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Hopf group-coalgebras

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Abstract

We study algebraic properties of Hopf group-coalgebras, recently introduced by Turaev. We show the existence of integrals and traces for such coalgebras, and we generalize the main properties of quasitriangular and ribbon Hopf algebras to the setting of Hopf group-coalgebras.

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0. Introduction

Recently, Turaev [19] introduced, for a group π, the notion of a modular crossed π-category and showed that such a category gives rise to a three-dimensional homotopy quantum field theory with target space $K(\pi,1)$. Examples of π-categories can be constructed from the so-called Hopf π-coalgebras also introduced in [19].

The notion of a Hopf π-coalgebra generalizes that of a Hopf algebra. Hopf π-coalgebras are used by the author in [20] to construct Hennings-like (see [4,6]) and Kuperberg-like (see [7]) invariants of principal π-bundles over link complements and over 3-manifolds. The aim of the present paper is to lay the algebraic foundations for [20], specifically to establish the existence of integrals and traces for a Hopf π-coalgebras.

Let us briefly recall some definitions of [19]. Given a (discrete) group π, a Hopf π-coalgebra is a family $H=\{H_\alpha\}_{\alpha\in\pi}$ of algebras (over a field $\Bbbk$) endowed with a comultiplication $\Delta=\{\Delta_{\alpha,\beta}: H_{\alpha\beta}\to H_\alpha\otimes H_\beta\}_{\alpha,\beta\in\pi}$, a counit $\varepsilon: H_1\to\Bbbk$, and an antipode $S=\{S_\alpha: H_\alpha\to H_{\alpha^{-1}}\}_{\alpha\in\pi}$ which verify some compatibility conditions. A crossing for

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H is a family of algebra isomorphisms $\varphi = \{\varphi_\beta : H_\alpha \to H_{\beta\alpha\beta^{-1}}\}_{\alpha,\beta\in\pi}$ which preserves the comultiplication and the counit, and which yields an action of π in the sense that $\varphi_\beta\varphi_{\beta'} = \varphi_{\beta\beta'}$. A crossed Hopf π-coalgebra H is quasitriangular (resp. ribbon) when it is endowed with an R-matrix $R = \{R_{\alpha,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ (resp. an R-matrix and a twist $\theta = \{\theta_\alpha \in H_\alpha\}_{\alpha\in\pi}$) verifying some axioms which generalize the classical ones given in [2] (resp. [16]). The case $\pi = 1$ is the standard setting of Hopf algebras. When π is commutative and φ is trivial, one recovers the definition of a quasitriangular or ribbon π-colored Hopf algebra given by Ohtsuki [12].

Basic notions of the theory of Hopf algebras can be extended to the setting of Hopf π-coalgebras. In particular, a (right) π-integral for a Hopf π-coalgebra H is a family of $\Bbbk$-forms $\lambda = (\lambda_\alpha : H_\alpha \to \Bbbk)_{\alpha\in\pi}$ such that $(\lambda_\alpha \otimes \mathrm{id}_{H_\beta})\Delta_{\alpha,\beta} = \lambda_{\alpha\beta} 1_\beta$ for all $\alpha, \beta \in \pi$. When H is crossed, a π-trace for H is a family of $\Bbbk$-forms $\mathrm{tr} = (\mathrm{tr}_\alpha : H_\alpha \to \Bbbk)_{\alpha\in\pi}$ which verifies $\mathrm{tr}_\alpha(xy) = \mathrm{tr}_\alpha(yx)$, $\mathrm{tr}_{\alpha^{-1}}(S_\alpha(x)) = \mathrm{tr}_\alpha(x)$, and $\mathrm{tr}_{\beta\alpha\beta^{-1}}(\varphi_\beta(x)) = \mathrm{tr}_\alpha(x)$ for all $\alpha, \beta \in \pi$ and $x, y \in H_\alpha$. These notions were introduced in [20] for topological purposes.

