T\big({}^\vee T(Y) \otimes X \otimes Y\big) \to TZ_T(X)\}_{Y \in \mathrm{Ob}(\mathcal{C})}$$

is universal. Therefore there exists a unique morphism $\Omega^T_X\colon TZ_T(X) \to Z_T T(X)$ such that, for any $Y \in \mathrm{Ob}(\mathcal{C})$,

$$\Omega^T_X T(i_{X,Y}) = i_{T(X),T(Y)}\big({}^\vee\mu_Y s^l_{T(Y)} T({}^\vee\mu_Y) \otimes T_2(X,Y)\big)\, T_2({}^\vee T(Y), X \otimes Y),$$

where μ and s^l are the product and the left antipode of T.

Theorem 3.9 ([10, Theorem 6.1])**.** $\Omega^T = \{\Omega^T_X\colon TZ_T(X) \to Z_T T(X)\}_{X \in \mathrm{Ob}(\mathcal{C})}$ *is an invertible comonoidal distributive law.*

We call Ω^T the *canonical distributive law of T over Z_T*. By Section 3.11, such a law allows to compose Z_T with T, giving the double of T (see Section 3.14 below), and to lift the monad Z_T to $\mathcal{C}^T$, leading to a description of the coend of $\mathcal{C}^T$ (see Section 3.15 below).

3.14. **The double of a Hopf monad.** Let T be a centralizable Hopf monad on an autonomous category $\mathcal{C}$, Z_T be its centralizer, and Ω^T be the canonical distributive law of T over Z_T. By Section 3.11, $D_T = Z_T \circ_{\Omega^T} T$ is a Hopf monad on $\mathcal{C}$. Let η and u be the units of T and Z_T respectively. For $X, Y \in \mathrm{Ob}(\mathcal{C})$, set

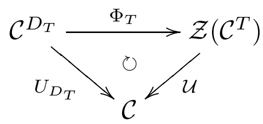
$$R_{X,Y} = \big(u_{T(Y)} \otimes Z_T(\eta_X)\big)\partial_{X,Y} \colon X \otimes Y \to D_T(Y) \otimes D_T(X),$$

where $\partial_{X,Y}$ is defined as in (11).

Theorem 3.10 ([10, Theorem 6.4]). *$R = \{R_{X,Y}\}_{X,Y \in \mathrm{Ob}(\mathcal{C})}$ is an* R*-matrix for the Hopf monad D_T.*

The quasitriangular Hopf monad D_T is called the *double of* T. This terminology is justified by the fact that the braided categories $Z(\mathcal{C}^T)$ and $\mathcal{C}^{D_T}$ are equivalent. More precisely, let $\mathcal{U}\colon \mathcal{Z}(\mathcal{C}^T) \to \mathcal{C}$ be the (strict monoidal) forgetful functor defined by $\mathcal{U}\big((M,r),\sigma\big) = M$ and $\mathcal{U}(f) = f$, and $\Phi_T\colon \mathcal{C}^{D_T} \to \mathcal{Z}(\mathcal{C}^T)$ be the functor defined by $\Phi_T(M,r) = \big((M, ru_{T(M)}),\sigma\big)$, with $\sigma_{(N,s)} = (s \otimes rZ_T(\eta_M))\partial_{M,N}$, and $\Phi_T(f) = f$. Then:

Theorem 3.11 ([10, Theorem 6.5]). *The functor $\Phi_T\colon \mathcal{C}^{D_T} \to \mathcal{Z}(\mathcal{C}^T)$ is a strict monoidal isomorphism of braided categories such that the following triangle of monoidal functors commutes:*



In particular, if $\mathcal{C}$ is an autonomous category such that the trivial Hopf monad $1_\mathcal{C}$ is centralizable, then the centralizer $Z = Z_{1_\mathcal{C}}$ of $1_\mathcal{C}$ coincides with the double of $1_\mathcal{C}$ and, by applying the results above to $T = 1_\mathcal{C}$, we obtain that Z is a quasitriangular Hopf monad on $\mathcal{C}$ and $\Phi = \Phi_{1_\mathcal{C}}\colon \mathcal{C}^Z \to \mathcal{Z}(\mathcal{C})$ is an isomorphism of braided categories.

3.15. **The coend of a category of modules.** Let T be a centralizable Hopf monad on an autonomous category $\mathcal{C}$, Z_T be its centralizer, and Ω^T be the canonical distributive law of T over Z_T. By Section 3.11, $\tilde{Z}_T = \tilde{Z}_T^{\Omega^T}$ is Hopf monad which is a lift of the Hopf monad Z_T to $\mathcal{C}^T$. Recall that $\tilde{Z}_T(M,r) = (Z_T(M), Z_T(r)\Omega^T_M)$ and $\tilde{Z}_T(f) = Z_T(f)$. For any T-modules (M,r) and (N,s), set

$$\iota_{(M,r),(N,s)} = i_{M,N}({}^\vee s \otimes \mathrm{id}_{M\otimes N})\colon {}^\vee(N,s) \otimes (M,r) \otimes (N,s) \to \tilde{Z}_T(M,r),$$

where i is the universal dinatural transformation associated with Z_T as in (10).

Theorem 3.12 ([10, Theorem 6.5]). *$\tilde{Z}_T$ is the centralizer of the trivial Hopf monad $1_{\mathcal{C}^T}$, with universal dinatural transformation ι.*

By the definition of a centralizer, $\tilde{Z}_T(\mathbb{1}, T_0) = \big(Z_T(\mathbb{1}), Z_T(T_0)\Omega^T_\mathbb{1}\big)$ is the coend of $\mathcal{C}^T$. It is a coalgebra in $\mathcal{C}^T$, with coproduct and counit given by

$$\Delta = (Z_T)_2(\mathbb{1},\mathbb{1})\colon Z_T(\mathbb{1}) \to Z_T(\mathbb{1}) \otimes Z_T(\mathbb{1}) \quad \text{and} \quad \varepsilon = (Z_T)_0\colon Z_T(\mathbb{1}) \to \mathbb{1}.$$

Assume now that T is furthermore quasitriangular, with R-matrix R, so that the autonomous category $\mathcal{C}^T$ is braided. The coend $\tilde{Z}_T(\mathbb{1}, T_0)$ of $\mathcal{C}^T$ becomes a Hopf algebra in $\mathcal{C}^T$ (see Section 2.5) endowed with a Hopf pairing (see Section 2.9). Its unit is $u = i_{\mathbb{1},\mathbb{1}}{}^\vee T_0\colon \mathbb{1} \to Z_T(\mathbb{1})$ and its product m, antipode S, and Hopf pairing ω are defined by the equalities of Figure 5, where $i_Y = i_{\mathbb{1},Y}$ for $Y \in \mathrm{Ob}(\mathcal{C})$. This gives an explicit description of the structural morphisms of the coend of $\mathcal{C}^T$.

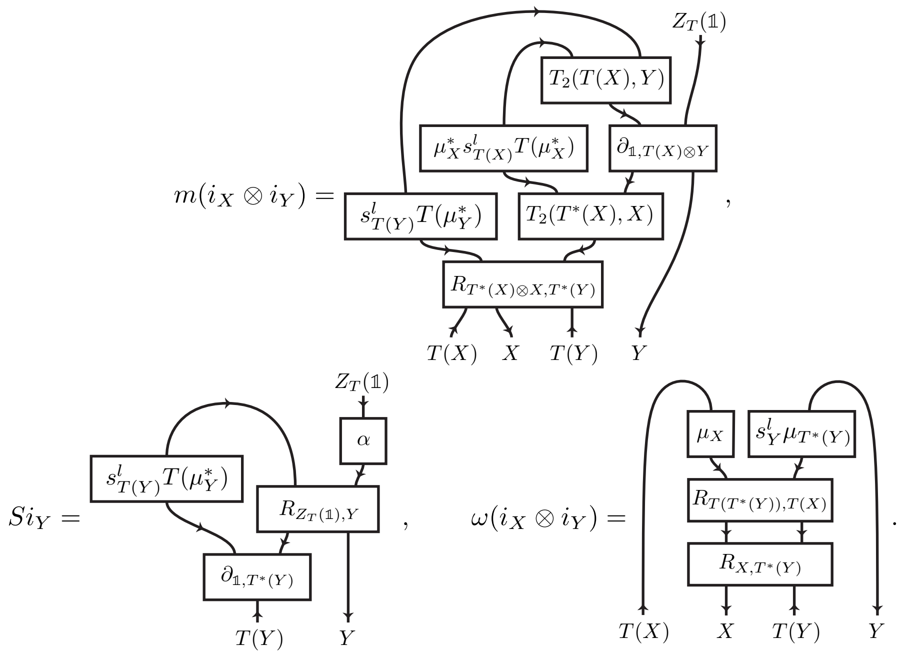


FIGURE 5. Hopf algebra structure of the coend of $\mathcal{C}^T$

3.16. **The case of Hopf algebras.** Let $\mathcal{B}$ be a braided autonomous category which admits a coend C and A be a Hopf algebra in $\mathcal{B}$. Then the Hopf monad $A\otimes?$ on $\mathcal{B}$ (see Section 3.4) is centralizable and we have:

$$Z_{A\otimes?} = {}^{\vee}A \otimes C\otimes? \quad \text{and} \quad D_{A\otimes?} = A \otimes {}^{\vee}A \otimes C\otimes?.$$

These Hopf monads are representable in $\mathcal{B}$ (see Corollary 3.2). Hence we get that $Z(A) = {}^{\vee}A\otimes C$ and $D(A) = A\otimes {}^{\vee}A\otimes C$ are Hopf algebras in $\mathcal{B}$. Furthermore $D(A)$ is quasitriangular, meaning that there exists a R-matrix

$$R\colon C \otimes C \to D(A) \otimes D(A)$$

verifying axioms generalizing (but not straightforwardly) the usual ones (when $\mathcal{B} = \mathrm{vect}_{\Bbbk}$, we have: $C = \Bbbk$ and $R \in D(A) \otimes D(A)$). This R-matrix makes the category $\mathrm{rep}_{\mathcal{B}}(D(A))$ of left $D(A)$- modules (in $\mathcal{B}$) braided in such a way that

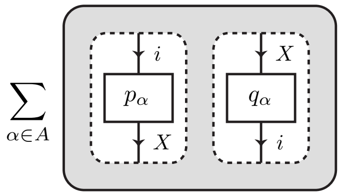
$$\mathcal{Z}(\mathrm{rep}_{\mathcal{B}}(A)) \simeq \mathrm{rep}_{\mathcal{B}}(D(A))$$

as braided categories. This generalizes the Drinfeld double of Hopf algebras over $\Bbbk$ to Hopf algebras in braided categories. We refer to [9, Section 8] for details.

3.17. **The case of fusion categories.** We apply the computations of the previous sections to a fusion category $\mathcal{C}$ over $\Bbbk$. Given a simple object i of $\mathcal{C}$, the *i-isotypical component* $X^{(i)}$ of an object X is the largest direct factor of X isomorphic to a direct sum of copies of i. The actual number of copies of i is $N_X^i = \mathrm{rank}_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(i, X)$. An *$i$-decomposition* of X is an explicit direct sum decomposition of $X^{(i)}$ into copies of i, that is, a family $(p_\alpha\colon X \to i, q_\alpha\colon i \to X)_{\alpha\in A}$ of pairs of morphisms in $\mathcal{C}$ such that $p_\alpha\, q_\beta = \delta_{\alpha,\beta}\, \mathrm{id}_i$ for all $\alpha, \beta \in A$ and the set the set A has N_X^i elements. Then the tensor

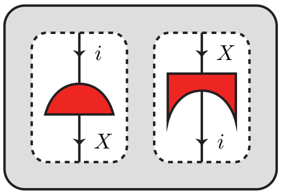
$$\sum_{\alpha\in A} p_\alpha \otimes_{\Bbbk} q_\alpha \in \mathrm{Hom}_{\mathcal{C}}(X, i) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(i, X)$$

does not depend on the choice of the i-decomposition $(p_\alpha, q_\alpha)_{\alpha\in A}$ of X. Consequently, a sum of the type

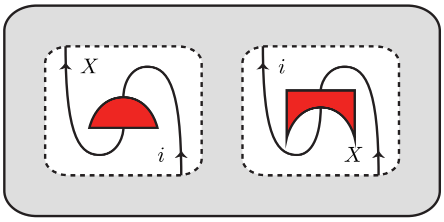


where $(p_\alpha, q_\alpha)_{\alpha\in A}$ is an i-decomposition of an object X and the gray area does not involve α, represents a morphism in $\mathcal{C}$ which is independent of the choice of the i-decomposition. We depict it as

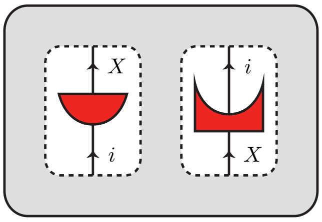
(12)



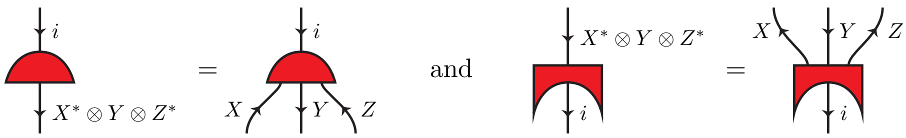
where the two curvilinear boxes should be shaded with the same color. If several such pairs of boxes appear in a picture, they must have different colors. We will also depict



as



Tensor products of objects may be depicted as bunches of strands. For example,



where the equality sign means that the pictures represent the same morphism of $\mathcal{C}$.

Fix a representative set I of simple objects of $\mathcal{C}$. Any $\Bbbk$-linear functor $T\colon \mathcal{C}\to\mathcal{C}$ is centralizable, and its centralizer $Z_T\colon \mathcal{C}\to\mathcal{C}$ is given, for all $X\in \mathrm{Ob}(\mathcal{C})$, by

(13) $$Z_T(X) = \bigoplus_{i\in I} T(i)^*\otimes X\otimes i.$$

The associated universal dinatural transformation is, or all $X, Y \in \mathrm{Ob}(\mathcal{C})$,

$$\rho_{X,Y} = \sum_{\alpha\in\Lambda_Y} T(q_Y^\alpha)^*\otimes \mathrm{id}_X\otimes p_Y^\alpha\colon T(Y)^*\otimes X\otimes Y\to Z_T(X),$$

where $(p_Y^\beta, q_Y^\beta)_\beta$ is any I-partition of Y.

The trivial Hopf monad $1_\mathcal{C}$, being $\Bbbk$-linear, is centralizable and its centralizer $Z = Z_{1_\mathcal{C}}\colon \mathcal{C}\to\mathcal{C}$ is the Hopf monad given by Formula (13) for $T = 1_\mathcal{C}$, that is,

$$Z(X) = \bigoplus_{i\in I} i^*\otimes X\otimes i.$$

The structural morphisms of Z are computed in Figure 6, see [9, Section 9]. (The dotted lines in the figure represent $\mathrm{id}_\mathbb{1}$ and can be removed without changing the

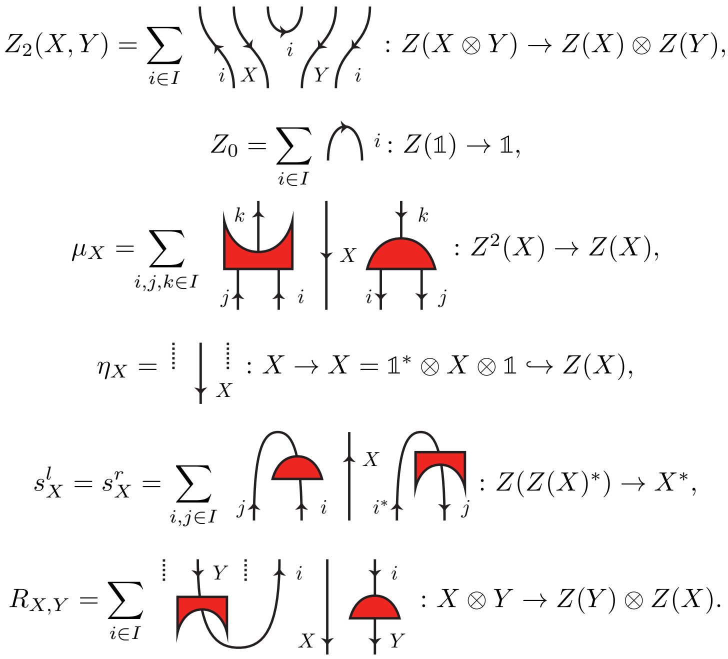


FIGURE 6. Structural morphisms of Z

morphisms; we depicted them in order to remember which factor of $Z(X)$ is concerned.)

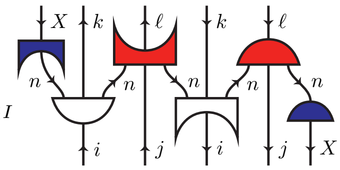
Being $\Bbbk$-linear, the Hopf monad Z is centralizable and its centralizer $Z_Z\colon \mathcal{C} \to \mathcal{C}$ is given by Formula (13) for $T = Z$, that is,

$$Z_Z(X) = \bigoplus_{j \in I} Z(j)^* \otimes X \otimes j \simeq \bigoplus_{i,j \in I} i^* \otimes j^* \otimes i \otimes X \otimes j.$$

Therefore, by Section 3.15, the category $\mathcal{C}^Z$ admits a coend. Now $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{C}^Z$ by Theorem 3.11 (applied with $T = 1_{\mathcal{C}}$). Hence $\mathcal{Z}(\mathcal{C})$ admits a coend (C, σ). We have:

$$C = Z_Z(\mathbb{1}) = \bigoplus_{i,j \in I} i^* \otimes j^* \otimes i \otimes j,$$

and $\sigma = \{\sigma_X\colon C \otimes X \to X \otimes C\}_{X \in \mathrm{Ob}(\mathcal{C})}$ is given by

(14) $\qquad \sigma_Y = \sum_{i,j,k,\ell,n \in I}$ 

for any $X \in \mathrm{Ob}(\mathcal{C})$. Recall from Section 2.5, that the coend (C, σ) of $\mathcal{Z}(\mathcal{C})$ is a Hopf algebra in $\mathcal{Z}(\mathcal{C})$. By Section 3.15, its structural morphisms can be expressed using only the category $\mathcal{C}$. Those needed to represent Hopf diagrams are depicted in Figure 7, together with $\tau_{(C,\sigma),(C,\sigma)} = \sigma_C$ depicted above.

Set

$$\Lambda = \sum_{i \in I} \dim(i)\, \mathrm{coev}_i\colon \mathbb{1} \to \bigoplus_{i \in I} i^* \otimes i = \bigoplus_{i \in I} \mathbb{1}^* \otimes i^* \otimes \mathbb{1} \otimes i \hookrightarrow C.$$

Theorem 3.13 ([13]). *The morphism $\Lambda\colon \mathbb{1} \to C$ is a $S_{(C,\sigma)}$-invariant integral of the coend (C, σ) of $\mathcal{Z}(\mathcal{C})$.*

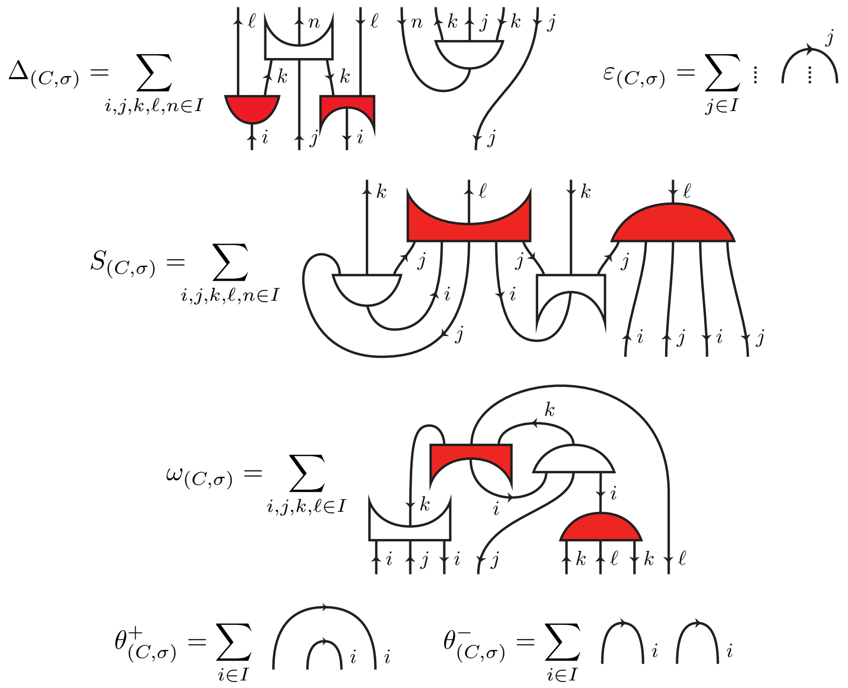


FIGURE 7. Structural morphisms of the coend of $\mathcal{Z}(\mathcal{C})$

From Theorem 3.13 we deduce that the Hopf pairing $\omega_{(C,\sigma)}$ associated with (C,σ) as in (9) is non-degenerate (see [13]). We recover from this fact that if $\Bbbk$ is an algebraically closed field and $\dim(\mathcal{C}) \neq 0$, then $\mathcal{Z}(\mathcal{C})$ is modular. This gives an alternative proof of Müger's Theorem 1.1.