In the first part of the paper (Sections 1–5), we mainly focus on Hopf π-coalgebras of finite type, that is Hopf π-coalgebras $H = \{H_\alpha\}_{\alpha\in\pi}$ with each H_α finite dimensional. The first main result is the existence and uniqueness (up to a scalar multiple) of a π-integral for such a Hopf π-coalgebra. To prove this result, we study rational π-graded modules, introduce the notion of a Hopf π-comodule, and generalize the fundamental theorem of Hopf modules (see [9]) to Hopf π-comodules.

As for Hopf algebras, any finite type Hopf π-coalgebra contains a distinguished π-grouplike element. Generalizing [15], we study the relationships between this element, the antipode, and the π-integrals. As a corollary, we give an upper bound for the order of $S_{\alpha^{-1}} S_\alpha$ whenever $\alpha \in \pi$ has a finite order.

The notions of semisimplicity and cosemisimplicity can be extended to the setting of Hopf π-coalgebras. We show that a finite type Hopf π-coalgebra $H = \{H_\alpha\}_{\alpha\in\pi}$ is semisimple (that is each H_α is semisimple) if and only if H_1 is semisimple. We define the cosemisimplicity for π-comodules and π-coalgebras, and we use π-integrals to give necessary and sufficient criteria for a Hopf π-coalgebra to be cosemisimple.

In the second part of the paper (Sections 6 and 7), we study quasitriangular Hopf π-coalgebras. The main result is the existence of π-traces for a semisimple (resp. cosemisimple) finite type unimodular ribbon Hopf π-coalgebra. To prove this result, we generalize the main properties of quasitriangular Hopf algebras (see [3,5,14]). In particular, we introduce and study the (generalized) Drinfeld elements of a quasitriangular Hopf π-coalgebra H, we compute the distinguished π-grouplike element of H by using the R-matrix, and we show that the twist of a ribbon Hopf π-coalgebra leads to a π-grouplike element which implements the square of the antipode by conjugation.

The paper is organized as follows. In Section 1, we review the basic definitions and properties of Hopf π-coalgebras. In Section 2, we discuss the notions of a rational π-graded module and of a Hopf π-comodule. In Section 3, we use these notions to establish the existence and uniqueness of π-integrals. Section 4 is devoted to the study of the distinguished π-grouplike element. In Section 5, we discuss the notion of a semisimple (resp. cosemisimple) Hopf π-coalgebra. In Section 6, we study crossed, quasitriangular, and ribbon Hopf π-coalgebras. Finally, we construct π-traces in Section 7.

1. Basic definitions

Throughout the paper, we let π be a discrete group (with neutral element 1) and $\Bbbk$ be a field (although much of what we do is valid over any commutative ring). We set $\Bbbk^* = \Bbbk \setminus \{0\}$. All algebras are supposed to be over $\Bbbk$, associative, and unitary. The tensor product $\otimes = \otimes_\Bbbk$ is always assumed to be over $\Bbbk$. If U and V are $\Bbbk$-spaces, $\sigma_{U,V}\colon U \otimes V \to V \otimes U$ will denote the flip map defined by $\sigma_{U,V}(u \otimes v) = v \otimes u$.

1.1. π-coalgebras

We recall the definition of a π-coalgebra, following [19, Section 11.2]. A *π-coalgebra* (over $\Bbbk$) is a family $C = \{C_\alpha\}_{\alpha\in\pi}$ of $\Bbbk$-spaces endowed with a family $\Delta = \{\Delta_{\alpha,\beta} : C_{\alpha\beta} \to C_\alpha \otimes C_\beta\}_{\alpha,\beta\in\pi}$ of $\Bbbk$-linear maps (the *comultiplication*) and a $\Bbbk$-linear map $\varepsilon : C_1 \to \Bbbk$ (the *counit*) such that