3.18. **Computing $\tau_{\mathcal{Z}(\mathcal{C})}(M^3)$ from $\mathcal{C}$.** Let $\mathcal{C}$ be a spherical fusion category over the commutative ring $\Bbbk$. The integral Λ of the coend (C,σ) of $\mathcal{Z}(\mathcal{C})$ given by Theorem 3.13 is a normalizable algebraic Kirby element such that $\theta_C^+\Lambda = 1_\Bbbk$ and $\theta_C^-\Lambda = 1_\Bbbk$. Since we have an explicit description of the structural morphisms of (C,σ) (see Section 3.17), we have a way to compute the 3-manifolds invariant $\tau_{\mathcal{Z}(\mathcal{C})}(M;\Lambda)$ through Hopf diagrams (see Section 2.10). For example, we get

$$\tau_{\mathcal{Z}(\mathcal{C})}(S^2 \times S^1;\Lambda) = \dim(\mathcal{C}).$$

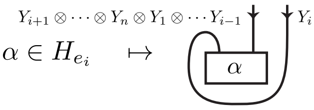
The invariant $\tau_{\mathcal{Z}(\mathcal{C})}(M;\Lambda)$ is well-defined even if $\dim \mathcal{C}$ is not invertible. When $\dim(\mathcal{C})$ is invertible and $\Bbbk$ is an algebraic closed field (so that $\mathcal{Z}(\mathcal{C})$ is a modular fusion category, see Theorem 1.1), the invariant $\tau_{\mathcal{Z}(\mathcal{C})}(M;\Lambda)$ is equal to the Reshetikhin-Turaev invariant $\tau_{\mathcal{Z}(\mathcal{C})}(M)$ (up to a different normalization, see Section 2.6). Hence we get a way to compute $\tau_{\mathcal{Z}(\mathcal{C})}(M)$ in terms of the structural morphisms of $\mathcal{C}$ (note that one cannot use the original algorithm of Reshetikhin-Turaev since the simple objects of $\mathcal{Z}(\mathcal{C})$ are unknown in general).

In the next section, we compare the Reshetikhin-Turaev invariant $\tau_{\mathcal{Z}(\mathcal{C})}(M)$ defined with $\mathcal{Z}(\mathcal{C})$ and the Turaev-Viro invariant $|M|_\mathcal{C}$ defined with $\mathcal{C}$: we show that $\tau_{\mathcal{Z}(\mathcal{C})}(M) = |M|_\mathcal{C}$ for any closed oriented 3-manifold M. As a corollary, the above method for computing $\tau_{\mathcal{Z}(\mathcal{C})}(M)$ in terms of Hopf diagrams and the structural morphisms of the coend (C,σ) gives rise to an alternative and efficient way to compute the sate-sum invariant $|M|_\mathcal{C}$.

4. On two approaches to 3-dimensional TQFTs

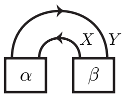
Our main goal in this section is to prove the conjecture (2) stated in the introduction, that is, to relate the Reshetikhin-Turaev invariant with the Turaev-Viro invariant through the categorical center. Given a spherical fusion category $\mathcal{C}$ with invertible dimension, we first define a new state sum on (non-generic) skeletons of 3-manifolds by means of an invariant of $\mathcal{C}$-colored graphs in the sphere. The 3-manifolds invariant $|M|_{\mathcal{C}}$ we obtain in this way is equal to the Turaev-Viro invariant, as revisited by Barrett and Westbury. Then we extend this invariant to a TQFT $|\cdot|_{\mathcal{C}}$ and show, when the ground ring $\Bbbk$ is an algebraically closed field, that the state sum TQFT $|\cdot|_{\mathcal{C}}$ is isomorphic to the Reshetikhin-Turaev TQFT $\tau_{\mathcal{Z}(\mathcal{C})}$ defined with the center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$. We refer to [12] for details.

4.1. **Symmetrized multiplicity modules.** Let $\mathcal{C}$ be a pivotal $\Bbbk$-category. A cyclic $\mathcal{C}$- set is a totally cyclically ordered finite set whose elements are labeled by objects of $\mathcal{C}$ and by signs in $\{+,-\}$. To any cyclic $\mathcal{C}$- set E we associate a $\Bbbk$-module $H(E)$ defined as follows. Let $e_1 < e_2 < \cdots < e_n < e_1$ be the elements of E ordered via the given cyclic order (here $n = \#E$ is the number of elements of E). Denote by $X_i \in \mathrm{Ob}(\mathcal{C})$ the color of e_i and by ε_i the sign of e_i. Let $Y_i = X_i$ if $\varepsilon_i = +$ and $Y_i = X_i^*$ if $\varepsilon_i = -$. Set $H_{e_i} = \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, Y_i \otimes \cdots \otimes Y_n \otimes Y_1 \otimes \cdots Y_{i-1})$. We identify the $\Bbbk$-modules $H_{e_1}, \ldots, H_{e_n}$ via the isomorphisms

$\alpha \in H_{e_i} \quad \mapsto$  $\in H_{e_{i+1}},$

which form a projective system. The projective limit of this system is a $\Bbbk$-module $H(E)$ which comes with a system of isomorphisms $\tau = \{\tau_e : H(E) \to H_e\}_{e \in E}$ called its universal cone.

A *duality* between two cyclic $\mathcal{C}$- sets E and E' is a bijection $E \to E'$ reversing the cyclic order, preserving the colors, and reversing the signs. Such a duality induces a $\Bbbk$-linear pairing $H(E) \otimes_{\Bbbk} H(E') \to \Bbbk$, defined as the map

$\alpha \otimes \beta \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, Y^* \otimes X) \otimes \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X^* \otimes Y) \mapsto$  $\in \mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \Bbbk.$

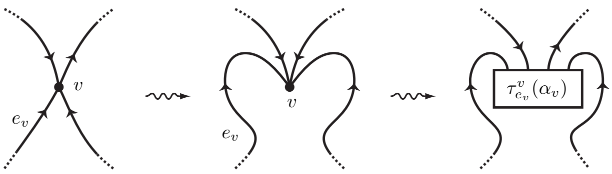
When the category $\mathcal{C}$ is fusion, this pairing is non-degenerate. In this case, the dual of the inverse of the pairing is a $\Bbbk$-homomorphism $H(E)^\star \otimes H(E')^\star \to \Bbbk$ called the *contraction*, where $H(E)^\star = \mathrm{Hom}_{\Bbbk}(H(E), \Bbbk)$. For more details, see [12, Section 2].

4.2. **Colored graphs in surfaces.** By a graph, we mean a finite graph without isolated vertices. Every edge of a graph connects two (possibly coinciding) vertices. We allow multiple edges with the same endpoints. A *$\mathcal{C}$-colored graph* in Σ is a graph embedded in Σ such that each edge is oriented and endowed with an object of $\mathcal{C}$ called the color of the edge. Given two $\mathcal{C}$-colored graphs G and G' in Σ, an isotopy of G into G' is an isotopy of G into G' in the class of $\mathcal{C}$-colored graphs in Σ preserving the vertices, the edges, and the orientation and the color of the edges.

Let G be a $\mathcal{C}$-colored graph in Σ. A vertex v of G determines a cyclic $\mathcal{C}$- set E_v as follows: E_v is the set of half-edges of G incident to v with cyclic order induced by the opposite orientation of Σ; each half-edge e is endowed with the color of the edge and with the sign $+$ if e is oriented towards v and $-$ otherwise. Set $H_v(G) = H(E_v)$ and $H(G) = \otimes_v H_v(G)$, where v runs over all vertices of G and $\otimes = \otimes_{\Bbbk}$ is the tensor product over $\Bbbk$. To stress the role of Σ, we shall sometimes write $H_v(G;\Sigma)$ for $H_v(G)$ and $H(G;\Sigma)$ for $H(G)$.

Denote by $-\Sigma$ the surface Σ with opposite orientation. A $\mathcal{C}$-colored graph G in Σ determines a $\mathcal{C}$-colored graph G^{op} in $-\Sigma$ obtained by reversing orientation in all edges of G and in Σ while keeping the colors of the edges. The cyclic $\mathcal{C}$-sets determined by a vertex v of G and G^{op} are dual. In particular, when $\mathcal{C}$ is fusion, we can conclude that $H_v(G^{\mathrm{op}};-\Sigma)^\star \simeq H_v(G;\Sigma)$.

4.3. **Invariants of colored graphs in the sphere.** Let $\mathcal{C}$ be a spherical $\Bbbk$-category. We orient the 2-sphere $S^2=\mathbb{R}^2\cup\{\infty\}$ in such a way it extends the counterclockwise orientation in $\mathbb{R}^2$. Let G be a $\mathcal{C}$-colored graph in S^2. Pushing, if necessary, G away from ∞, we can assume that G lies in $\mathbb{R}^2$. For each vertex v of G, we choose a half-edge e_v incident to v and isotope G near v so that the half-edges incident to v lie above v with respect to the second coordinate on $\mathbb{R}^2$ and e_v is the leftmost of them. Pick any $\alpha_v \in H_v(G)$ and replace v by a box colored with $\tau^v_{e_v}(\alpha_v)$, where τ^v is the universal cone of $H_v(G)$ (see Section 4.1):

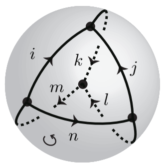


.

This transforms G into a planar diagram which determines, by the Penrose calculus, an element of $\mathrm{End}_{\mathcal{C}}(\mathbb{1})=\Bbbk$ denoted $\mathbb{F}_{\mathcal{C}}(G)(\otimes_v \alpha_v)$. By linear extension, this procedure defines a vector $\mathbb{F}_{\mathcal{C}}(G)\in H(G)^\star=\mathrm{Hom}_{\Bbbk}(H(G),\Bbbk)$.

Proposition 4.1 ([12, Section 3])**.** *The vector $\mathbb{F}_{\mathcal{C}}(G)\in H(G)^\star$ is a well-defined isotopy invariant of a $\mathcal{C}$-colored graph G in S^2.*

For example, consider for $i,j,k,l,m,n\in \mathrm{Ob}(\mathcal{C})$ the following $\mathcal{C}$-colored graph in S^2:

$\Gamma =$ 

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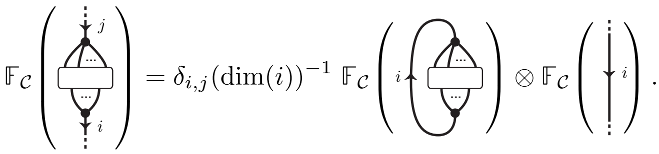
Here $H(\Gamma)$ is the tensor product of four modules isomorphic to $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, m\otimes i^*\otimes n^*)$, $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, j\otimes i\otimes k^*)$, $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, n\otimes j^*\otimes l^*)$, and $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, l\otimes k\otimes m^*)$. The vector $\mathbb{F}_{\mathcal{C}}(\Gamma)\in H(\Gamma)^\star$ and similar vectors associated with other orientations of the edges of Γ form a family of $2^6=64$ tensors called *$6j$-symbols* associated with i,j,k,l,m,n. For more on this, see [12, Appendix].

The invariant $\mathbb{F}_{\mathcal{C}}$ has several properties (see [12]). In particular, if $\mathcal{C}$ is a spherical fusion category and I is a representative set of simple objects of $\mathcal{C}$, then:

(a) For any $i,j\in I$,



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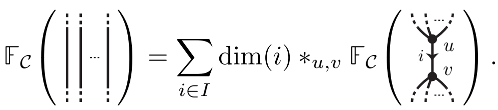
(b)



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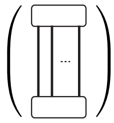
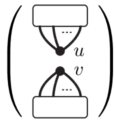
(c) $\mathbb{F}_{\mathcal{C}}$  $= *_{u,v}\, \mathbb{F}_{\mathcal{C}}$ 

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In (a) and (c) the empty rectangles stand for pieces of $\mathcal{C}$-colored graphs sitting inside the rectangles. The same $\mathcal{C}$-colored graphs appear on both sides of the equalities. In (b) and (c), the duality between the cyclic $\mathcal{C}$- sets E_u and E_v associated with the vertices u and v is induced by the symmetry with respect to a horizontal line and gives rise to the contraction map $*_{u,v}\colon H_u(G)^* \otimes H_v(G)^* = H(E_u)^* \otimes H(E_v)^* \to \Bbbk$, see Section 4.1.

4.4. **Skeletons of 3-manifolds.** By a *2-polyhedron*, we mean a compact topological space that can be triangulated using only simplices of dimension ≤ 2. For a 2-polyhedron P, denote by $\mathrm{Int}(P)$ the subspace of P consisting of all points having a neighborhood homeomorphic to $\mathbb{R}^2$. Clearly, $\mathrm{Int}(P)$ is an (open) 2-manifold without boundary. By an *arc* in P, we mean the image of a path $\alpha\colon [0,1] \to P$ which is an embedding except that possibly $\alpha(0) = \alpha(1)$. (Thus, arcs may be loops.)

To work with polyhedra, we will use the language of stratifications as follows. Consider a 2-polyhedron P endowed with a finite set of arcs E such that

(a) different arcs in E may meet only at their endpoints;
(b) $P \setminus \cup_{e\in E}\, e \subset \mathrm{Int}(P)$ and $P \setminus \cup_{e\in E}\, e$ is dense in P.

The arcs of E are called *edges* of P and their endpoints are called *vertices* of P. The vertices and edges of P form a graph $P^{(1)} = \cup_{e\in E}\, e$. Since all vertices of P are endpoints of the edges, $P^{(1)}$ has no isolated vertices. Cutting P along $P^{(1)}$, we obtain a compact surface $\widetilde{P}$ with interior $P \setminus P^{(1)}$. The polyhedron P can be recovered by gluing $\widetilde{P}$ to $P^{(1)}$ along a map $p\colon \partial\widetilde{P} \to P^{(1)}$. Condition (b) ensures the surjectivity of p. We call the pair (P, E) (or, shorter, P) a *stratified 2-polyhedron* if the set p^{-1}(the set of vertices of P) is finite and each component of the complement of this set in $\partial\widetilde{P}$ is mapped homeomorphically onto the interior of an edge of P.

A 2-polyhedron P can be stratified if and only if $\mathrm{Int}(P)$ is dense in P. For such a P, the edges of any triangulation form a stratification. Another example: a closed surface with an empty set of edges is a stratified 2-polyhedron.

For a stratified 2-polyhedron P, the connected components of $\widetilde{P}$ are called *regions* of P. Clearly, the set $\mathrm{Reg}(P)$ of the regions of P is finite. For a vertex x of P, a *branch* of P at x is a germ at x of a region of P adjacent to x. The set of branches of P at x is finite and non-empty. Similarly, for an edge e of P, a *branch* of P at e is a germ at e of a region of P adjacent to e. The set of branches of P at e is denoted P_e. This set is finite and non-empty. The number of elements of P_e is the *valence* of e. The edges of P of valence 1 and their vertices form a graph called the *boundary* of P and denoted ∂P. We say that P is *orientable* (resp. *oriented*) if all regions of P are orientable (resp. oriented).

A *skeleton* of a closed 3-manifold M is an oriented stratified 2-polyhedron $P \subset M$ such that $\partial P = \emptyset$ and $M \setminus P$ is a disjoint union of open 3-balls. An example of a skeleton of M is provided by the (oriented) 2-skeleton $t^{(2)}$ of a triangulation t of M, where the edges of $t^{(2)}$ are the edges of t.

We define four moves $T_1, \ldots, T_4$ on a skeleton P of M transforming P into a new skeleton of M, see Figure 8. The "phantom edge move" T_1 keeps P as a polyhedron and adds one new edge connecting distinct vertices of P (this edge is an arc in P meeting $P^{(1)}$ solely at its endpoints and has the valence 2). The "contraction move" T_2 collapses into a point an edge e of P with distinct endpoints. This move is allowed only when at least one endpoint of e is the endpoint of some other edge. The "percolation move" T_3 pushes a branch b of P through a vertex x of P. The

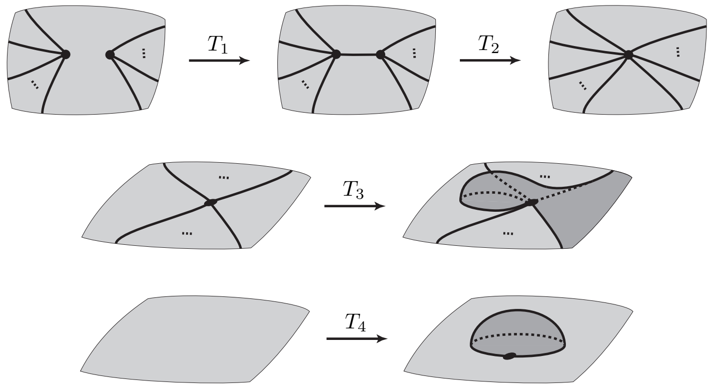

FIGURE 8. Local moves on skeletons

branch b is pushed across a small disk D lying in another branch of P at x so that $D \cap P^{(1)} = \partial D \cap P^{(1)} = \{x\}$ and both these branches are adjacent to the same component of $M \setminus P$. The "bubble move" T_4 adds to P an embedded disk $D_+ \subset M$ such that $D_+ \cap P = \partial D_+ \subset P \setminus P^{(1)}$, the circle ∂D_+ bounds a disk D_- in $P \setminus P^{(1)}$, and the 2-sphere $D_+ \cup D_-$ bounds a ball in M meeting P precisely along D_-. A point of the circle ∂D_+ is chosen as a vertex and the circle itself is viewed as an edge of the resulting skeleton. The orientation of the skeletons produced by the moves $T_1, \ldots, T_4$ on P is induced by the orientation of P except for the small disk regions created by T_3, T_4 whose orientation is chosen arbitrarily.

The moves $T_1, \ldots, T_4$ have obvious inverses. The move T_1^{-1} deletes a 2-valent edge e with distinct endpoints; this move is allowed only when both endpoints of e are endpoints of some other edges and the orientations on both sides of e are compatible. We call the moves $T_1, \ldots, T_4$ and their inverses *primary moves*. In the sequel, we tacitly assume the right to use ambient isotopies of skeletons in M. In other words, ambient isotopies are treated as primary moves.

Theorem 4.2 ([12, Section 7]). *Any two skeletons of M can be related by primary moves.*

We prove Theorem 4.2 by showing that any skeleton of M can be transformed via the primary moves into a so called special skeleton, and then using the theory of special skeletons due to Casler, Matveev, and Piergallini.

4.5. **State sums on skeletons of 3-manifolds.** Let $\mathcal{C}$ be a spherical fusion category over $\Bbbk$ whose dimension is invertible in $\Bbbk$. Fix a (finite) representative set I of simple objects of $\mathcal{C}$. For each closed oriented 3-manifold M, we define a topological invariant $|M|_{\mathcal{C}} \in \Bbbk$.

Pick a skeleton P of M and a map $c\colon \mathrm{Reg}(P) \to I$. For each oriented edge e of P, we define a $\Bbbk$-module $H_c(e)$ as follows. The orientations of e and M determine a positive direction on a small loop in M encircling e; this direction determines a cyclic order on the set P_e of all branches of P at e. To each branch $b \in P_e$ we assign the object $c(r) \in I$ (where r is the region of P containing b) and a sign equal to $+$ if the orientations of b and e are compatible and to $-$ otherwise. (The orientations of b and e are compatible if each pair (a tangent vector directed outward b at a point of e, a positive tangent vector of e) is positively oriented in b.) In this way, P_e becomes a cyclic $\mathcal{C}$-set. Set $H_c(e) = H(P_e)$. If e^{op} is the same edge with opposite orientation, then $P_{e^{\mathrm{op}}}$ and P_e are in duality. This induces a duality between the modules $H_c(e)$, $H_c(e^{\mathrm{op}})$ and a contraction $*_e\colon H_c(e^{\mathrm{op}})^\star \otimes H_c(e)^\star \to \Bbbk$, see Section 4.1. Note that the contractions $*_e$ and $*_{e^{\mathrm{op}}}$ are equal up to permutation of the tensor factors.