Δ is coassociative in the sense that, for any $\alpha, \beta, \gamma \in \pi$,

$$(\Delta_{\alpha,\beta} \otimes \mathrm{id}_{C_\gamma})\Delta_{\alpha\beta,\gamma} = (\mathrm{id}_{C_\alpha} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma}, \tag{1.1}$$

$$\text{for all } \alpha, \beta \in \pi, \quad (\mathrm{id}_{C_\alpha} \otimes \varepsilon)\Delta_{\alpha,1} = \mathrm{id}_{C_\alpha} = (\varepsilon \otimes \mathrm{id}_{C_\alpha})\Delta_{1,\alpha}. \tag{1.2}$$

Note that $(C_1, \Delta_{1,1}, \varepsilon)$ is a coalgebra in the usual sense of the word.

Sweedler's notation. We extend the Sweedler notation for a comultiplication in the following way: for any $\alpha, \beta \in \pi$ and $c \in C_{\alpha\beta}$, we write

$$\Delta_{\alpha,\beta}(c) = \sum_{(c)} c_{(1,\alpha)} \otimes c_{(2,\beta)} \in C_\alpha \otimes C_\beta,$$

or shortly, if we leave the summation implicit, $\Delta_{\alpha,\beta}(c) = c_{(1,\alpha)} \otimes c_{(2,\beta)}$. The coassociativity axiom (1.1) gives that, for any $\alpha, \beta, \gamma \in \pi$ and $c \in C_{\alpha\beta\gamma}$,

$$c_{(1,\alpha\beta)(1,\alpha)} \otimes c_{(1,\alpha\beta)(2,\beta)} \otimes c_{(2,\gamma)} = c_{(1,\alpha)} \otimes c_{(2,\beta\gamma)(1,\beta)} \otimes c_{(2,\beta\gamma)(2,\gamma)}.$$

This element of $C_\alpha \otimes C_\beta \otimes C_\gamma$ is written as $c_{(1,\alpha)} \otimes c_{(2,\beta)} \otimes c_{(3,\gamma)}$. By iterating the procedure, we define inductively $c_{(1,\alpha_1)} \otimes \cdots \otimes c_{(n,\alpha_n)}$ for any $c \in C_{\alpha_1\cdots\alpha_n}$.

1.2. Convolution algebras

Let $C = (\{C_\alpha\}, \Delta, \varepsilon)$ be a π-coalgebra and A be an algebra with multiplication m and unit element 1_A. For any $f \in \mathrm{Hom}_\Bbbk(C_\alpha, A)$ and $g \in \mathrm{Hom}_\Bbbk(C_\beta, A)$, we define their *convolution product* by

$$f * g = m(f \otimes g)\Delta_{\alpha,\beta} \in \mathrm{Hom}_\Bbbk(C_{\alpha\beta}, A).$$

Using (1.1) and (1.2), one verifies that the $\Bbbk$-space

$$\mathrm{Conv}(C,A)=\bigoplus_{\alpha\in\pi}\mathrm{Hom}_{\Bbbk}(C_\alpha,A)$$

endowed with the convolution product $*$ and the unit element $\varepsilon 1_A$, is a π-graded algebra, called *convolution algebra*.

In particular, for $A=\Bbbk$, the π-graded algebra $\mathrm{Conv}(C,\Bbbk)=\bigoplus_{\alpha\in\pi}C_\alpha^*$ is called *dual* to C and is denoted by C^*.

1.3. Hopf π-coalgebras

Following [19, Section 11.2], a *Hopf π-coalgebra* is a π-coalgebra $H=(\{H_\alpha\},\Delta,\varepsilon)$ endowed with a family $S=\{S_\alpha : H_\alpha\to H_{\alpha^{-1}}\}_{\alpha\in\pi}$ of $\Bbbk$-linear maps (the *antipode*) such that

$$\text{each } H_\alpha \text{ is an algebra with multiplication } m_\alpha \text{ and unit element } 1_\alpha\in H_\alpha, \tag{1.3}$$

$$\varepsilon : H_1\to\Bbbk \text{ and } \Delta_{\alpha,\beta} : H_{\alpha\beta}\to H_\alpha\otimes H_\beta \text{ (for all } \alpha,\beta\in\pi) \text{ are algebra homomorphisms,} \tag{1.4}$$

$$\text{for any } \alpha\in\pi,\quad m_\alpha(S_{\alpha^{-1}}\otimes \mathrm{id}_{H_\alpha})\Delta_{\alpha^{-1},\alpha}=\varepsilon 1_\alpha=m_\alpha(\mathrm{id}_{H_\alpha}\otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}}. \tag{1.5}$$

We remark that the notion of a Hopf π-coalgebra is not self-dual and that $(H_1,m_1,1_1,\Delta_{1,1},\varepsilon,S_1)$ is a (classical) Hopf algebra.

The Hopf π-coalgebra H is said to be of *finite type* if, for all $\alpha\in\pi$, H_α is finite dimensional (over $\Bbbk$). Note that it does not mean that $\bigoplus_{\alpha\in\pi}H_\alpha$ is finite-dimensional (unless $H_\alpha=0$ for all but a finite number of $\alpha\in\pi$).

The antipode $S=\{S_\alpha\}_{\alpha\in\pi}$ of H is said to be *bijective* if each S_α is bijective. Unlike [19, Section 11.2], we do not suppose that the antipode of a Hopf π-coalgebra H is bijective. However, we will show that it is bijective whenever H is of finite type (see Corollary 3.7(a)) or quasitriangular (see Lemma 6.5(c)).

A useful remark is that if $H=\{H_\alpha\}_{\alpha\in\pi}$ is a Hopf π-coalgebra with antipode $S=\{S_\alpha\}_{\alpha\in\pi}$, then axiom (1.5) says that S_α is the inverse of $\mathrm{id}_{H_{\alpha^{-1}}}$ in the convolution algebra $\mathrm{Conv}(H,H_{\alpha^{-1}})$ for all $\alpha\in\pi$.

In the next lemma, generalizing [17, Proposition 4.0.1], we show that the antipode of a Hopf π-coalgebra is anti-multiplicative and anti-comultiplicative.

Lemma 1.1. *Let $H=(\{H_\alpha,m_\alpha,1_\alpha\},\Delta,\varepsilon,S)$ be a Hopf π-coalgebra. Then*
(a) $S_\alpha(ab)=S_\alpha(b)S_\alpha(a)$ *for any* $\alpha\in\pi$ *and* $a,b\in H_\alpha$;
(b) $S_\alpha(1_\alpha)=1_{\alpha^{-1}}$ *for any* $\alpha\in\pi$;
(c) $\Delta_{\beta^{-1},\alpha^{-1}}S_{\alpha\beta}=\sigma_{H_{\alpha^{-1}},H_{\beta^{-1}}}(S_\alpha\otimes S_\beta)\Delta_{\alpha,\beta}$ *for any* $\alpha,\beta\in\pi$;
(d) $\varepsilon S_1=\varepsilon$.

Proof. The proof is essentially the same as in the Hopf algebra setting. For example, to show part (c), fix $\alpha,\beta\in\pi$ and consider the algebra $\mathrm{Conv}(H,H_{\beta^{-1}}\otimes H_{\alpha^{-1}})$ with convolution product $*$ and unit element $e=\varepsilon 1_{\beta^{-1}}\otimes 1_{\alpha^{-1}}$. Using axioms (1.2), (1.4), and (1.5), one easily checks that $\Delta_{\beta^{-1},\alpha^{-1}}S_{\alpha\beta}*\Delta_{\beta^{-1},\alpha^{-1}}=e$ and $\Delta_{\beta^{-1},\alpha^{-1}}*\sigma_{H_{\alpha^{-1}},H_{\beta^{-1}}}(S_\alpha\otimes S_\beta)\Delta_{\alpha,\beta}=e$. Hence we can conclude that $\Delta_{\beta^{-1},\alpha^{-1}}S_{\alpha\beta}=\sigma_{H_{\alpha^{-1}},H_{\beta^{-1}}}(S_\alpha\otimes S_\beta)\Delta_{\alpha,\beta}$. □