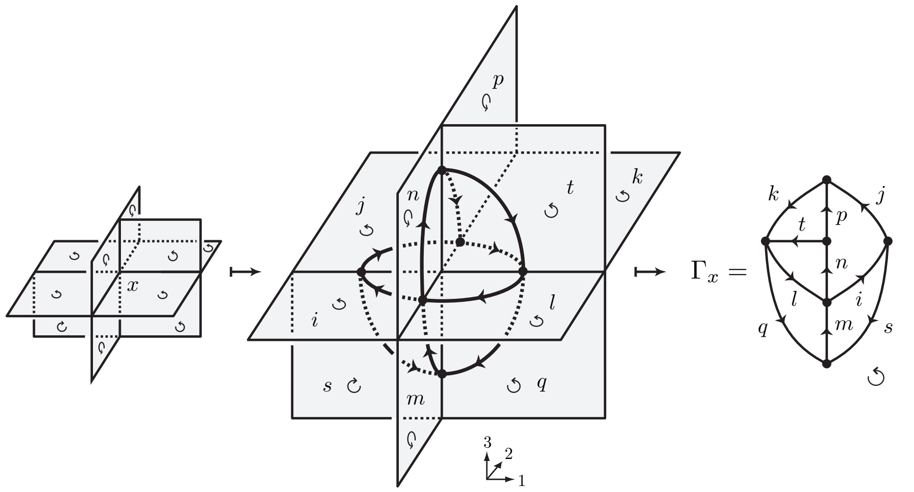

FIGURE 9. The graph $\Gamma_x \subset S^2$ associated with a vertex x

Any vertex x of a skeleton $P \subset M$ has a closed ball neighborhood $B_x \subset M$ such that $\Gamma_x = P \cap \partial B_x$ is a finite non-empty graph and $P \cap B_x$ is the cone over Γ_x. The vertices of Γ_x are the intersections of ∂B_x with the half-edges of P incident to x; the edges of Γ_x are the intersections of ∂B_x with the branches of P at x. We color every edge α of Γ_x with $c(r_\alpha) \in I$, where r_α is the region of P containing the branch b such that $\alpha = b \cap \partial B_x$, and endow α with the orientation induced by that of $r_\alpha \setminus \mathrm{Int}(B_x)$. We identify ∂B_x with the standard 2-sphere S^2 via an orientation preserving homeomorphism, where the orientation of ∂B_x is induced by that of M restricted to $M \setminus \mathrm{Int}(B_x)$. In this way, Γ_x becomes a $\mathcal{C}$-colored graph in S^2. Section 4.3 yields a tensor $\mathbb{F}_\mathcal{C}(\Gamma_x) \in H_c(\Gamma_x)^*$. See Figure 9 for an example. By definition, $H_c(\Gamma_x) = \otimes_e H_c(e)$, where e runs over all edges of P incident to x and oriented away from x (an edge with both endpoints in x appears in this tensor product twice with opposite orientations). The tensor product $\otimes_x \mathbb{F}_\mathcal{C}(\Gamma_x)$ over all vertices x of P is a vector in $\otimes_e H_c(e)^\star$, where e runs over all oriented edges of P. Set $*_P = \otimes_e *_e \colon \otimes_e H_c(e)^\star \to \Bbbk$ and

$$|M|_\mathcal{C} = (\dim(\mathcal{C}))^{-|P|} \sum_c \left(\prod_{r \in \mathrm{Reg}(P)} (\dim c(r))^{\chi(r)} \right) *_P (\otimes_x \mathbb{F}_\mathcal{C}(\Gamma_x)) \in \Bbbk, \tag{15}$$

where $|P|$ is the number of components of $M \setminus P$, c runs over all maps $\mathrm{Reg}(P) \to I$, and $\chi(r)$ is the Euler characteristic of r.

Theorem 4.3. [12, Section 5] $|M|_\mathcal{C}$ *is a topological invariant of* M*. This invariant does not depend on the choice of* I*.*

We prove Theorem 4.3 by showing that this construction in independent of the choice of skeleton P of M: we verify that the right hand side of (15) remains unchanged when applying a primary move (see Theorem 4.2), thanks to the properties of the invariant $\mathbb{F}_\mathcal{C}$ of $\mathcal{C}$-colored graphs in S^2 (see Section 4.3).

The state sum invariant $|M|_\mathcal{C}$ generalizes the state sums of Turaev-Viro [TV] and Barrett-Westbury [BW1]. Indeed, when P is the oriented 2-skeleton of the cellular subdivision of M dual to a triangulation t of M, and the orientation of P is induced by that of M and a total order on the set of vertices of t, Formula (15) is equivalent to the state sum on t given in [BW1]. In particular $|M|_\mathcal{C}$ is equal to Turaev-Viro-Barrett-Westbury invariant.

It is clear from the definitions that $|M \amalg N|_{\mathcal{C}} = |M|_{\mathcal{C}}\, |N|_{\mathcal{C}}$ for any oriented closed 3-manifolds M, N. One can show that $|-M|_{\mathcal{C}} = |M|_{\mathcal{C}^{\mathrm{op}}}$, where $-M$ is M with opposite orientation and $\mathcal{C}^{\mathrm{op}}$ is the category opposite to $\mathcal{C}$. We have: $|S^3|_{\mathcal{C}} = (\dim(\mathcal{C}))^{-1}$ and $|S^1 \times S^2|_{\mathcal{C}} = 1$.

Previous work done on an extension of the 3-manifolds invariant $|\cdot|_{\mathcal{C}}$ to a TQFT was little conclusive. The original paper of Turaev and Viro produced a TQFT associated with the categories of representations of $U_q(sl_2(\mathbb{C}))$ at roots of unity. This was generalized to modular categories in [Tu1]. It was natural to expect a further generalization to spherical fusion categories in spirit of the Barrett-Westbury construction [BW1]. However, Barrett and Westbury did not construct a TQFT (though they claimed that it was feasible under additional assumptions on the category). Subsequent papers on this subject have left this question open. The reason may lie in technical difficulties encountered in this direction. In Section 4.9, we extend the invariant $|M|_{\mathcal{C}}$ to a TQFT. This extension is based on a skeleton presentation of 3-manifolds with boundary.

4.6. **Skeletons in the relative case.** Let M be a compact 3-manifold (with boundary). Let G be an oriented graph in ∂M such that all vertices of G have valence ≥ 2. (A graph is oriented, if all its edges are oriented.) A *skeleton* of the pair (M, G) is an oriented stratified 2-polyhedron $P \subset M$ such that

(i) $P \cap \partial M = \partial P = G$;
(ii) every vertex v of G is an endpoint of a unique edge d_v of P not contained in ∂M; moreover, $d_v \cap \partial M = \{v\}$ and d_v is not a loop;
(iii) every edge a of G is an edge of P of valence 1; the only region D_a of P adjacent to a is a closed 2-disk, $D_a \cap \partial M = a$, and the orientation of D_a is compatible with that of a (see Section 4.5 for compatibility of orientations);
(iv) $M \setminus P$ is a disjoint union of a finite collection of open 3-balls and a 3-manifold homeomorphic to $(\partial M \setminus G) \times [0, 1)$ through a homeomorphism extending the identity map $\partial(M \setminus P) = \partial M \setminus G = (\partial M \setminus G) \times \{0\}$.

Conditions (i)–(iii) imply that in a neighborhood of ∂M, a skeleton of (M, G) is a copy of $G \times [0, 1]$. The primary moves $T_1^{\pm 1}, \dots, T_4^{\pm 1}$ on skeletons of closed 3-manifolds extend to skeletons P of (M, G) in the obvious way. These moves keep $\partial P = G$ and preserve the skeletons in a neighborhood of their boundary G. In particular, the move T_1 adds an edge with both endpoints in $\mathrm{Int}(M)$, the move T_2 collapses an edge contained in $\mathrm{Int}(M)$, etc. Ambient isotopies of skeletons in M keeping the boundary pointwise are also viewed as primary moves.

Every pair (a compact orientable 3-manifold M, an oriented graph G in ∂M such that all vertices of G have valence ≥ 2) has a skeleton. Theorem 4.2 has the following relative version:

Theorem 4.4 ([12, Section 8]). *Any two skeletons of* (M, G) *can be related by primary moves in* M.

4.7. **Invariants of I-colored graphs.** Fix up to the end of Section 4.9 a spherical fusion category $\mathcal{C}$ over $\Bbbk$ such that $\dim(\mathcal{C})$ is invertible in $\Bbbk$. Fix a representative set I of simple objects of $\mathcal{C}$. We shall derive from $\mathcal{C}$ and I a 3-dimensional TQFT.

By an *I-colored graph* in a surface, we mean a $\mathcal{C}$-colored graph such that the colors of all edges belong to I and all vertices have valence ≥ 2. For any compact oriented 3-manifold M and any I-colored graph G in ∂M, we define a topological invariant $|M, G| \in \Bbbk$ as follows. Pick a skeleton $P \subset M$ of the pair (M, G). Pick a map $c\colon \mathrm{Reg}(P) \to I$ extending the coloring of G in the sense that for every edge a of G, the value of c on the region of P adjacent to a is the $\mathcal{C}$-color of a. For every oriented edge e of P, consider the $\Bbbk$-module $H_c(e) = H(P_e)$, where P_e is the set of branches of P at e turned into a cyclic $\mathcal{C}$-set as in Section 4.5. Let E_0 be the set of oriented edges of P with both endpoints in $\mathrm{Int}(M)$, and let E_∂ be the set of edges of P with exactly one endpoint in ∂M oriented towards this endpoint. Note that every vertex v of G is incident to a unique edge e_v belonging to E_∂ and $H_c(e_v) = H_v(G^{\mathrm{op}}; -\partial M)$, where the orientation of ∂M is induced by that of M. Therefore

$$\otimes_{e\in E_\partial} H_c(e)^\star = \otimes_v H_v(G^{\mathrm{op}}; -\partial M)^\star = H(G^{\mathrm{op}}; -\partial M)^\star.$$

For $e \in E_0$, the equality $P_{e^{\mathrm{op}}} = (P_e)^{\mathrm{op}}$ induces a duality between the modules $H_c(e)$, $H_c(e^{\mathrm{op}})$ and a contraction $H_c(e)^\star \otimes H_c(e^{\mathrm{op}})^\star \to \Bbbk$. This contraction does not depend on the orientation of e up to permutation of the factors. Applying these contractions, we obtain a homomorphism

$$*_P\colon \otimes_{e\in E_0\cup E_\partial} H_c(e)^\star \longrightarrow \otimes_{e\in E_\partial} H_c(e)^\star = H(G^{\mathrm{op}}; -\partial M)^\star.$$

As in Section 4.5, any vertex x of P lying in $\mathrm{Int}(M)$ determines an oriented graph Γ_x in S^2, and the mapping c turns Γ_x into a $\mathcal{C}$-colored graph. Section 4.3 yields a tensor $\mathbb{F}_{\mathcal{C}}(\Gamma_x) \in H_c(\Gamma_x)^*$. Here $H_c(\Gamma_x) = \otimes_e H_c(e)$, where e runs over all edges of P incident to x and oriented away from x. The tensor product $\otimes_x \mathbb{F}_{\mathcal{C}}(\Gamma_x)$ over all vertices x of P lying in $\mathrm{Int}(M)$ is a vector in $\otimes_{e\in E_0\cup E_\partial} H_c(e)^\star$. Set

$$|M,G| = (\dim(\mathcal{C}))^{-|P|} \sum_c \left(\prod_{r\in \mathrm{Reg}(P)} (\dim c(r))^{\chi(r)} \right) *_P(\otimes_x \mathbb{F}_{\mathcal{C}}(\Gamma_x)),$$

where $|P|$ is the number of components of $M \setminus P$, c runs over all maps $\mathrm{Reg}(P) \to I$ extending the coloring of G, and χ is the Euler characteristic.