Corollary 1.2. *Let $H=\{H_\alpha\}_{\alpha\in\pi}$ be a Hopf π-coalgebra. Then $\{\alpha\in\pi\,|\,H_\alpha\neq 0\}$ is a subgroup of π.*

Proof. Set $G=\{\alpha\in\pi\,|\,H_\alpha\neq 0\}$. Firstly $1_1\neq 0$ (since $\varepsilon(1_1)=1\neq 0$) and so $1\in G$. Then let $\alpha,\beta\in G$. Using (1.4), $\Delta_{\alpha,\beta}(1_{\alpha\beta})=1_\alpha\otimes 1_\beta\neq 0$. Therefore $1_{\alpha\beta}\neq 0$ and so $\alpha\beta\in G$. Finally, let $\alpha\in G$. By Lemma 1.1(b), $S_{\alpha^{-1}}(1_{\alpha^{-1}})=1_\alpha\neq 0$. Thus $1_{\alpha^{-1}}\neq 0$ and hence $\alpha^{-1}\in G$. □

1.3.1. Opposite Hopf π-coalgebra

Let $H=\{H_\alpha\}_{\alpha\in\pi}$ be a Hopf π-coalgebra. Suppose that the antipode $S=\{S_\alpha\}_{\alpha\in\pi}$ of H is bijective. For any $\alpha\in\pi$, let H_α^{op} be the opposite algebra to H_α. Then $H^{\mathrm{op}}=\{H_\alpha^{\mathrm{op}}\}_{\alpha\in\pi}$, endowed with the comultiplication and counit of H and with the antipode $S^{\mathrm{op}}=\{S_\alpha^{\mathrm{op}}=S_{\alpha^{-1}}^{-1}\}_{\alpha\in\pi}$, is a Hopf π-coalgebra called *opposite to H*.

1.3.2. Coopposite Hopf π-coalgebra

Let $C=(\{C_\alpha\},\Delta,\varepsilon)$ be a π-coalgebra. Set

$$C_\alpha^{\mathrm{cop}}=C_{\alpha^{-1}}\quad\text{and}\quad \Delta_{\alpha,\beta}^{\mathrm{cop}}=\sigma_{C_{\beta^{-1}},C_{\alpha^{-1}}}\Delta_{\beta^{-1},\alpha^{-1}}.$$

Then $C^{\mathrm{cop}}=(\{C_\alpha^{\mathrm{cop}}\},\ \Delta^{\mathrm{cop}},\varepsilon)$ is a π-coalgebra, called *coopposite* to C. If H is a Hopf π-coalgebra whose antipode $S=\{S_\alpha\}_{\alpha\in\pi}$ is bijective, then the coopposite π-coalgebra H^{cop}, where $H_\alpha^{\mathrm{cop}}=H_{\alpha^{-1}}$ as an algebra, is a Hopf π-coalgebra with antipode $S^{\mathrm{cop}}=\{S_\alpha^{\mathrm{cop}}=S_\alpha^{-1}\}_{\alpha\in\pi}$.

1.3.3. Opposite and coopposite Hopf π-coalgebra

Let $H=(\{H_\alpha\},\Delta,\varepsilon,S)$ be a Hopf π-coalgebra. Even if the antipode of H is not bijective, one can always define a Hopf π-coalgebra *opposite and coopposite to H* by setting $H_\alpha^{\mathrm{op,cop}}=H_{\alpha^{-1}}^{\mathrm{op}}$, $\Delta_{\alpha,\beta}^{\mathrm{op,cop}}=\Delta_{\alpha,\beta}^{\mathrm{cop}}$, $\varepsilon^{\mathrm{op,cop}}=\varepsilon$, and $S_\alpha^{\mathrm{op,cop}}=S_{\alpha^{-1}}$.

1.3.4. The dual Hopf algebra

Let $H=(\{H_\alpha,m_\alpha,1_\alpha\},\Delta,\varepsilon,S)$ be a finite type Hopf π-coalgebra. The π-graded algebra $H^*=\bigoplus_{\alpha\in\pi}H_\alpha^*$ dual to H (see Section 1.2) inherits a structure of a Hopf algebra by setting, for all $\alpha\in\pi$ and $f\in H_\alpha^*$,

$$\Delta(f)=m_\alpha^*(f)\in(H_\alpha\otimes H_\alpha)^*\cong H_\alpha^*\otimes H_\alpha^*,$$

$\varepsilon(f)=f(1_\alpha)$, and $S(f)=f\circ S_{\alpha^{-1}}$. Note that if $H_\alpha\neq 0$ for infinitely many $\alpha\in\pi$, then H^* is infinite dimensional.

1.3.5. The case π finite

Let us first remark that, when π is a finite group, there is a one-to-one correspondence between (isomorphic classes of) π-coalgebras and (isomorphic classes of) π-graded coalgebras. Recall that a coalgebra (C,Δ,ε) is π-*graded* if C admits a decomposition as a direct sum of $\Bbbk$-spaces $C=\bigoplus_{\alpha\in\pi} C_\alpha$ such that, for any $\alpha\in\pi$,

$$\Delta(C_\alpha)\subset\sum_{\beta\gamma=\alpha} C_\beta\otimes C_\gamma \quad\text{and}\quad \varepsilon(C_\alpha)=0 \quad \text{if } \alpha\neq 1.$$

Let us denote by $p_\alpha : C\to C_\alpha$ the canonical projection. Then $\{C_\alpha\}_{\alpha\in\pi}$ is a π-coalgebra with comultiplication $\{(p_\alpha\otimes p_\beta)\Delta|_{C_{\alpha\beta}}\}_{\alpha,\beta\in\pi}$ and counit $\varepsilon|_{C_1}$. Conversely, if $C=(\{C_\alpha\},\Delta,\varepsilon)$ is a π-coalgebra, then $\tilde{C}=\bigoplus_{\alpha\in\pi} C_\alpha$ is a π-graded coalgebra with comultiplication $\tilde{\Delta}$ and counit $\tilde{\varepsilon}$ given on the summands by

$$\tilde{\Delta}|_{C_\alpha}=\sum_{\beta\gamma=\alpha}\Delta_{\beta,\gamma} \quad\text{and}\quad \tilde{\varepsilon}|_{C_\alpha}=\begin{cases}\varepsilon & \text{if } \alpha=1,\\ 0 & \text{if } \alpha\neq 1.\end{cases}$$

Let now $H=(\{H_\alpha,m_\alpha,1_\alpha\},\Delta,\varepsilon,S)$ be a Hopf π-coalgebra, where π is a finite group. Then the coalgebra $(\tilde{H},\tilde{\Delta},\tilde{\varepsilon})$, defined as above, is a Hopf algebra with multiplication $\tilde{m}$, unit element $\tilde{1}$, and antipode $\tilde{S}$ given by

$$\tilde{m}|_{H_\alpha\otimes H_\beta}=\begin{cases} m_\alpha & \text{if } \alpha=\beta,\\ 0 & \text{if } \alpha\neq\beta,\end{cases} \qquad \tilde{1}=\sum_{\alpha\in\pi}1_\alpha, \quad\text{and}\quad \tilde{S}=\sum_{\alpha\in\pi}S_\alpha.$$

When H is of finite type and π is finite, the Hopf algebra H^* (see Section 1.3.4) is simply the dual Hopf algebra $\tilde{H}^*$.