Theorem 4.5 ([12, Section 9])**.** *$|M,G| \in H(G^{\mathrm{op}}; -\partial M)^\star$ is a topological invariant of the pair (M,G).*

Though there is a canonical isomorphism $H(G^{\mathrm{op}}; -\partial M)^\star \simeq H(G; \partial M)$ (see Section 4.2), we view $|M,G|$ as an element of $H(G^{\mathrm{op}}; -\partial M)^\star$.

We prove Theorem 4.5 by showing that the sum defining $|M,G|$ does not depend on the choice of P: it remains unchanged when applying a primary move to P (see Theorem 4.4).

Taking $G = \emptyset$, the scalar topological invariant $|M|_{\mathcal{C}} = |M,\emptyset| \in H(\emptyset)^\star = \Bbbk$ of M is equal to the invariant $|M|_{\mathcal{C}}$ of Theorem 4.3. In Section 4.9, we use $|M,G|$ to extend $|M|_{\mathcal{C}}$ to a TQFT.

4.8. **Preliminaries on TQFTs.** For convenience of the reader, we outline a definition of a 3-dimensional Topological Quantum Field Theory (TQFT) referring for details to [At]. We first define a category Cob_3 as follows. Objects of Cob_3 are closed oriented surfaces. A morphism $\Sigma_0 \to \Sigma_1$ in Cob_3 is represented by a pair (M,h), where M is a compact oriented 3-manifold and h is an orientation-preserving homeomorphism $(-\Sigma_0) \sqcup \Sigma_1 \simeq \partial M$. Two such pairs $(M, h\colon (-\Sigma_0) \sqcup \Sigma_1 \to \partial M)$ and $(M', h'\colon (-\Sigma_0) \sqcup \Sigma_1 \to \partial M')$ represent the same morphism $\Sigma_0 \to \Sigma_1$ if there is an orientation-preserving homeomorphism $F\colon M \to M'$ such that $h' = Fh$. The identity morphism of a surface Σ is represented by the cylinder $\Sigma \times [0,1]$ with the product orientation and the tautological identification of the boundary with $(-\Sigma)\sqcup\Sigma$. Composition of morphisms in Cob_3 is defined as follows: the composition of morphisms $(M_0,h_0)\colon \Sigma_0 \to \Sigma_1$ and $(M_1,h_1)\colon \Sigma_1 \to \Sigma_2$ is represented by the pair (M,h), where M is the result of gluing M_0 to M_1 along $h_1 h_0^{-1}\colon h_0(\Sigma_1) \to h_1(\Sigma_1)$ and $h = h_0|_{\Sigma_0} \sqcup h_1|_{\Sigma_2}\colon (-\Sigma_0) \sqcup \Sigma_2 \simeq \partial M$. The category Cob_3 is a symmetric monoidal category with tensor product given by disjoint union. The unit object of Cob_3 is the empty surface $\emptyset$ (which by convention has a unique orientation).

A *3-dimensional TQFT* is a symmetric monoidal functor $Z\colon \mathrm{Cob}_3 \to \mathrm{vect}_\Bbbk$, where $\mathrm{vect}_\Bbbk$ is the category of finitely generated projective $\Bbbk$-modules . In particular,

$Z(\emptyset) = \Bbbk$, $Z(\Sigma \sqcup \Sigma') = Z(\Sigma) \otimes Z(\Sigma')$ for any closed oriented surfaces Σ, Σ', and similarly for morphisms.

Each compact oriented 3-manifold M determines two morphisms $\emptyset \to \partial M$ and $-\partial M \to \emptyset$ in Cob_3. The associated homomorphisms $Z(\emptyset) = \Bbbk \to Z(\partial M)$ and $Z(-\partial M) \to Z(\emptyset) = \Bbbk$ are denoted $Z(M, \emptyset, \partial M)$ and $Z(M, -\partial M, \emptyset)$, respectively. If $\partial M = \emptyset$, then $Z(M, \emptyset, \partial M) = Z(M, -\partial M, \emptyset) \colon \Bbbk \to \Bbbk$ is multiplication by an element of $\Bbbk$ denoted $Z(M)$.

An *isomorphism* of 3-dimensional TQFTs $Z_1 \to Z_2$ is a natural monoidal isomorphism of functors. In particular, if two TQFTs Z_1, Z_2 are isomorphic, then $Z_1(M) = Z_2(M)$ for any closed oriented 3-manifold M.

4.9. **The state sum TQFT.** By a *3-cobordism* we mean a triple (M, Σ_0, Σ_1), where M is a compact oriented 3-manifold and Σ_0, Σ_1 are disjoint closed oriented surfaces contained in ∂M such that $\partial M = (-\Sigma_0) \sqcup \Sigma_1$ in the category of oriented manifolds. Note that the pair $(M, \mathrm{id}_{\partial M})$ represents a morphism in Cob_3.

Consider a 3-cobordism (M, Σ_0, Σ_1) and an I-colored graph $G_i \subset \Sigma_i$ for $i = 0, 1$. Theorem 4.5 yields a vector

$$|M, G_0^{\mathrm{op}} \cup G_1| \in H(G_0 \cup G_1^{\mathrm{op}}, -\partial M)^\star = H(G_0, \Sigma_0)^\star \otimes H(G_1^{\mathrm{op}}, -\Sigma_1)^\star.$$

The isomorphism $H(G_1^{\mathrm{op}}, -\Sigma_1)^\star \simeq H(G_1, \Sigma_1)$ given in Section 4.2 induces an isomorphism $\Upsilon \colon H(G_0, \Sigma_0)^\star \otimes H(G_1^{\mathrm{op}}, -\Sigma_1)^\star \to \mathrm{Hom}_\Bbbk\big(H(G_0, \Sigma_0), H(G_1, \Sigma_1)\big)$. Set

$$|M, \Sigma_0, G_0, \Sigma_1, G_1| = \frac{(\dim(\mathcal{C}))^{|G_1|}}{\dim(G_1)}\, \Upsilon\big(|M, G_0^{\mathrm{op}} \cup G_1|\big) \colon H(G_0; \Sigma_0) \to H(G_1; \Sigma_1),$$

where for an I-colored graph G in a surface Σ, the symbol $|G|$ denotes the number of components of $\Sigma \setminus G$ and $\dim(G)$ denotes the product of the dimensions of the objects of $\mathcal{C}$ associated with the edges of G.

By *skeleton* of a closed surface Σ we mean an oriented graph $G \subset \Sigma$ such that all vertices of G have valence ≥ 2 and all components of $\Sigma \setminus G$ are open disks. For example, the vertices and the edges of a triangulation t of Σ (with an arbitrary orientation of the edges) form a skeleton of Σ.

For a skeleton G of a closed oriented surface Σ, denote by $\mathrm{col}(G)$ the set of all maps from the set of edges of G to I and set $|G; \Sigma|^\circ = \oplus_{c \in \mathrm{col}(G)} H((G, c); \Sigma)$. Given a 3-cobordism (M, Σ_0, Σ_1), we define for any skeletons $G_0 \subset \Sigma_0$ and $G_1 \subset \Sigma_1$ a homomorphism $|M, \Sigma_0, G_0, \Sigma_1, G_1|^\circ \colon |G_0; \Sigma_0|^\circ \to |G_1; \Sigma_1|^\circ$ by

$$(16) \qquad |M, \Sigma_0, G_0, \Sigma_1, G_1|^\circ = \sum_{\substack{c_0 \in \mathrm{col}(G_0) \\ c_1 \in \mathrm{col}(G_1)}} |M, \Sigma_0, (G_0, c_0), \Sigma_1, (G_1, c_1)|.$$

If $(M_0, \Sigma_0, \Sigma_1)$, $(M_1, \Sigma_1, \Sigma_2)$ are two 3-cobordisms and (M, Σ_0, Σ_2) is the 3-cobordism obtained by gluing M_0 and M_1 along Σ_1, and if G_i is skeleton of Σ_i with $i = 0, 1, 2$, then

$$(17) \qquad |M, \Sigma_0, G_0, \Sigma_2, G_2|^\circ = |M, \Sigma_1, G_1, \Sigma_2, G_2|^\circ \circ |M, \Sigma_0, G_0, \Sigma_1, G_1|^\circ.$$

The constructions above assign a finitely generated free module to every closed oriented surface with distinguished skeleton and a homomorphism of these modules to every 3-cobordism whose bases are endowed with skeletons. This data satisfies the axioms of a TQFT except one: the homomorphism associated with the cylinder over a surface, generally speaking, is not the identity. There is a standard procedure which transforms such a "pseudo-TQFT" into a genuine TQFT and gets rid of the skeletons of surfaces at the same time. The idea is that if G_0, G_1 are two skeletons of a closed oriented surface Σ, then the cylinder cobordism $M = \Sigma \times [0,1]$ gives a homomorphism

$$p(G_0, G_1) = |M, \Sigma \times \{0\}, G_0 \times \{0\}, \Sigma \times \{1\}, G_1 \times \{1\}|^\circ \colon |G_0; \Sigma|^\circ \to |G_1; \Sigma|^\circ\,.$$

Formula (17) implies that $p(G_0, G_2) = p(G_1, G_2)\, p(G_0, G_1)$ for any skeletons G_0, G_1, G_2 of Σ. Taking $G_0 = G_1 = G_2$ we obtain that $p(G_0, G_0)$ is a projector onto a direct summand $|G_0; \Sigma|$ of $|G_0; \Sigma|^\circ$. Moreover, $p(G_0, G_1)$ maps $|G_0; \Sigma|$ isomorphically onto $|G_1; \Sigma|$. The finitely generated projective $\Bbbk$-modules $\{|G; \Sigma|\}_G$, where G runs over all skeletons of Σ, and the homomorphisms $\{p(G_0, G_1)\}_{G_0,G_1}$ form a projective system. The projective limit of this system, denoted $|\Sigma|_{\mathcal{C}}$, is a $\Bbbk$-module independent of the choice of a skeleton of Σ. For each skeleton G of Σ, we have a "cone isomorphism" of $\Bbbk$-modules $|G; \Sigma| \cong |\Sigma|_{\mathcal{C}}$. For example, we have: $|S^2|_{\mathcal{C}} \simeq \Bbbk$. By convention, the empty surface $\emptyset$ has a unique (empty) skeleton and $|\emptyset|_{\mathcal{C}} = \Bbbk$.

Any 3-cobordism (M, Σ_0, Σ_1) splits as a product of a 3-cobordism with a cylinder over Σ_1. Using this splitting and Formula (17), we obtain that the homomorphism (16) carries $|\Sigma_0|_{\mathcal{C}} \cong |G_0; \Sigma_0| \subset |G_0; \Sigma_0|^\circ$ into $|\Sigma_1|_{\mathcal{C}} \cong |G_1; \Sigma_1| \subset |G_1; \Sigma_1|^\circ$ for any skeletons G_0, G_1 of Σ_0, Σ_1, respectively. This gives a homomorphism

$$|M, \Sigma_0, \Sigma_1|_{\mathcal{C}} \colon |\Sigma_0|_{\mathcal{C}} \to |\Sigma_1|_{\mathcal{C}}$$

independent of the choice of G_0, G_1.

An orientation preserving homeomorphism of closed oriented surfaces $f \colon \Sigma \to \Sigma'$ induces an isomorphism $|f|_{\mathcal{C}} \colon |\Sigma|_{\mathcal{C}} \to |\Sigma'|_{\mathcal{C}}$ as follows. Pick a skeleton G of Σ. Then $G' = f(G)$ is a skeleton of Σ', and $|f|_{\mathcal{C}}$ is the composition of the isomorphisms

$$|\Sigma|_{\mathcal{C}} \cong |G; \Sigma| \cong |G'; \Sigma'| \cong |\Sigma'|_{\mathcal{C}}\,.$$

Here the first and the third isomorphisms are the cone isomorphisms and the middle isomorphism is induced by the homeomorphism of pairs $f \colon (\Sigma, G) \to (\Sigma', G')$. The homeomorphism $|f|_{\mathcal{C}}$ does not depend on the choice of G.

Finally let $\varphi \colon \Sigma_0 \to \Sigma_1$ be a morphism in Cob_3. Represent φ by a pair (M, h) where h is an orientation-preserving homeomorphism $(-\Sigma_0) \sqcup \Sigma_1 \simeq \partial M$. For $i = 0, 1$ denote by Σ'_i the surface $h(\Sigma_i) \subset \partial M$ with orientation induced by the one in Σ_i. The 3-cobordism $(M, \Sigma'_0, \Sigma'_1)$ yields a homomorphism $|M, \Sigma'_0, \Sigma'_1|_{\mathcal{C}} \colon |\Sigma'_0|_{\mathcal{C}} \to |\Sigma'_1|_{\mathcal{C}}$. The homeomorphism $h \colon \Sigma_i \to \Sigma'_i$ induces an isomorphism $|\Sigma_i|_{\mathcal{C}} \cong |\Sigma'_i|_{\mathcal{C}}$ for $i = 0, 1$. Composing these three homomorphisms we obtain the homomorphism $|\varphi|_{\mathcal{C}} \colon |\Sigma_0|_{\mathcal{C}} \to |\Sigma_1|_{\mathcal{C}}$. This homomorphism does not depend on the choice of the representative pair (M, h).

Theorem 4.6 ([12, Section 9]). *Let $\mathcal{C}$ be a spherical fusion category with invertible dimension. Then $|\cdot|_{\mathcal{C}}$ is a 3-dimensional TQFT.*

Considered up to isomorphism, the TQFT $|\cdot|_{\mathcal{C}}$ does not depend on the choice of the representative set I of simple objects of $\mathcal{C}$. For any closed oriented 3-manifold M, the invariant $|M|_{\mathcal{C}} \in \Bbbk$ produced by this TQFT coincides with the invariant of Section 4.5.

4.10. **Comparison of the RT and TV invariants.** The Reshetikhin-Turaev construction (see [RT, Tu1]) derives from any modular category $\mathcal{B}$ over $\Bbbk$ equipped with a distinguished square root of $\dim(\mathcal{B})$ a 3-dimensional "extended TQFT" $\tau_{\mathcal{B}}$. The latter is a functor from a certain extension of the category Cob_3 to $\mathrm{vect}_{\Bbbk}$; the extension in question is formed by surfaces with a Lagrangian subspace in the real 1-homology. For an anomaly free $\mathcal{B}$ (see Section 1.10), we take the element $\Delta = \Delta_{\pm} \in \Bbbk$ as the distinguished square root of $\dim(\mathcal{B})$. The corresponding extended TQFT $\tau_{\mathcal{B}}$ does not involve Lagrangian spaces and is a TQFT in the sense of Section 4.8.

We recall the definition of $\tau_{\mathcal{B}}(M) \in \Bbbk$ for a closed oriented 3-manifold M and anomaly free $\mathcal{B}$. Pick a representative set $\mathcal{J}$ of simple objects of $\mathcal{B}$. Present M by surgery on S^3 along a framed link $L = L_1 \cup \cdots \cup L_N$. Denote $\mathrm{col}(L)$ the set of maps

$\{1,\ldots,N\} \to \mathcal{J}$ and, for $\lambda \in \mathrm{col}(L)$, denote L_λ the framed link L whose component L_q is oriented in an arbitrary way and colored by $\lambda(q)$ for all $q = 1,...,N$. Then

$$\tau_{\mathcal{B}}(M) = \Delta^{-N-1} \sum_{\lambda\in\mathrm{col}(L)} \left(\prod_{q=1}^{N} \dim\bigl(\lambda(q)\bigr)\right) F_{\mathcal{B}}(L_\lambda) \tag{18}$$

where $F_{\mathcal{B}}$ is the invariant of $\mathcal{B}$-colored framed oriented links in S^3 discussed in Section 1.8. Recall that Hopf diagrams provide an alternative way for computing $\tau_{\mathcal{B}}(M)$ in terms of the coend of $\mathcal{B}$ (see Section 2).

In particular, we can apply these results to the anomaly free modular category $\mathcal{B} = \mathcal{Z}(\mathcal{C})$ provided by Theorem 1.1. In the following theorem, we prove a conjecture, formulated by Turaev in 1995, relating the Reshetikhin-Turaev and Turaev-Viro invariants via the categorical center.

Theorem 4.7 ([12, Theorem 11.1])**.** *Let $\mathcal{C}$ be a spherical fusion category over an algebraically closed field such that* $\dim\mathcal{C} \neq 0$. *Then* $|M|_{\mathcal{C}} = \tau_{\mathcal{Z}(\mathcal{C})}(M)$ *for any closed oriented 3-manifold M.*

This equality extends to an isomorphism of TQFTs as follows.

Theorem 4.8 ([12, Theorem 11.2])**.** *Under the conditions of Theorem 4.7, the TQFTs* $|\cdot|_{\mathcal{C}}$ *and* $\tau_{\mathcal{Z}(\mathcal{C})}$ *are isomorphic.*

We present the ideas of the proof of Theorems 4.7 and 4.8 in Section 4.11.

Let us give some corollaries of these theorems. From Theorem 4.7 and the results of Sections 2 and 3, we obtain an alternative and efficient method for computing the state sum $|M|_{\mathcal{C}}$ in terms of Hopf diagrams and the structural morphisms of the coend of $\mathcal{Z}(\mathcal{C})$, see Section 3.18 for more details.

Also Theorem 4.7 allows us to clarify relationships between invariants of 3-manifolds derived from involutory Hopf algebras. Let H be a finite-dimensional involutory Hopf algebra over an algebraically closed field $\Bbbk$ such that the characteristic of $\Bbbk$ does not divide $\dim(H)$. By a well-known theorem of Radford, H is semisimple, so that the category of finite-dimensional left H- modules ${}_H\mathrm{mod}$ is a spherical fusion category. The category of finite-dimensional left $D(H)$-modules ${}_{D(H)}\mathrm{mod}$, where $D(H)$ is the Drinfeld double of H, is a modular category (see [EG, Mü2]). Denote by Ku_H the Kuperberg invariant of 3-manifolds [Ku] derived from H and by $\mathrm{HKR}_{D(H)}$ the Hennings-Kauffman-Radford invariant of 3-manifolds [He, KR] derived from $D(H)$.

Corollary 4.9. *For any closed oriented 3-manifold M,*

$$\tau_{{}_{D(H)}\mathrm{mod}}(M) = |M|_{{}_H\mathrm{mod}} = (\dim(H))^{-1}\,\mathrm{Ku}_H(M) = (\dim(H))^{-1}\,\mathrm{HKR}_{D(H)}(M).$$

We say that two fusion categories are *equivalent* if their centers are braided equivalent. For example, two fusion categories weakly Morita equivalent in the sense of Müger [Mü1] are equivalent in our sense. Theorem 4.8 implies:

Corollary 4.10. *Equivalent spherical fusion categories of non-zero dimension over an algebraically closed field give rise to isomorphic TQFTs.*

A *unitary fusion category* is a fusion category $\mathcal{C}$ over $\mathbb{C}$ endowed with an Hermitian structure $\{f \in \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \bar{f} \in \mathrm{Hom}_{\mathcal{C}}(Y,X)\}_{X,Y\in\mathrm{Ob}(\mathcal{C})}$ such that $\mathrm{tr}(f\bar{f}) > 0$ for any non-zero morphism f in $\mathcal{C}$.

Corollary 4.11. *The TQFT* $|\cdot|_{\mathcal{C}}$ *associated with a unitary fusion category $\mathcal{C}$ is unitary in the sense of* [Tu1, Chapter III]. *In particular* $|-M|_{\mathcal{C}} = \overline{|M|_{\mathcal{C}}}$ *for any closed oriented 3-manifold M.*

From Corollary 4.11, Theorem 4.8, and [Tu1, Theorem 11.5], we deduce that if $\mathcal{C}$ is a unitary fusion category, then

$$||M|_{\mathcal{C}}| \leq (\dim(\mathcal{C}))^{g(M)-1}$$

for any closed oriented 3-manifold M, where $g(M)$ is the Heegaard genus of M.

4.11. **Sketch of proof of Theorems 4.7 and 4.8.** The proofs of Theorems 4.7 and 4.8 are based on the following key lemma:

Lemma 4.12. *Let $\mathcal{C}$ be a spherical fusion category over a commutative ring $\Bbbk$ such that $\dim(\mathcal{C})$ is invertible in $\Bbbk$. Then for any closed connected oriented surface Σ of genus $g \geq 0$, the $\Bbbk$-module $|\Sigma|_{\mathcal{C}}$ is isomorphic to $\mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(\mathbb{1}_{\mathcal{Z}(\mathcal{C})}, (C,\sigma)^{\otimes g})$, where (C,σ) is the coend of $\mathcal{Z}(\mathcal{C})$.*

The technical proof of Lemma 4.12 involves two main ingredients: firstly the fact that we allow non-generic skeletons (i.e., skeletons with edges incident to ≥ 4 regions, see Section 4.6), which provide us 'workable' skeletons of $\Sigma \times [0,1]$ for computing the projector whose image is $|\Sigma|_{\mathcal{C}}$ (see Section 4.9), and secondly the description of the coend (C,σ) of $Z(\mathcal{C})$ in terms of $\mathcal{C}$ provided by the theory of Hopf monads (see Section 3.17).

Assume now that $\Bbbk$ is an algebraically closed field, so that the category $\mathcal{Z}(\mathcal{C})$ is modular (see Theorem 1.1). Let us outline very roughly the proof of Theorem 4.7, referring to [12] for details. It proceeds in several steps:

Firstly, we extend the TQFT $|\cdot|_{\mathcal{C}}$ to a TQFT based on 3-cobordisms with $\mathcal{Z}(\mathcal{C})$- colored framed oriented links in their interior. This technical part proceeds in extending $\mathbb{F}_{\mathcal{C}}$ to so-called knotted $\mathcal{C}$- colored graphs in S^2, which allows define a state sum on skeleton with $\mathcal{Z}(\mathcal{C})$-colored link diagrams inside. The resulting link TQFT is also denoted $|\cdot|_{\mathcal{C}}$. This TQFT has the following property: if L is a $\mathcal{Z}(\mathcal{C})$-colored framed oriented link in S^3, then

$$|S^3, L|_{\mathcal{C}} = \dim(\mathcal{C})^{-1} F_{\mathcal{Z}(\mathcal{C})}(L),$$

where $F_{\mathcal{Z}}(\mathcal{C})$ is the invariant of $\mathcal{Z}(\mathcal{C})$-colored framed oriented links in S^3 discussed in Section 1.8.

Secondly, we establish a surgery formula for the value of $|\cdot|_{\mathcal{C}}$ on closed oriented 3-manifolds. Let L be a framed oriented link in S^3 with N components. For any $y_1, \ldots, y_N \in A = |S^1 \times S^1|_{\mathcal{C}}$, set

$$|L; y_1, \ldots, y_N|_{\mathcal{C}} = |E_L, -\partial E_L, \emptyset|_{\mathcal{C}} \circ |f|_{\mathcal{C}}(y_1 \otimes \cdots \otimes y_N) \in \Bbbk,$$

where E_L is the exterior of L (i.e., the complement in S^3 of an open regular neighborhood of L) and $f\colon \amalg_{q=1}^N (S^1 \times S^1)_q \to -\partial E_L$ is an orientation preserving homeomorphism induced by the framing of L. Set $w = |V, \emptyset, \partial V|_{\mathcal{C}}(1_{\Bbbk}) \in A$, where $V = -(S^1 \times D^2)$. Pick an arbitrary basis Y of the vector space A and expand $w = \sum_{y \in Y} w_y y$ where $w_y \in \Bbbk$. Denote by M the 3-manifold obtained by surgery on S^3 along L. Then, using the axioms of a TQFT, we get:

$$|M|_{\mathcal{C}} = \sum_{y_1, \ldots, y_N \in Y} \left(\prod_{q=1}^N w_{y_q} \right) |L; y_1, \ldots, y_N|_{\mathcal{C}}. \tag{19}$$

Then pick a representative set $\mathcal{J}$ of simple objects of $\mathcal{Z}(\mathcal{C})$. For $j \in \mathcal{J}$, set $y^j = |U^j, \emptyset, \partial U^j|_{\mathcal{C}} \in A$, where U^j is the solid torus $D^2 \times S^1$ endowed with the j-colored framed oriented knot $\{0\} \times S^1$ whose orientation is induced by that of S^1 and whose framing is constant. The modularity of $\mathcal{Z}(\mathcal{C})$ and Lemma 4.12 (for $\Sigma = S^1 \times S^1$)

allows to show that $Y = (y^j)_{j\in\mathcal{J}}$ is a basis of the vector space $A = |S^1 \times S^1|_{\mathcal{C}}$. Furthermore the vector w expands as

$$w = (\dim(\mathcal{C}))^{-1} \sum_{j\in\mathcal{J}} \dim(j)\, y^j. \tag{20}$$

Finally let M be a closed oriented 3-manifold M presented by surgery along a framed link $L = L_1 \cup \cdots \cup L_N \subset S^3$. Orient L arbitrarily. For any $j_1, ..., j_N \in \mathcal{J}$, denote by $L_{(j_1,...,j_N)}$ the framed oriented link L whose components $L_1, \ldots, L_N$ are colored with $j_1, \ldots, j_N$. We have:

$$|L; j_1, \ldots, j_N|_{\mathcal{C}} = |S^3, L_{(j_1,\ldots,j_N)}|_{\mathcal{C}} = \dim(\mathcal{C})^{-1} F_{\mathcal{Z}(\mathcal{C})}(L_{(j_1,\ldots,j_N)}).$$

Therefore Formulas (19) and (20) give that:

$$\begin{aligned}
|M|_{\mathcal{C}} &= \sum_{j_1,\ldots,j_N\in\mathcal{J}} \left(\prod_{q=1}^{N} \frac{\dim(j_q)}{\dim(\mathcal{C})}\right) (\dim(\mathcal{C}))^{-1} F_{\mathcal{Z}(\mathcal{C})}(L_{(j_1,\ldots,j_N)}) \\
&= (\dim(\mathcal{C}))^{-N-1} \sum_{j_1,\ldots,j_N\in\mathcal{J}} \left(\prod_{q=1}^{N} \dim(j_q)\right) F_{\mathcal{Z}(\mathcal{C})}(L_{(j_1,\ldots,j_N)}) \\
&= \tau_{\mathcal{Z}(\mathcal{C})}(M),
\end{aligned}$$

where the last equality is the definition of $\tau_{\mathcal{Z}(\mathcal{C})}(M)$, see Formula (18).

The proof of Theorem 4.8 goes by extending the TQFT $|\cdot|_{\mathcal{C}}$ to a TQFT based on 3-cobordisms with $\mathcal{Z}(\mathcal{C})$- colored ribbon graphs in their interior. The TQFT $\tau_{\mathcal{Z}(\mathcal{C})}$ also extends to a graph TQFT which is non-degenerate (see [Tu1, Chapter IV]). From Theorem 4.8 and Lemma 4.12, we show that there is an isomorphism of TQFTs between $|\cdot|_{\mathcal{C}}$ and $\tau_{\mathcal{Z}(\mathcal{C})}$ by using a general criterion: if at least one of two TQFTs is non-degenerate, the values of these TQFTs on closed 3-manifolds are equal, and the vector spaces associated by these TQFTs with any closed oriented surface have equal dimensions, then these TQFTs are isomorphic.

5. Other works and perspectives

1. In [14], we extend the notion of ambidextrous trace on ideal developed in [GKP] to the setting of a pivotal category. We show that under some conditions, these traces lead to invariants of colored spherical graphs and modified 6j-symbols. The categories involved are non semisimple (simple objects may have zero dimension and be infinitely many). In [GPT], modified 6j-symbols are used to produce a state sum invariant of 3-manifolds (in the spirit of Turaev-Viro).

As explained in Section 3, the state sum approach of quantum invariants of 3-manifolds is closely related to the surgical approach (through the categorical center). I plan to use the theory of Kirby elements and Hopf diagrams (developed in Section 2) to have a surgical point of view on the state sum invariants defined in [GPT].

2. Let G be a group. The notion of (ribbon) Hopf G-coalgebra is the prototype of the algebraic structure whose category of representation is a (ribbon) G-category. Recall that such categories are of special interest to construct invariants of 3-dimensional G-manifolds and 3-dimensional homotopy quantum field theories (HQFT) with target $K(G,1)$, see [Tu2] and [1, 2, 3, 4]. In [5] I give a method for constructing a quasitriangular Hopf G-coalgebra starting from a Hopf algebra endowed with an action of a group G by Hopf automorphisms. This leads to non-trivial examples of quasitriangular and ribbon Hopf group-coalgebras for any finite group and for infinite groups such as linear groups. In particular, we define the graded quantum groups.

I project to extend to this 'G-graded case' the work presented here as well from the algebraic point of view (study of graded Hopf monads) as from the topological point of view (generalization of our results on 3-dimensional TQFTs to 3-dimensions HQFT with target $K(G,1)$).

3. Recall from Section 2 that quantum 3-manifolds invariants defined via surgery presentation have a 'universal Hopf algebraic expression': they may be computed by evaluating, with a Kirby element, universal forms obtained from Hopf diagrams and the Hopf algebra structural morphisms of the coend (see Section 2.10). It would be very interesting to have a similar universal construction for the state sum invariants. This should go by constructing universal $6j$-symbols and, more generally, a universal invariant of colored graphs in the sphere. I expect to obtain such an invariant by using centralizers of Hopf monads (see Section 3.12).

Author's publications

PhD Thesis and papers derived from it

[1] **A. Virelizier**, *Algèbres de Hopf graduées et fibrés plats sur les 3-variétés*, PhD thesis (153 pages), Prépublication de l'Institut de Recherche Mathématique Avancée de Strasbourg **44** (2001).

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[13] **A. Bruguières and A. Virelizier**, *On the center of fusion categories.*

[14] **N. Geer, B. Patureau-Mirand, and A. Virelizier**, *Traces and ideals in pivotal categories.*

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