Remark 1.3. When π is finite, the structure of π-comodules over a π-coalgebra C (Theorem 2.2), the existence of π-integrals for a finite type Hopf π-coalgebra H (Theorem 3.6) and their relations with the distinguished π-group-like element (Theorem 4.2) can be deduced from the classical theory of coalgebras and Hopf algebras by using $\tilde{C}$ or $\tilde{H}$ (defined as in Section 1.3.5). Nevertheless, for the general case, self-contained proofs must be given.

In general, the results relating to a quasitriangular Hopf π-coalgebra (see Sections 6 and 7) cannot be deduced from the classical theory of quasitriangular Hopf algebras. Indeed, even if π is finite, an R-matrix for a Hopf π-coalgebra H (whose definition involves an action of π, see Section 6.2) does not necessarily lead to a usual R-matrix for the Hopf algebra $\tilde{H}$.

2. Modules and comodules

In this section, we introduce and discuss the notions of π-comodules, rational π-graded modules, and Hopf π-comodules. They are used in Section 3 to show the existence of integrals.

2.1. π-comodules

Let $C=(\{C_\alpha\},\Delta,\varepsilon)$ be a π-coalgebra. A *right π-comodule over C* is a family $M=\{M_\alpha\}_{\alpha\in\pi}$ of $\Bbbk$-spaces endowed with a family $\rho=\{\rho_{\alpha,\beta}:M_{\alpha\beta}\to M_\alpha\otimes C_\beta\}_{\alpha,\beta\in\pi}$ of $\Bbbk$-linear maps (the *structure maps*) such that

for any $\alpha,\beta,\gamma\in\pi$,

$$(\rho_{\alpha,\beta}\otimes \mathrm{id}_{C_\gamma})\rho_{\alpha\beta,\gamma}=(\mathrm{id}_{M_\alpha}\otimes\Delta_{\beta,\gamma})\rho_{\alpha,\beta\gamma}, \tag{2.1}$$

$$\text{for any } \alpha\in\pi,\ (\mathrm{id}_{M_\alpha}\otimes\varepsilon)\rho_{\alpha,1}=\mathrm{id}_{M_\alpha}. \tag{2.2}$$

Note that M_1 endowed with the structure map $\rho_{1,1}$ is a (usual) right comodule over the coalgebra C_1.

If π is finite and $\tilde{C}=\bigotimes_{\alpha\in\pi}C_\alpha$ is the π-graded coalgebra defined as in Section 1.3.5, then M leads to a π-graded right comodule $\tilde{M}=\bigotimes_{\alpha\in\pi}M_\alpha$ over $\tilde{C}$ with comodule map $\tilde{\rho}=\sum_{\alpha,\beta\in\pi}\rho_{\alpha,\beta}$ (see [10]).

A *π-subcomodule of M* is a family $N=\{N_\alpha\}_{\alpha\in\pi}$, where N_α is a $\Bbbk$-subspace of M_α, such that $\rho_{\alpha,\beta}(N_{\alpha\beta})\subset N_\alpha\otimes C_\beta$ for all $\alpha,\beta\in\pi$. Then N is a right π-comodule over C with induced structure maps.

A *π-comodule morphism* between two right π-comodules M and M' over C (with structure maps ρ and ρ') is a family $f=\{f_\alpha:M_\alpha\to M'_\alpha\}_{\alpha\in\pi}$ of $\Bbbk$-linear maps such that $\rho'_{\alpha,\beta}f_{\alpha\beta}=(f_\alpha\otimes\mathrm{id}_{C_\beta})\rho_{\alpha,\beta}$ for all $\alpha,\beta\in\pi$.

Sweedler's notation. We extend the notation of Section 1.1 by setting, for any $\alpha,\beta\in\pi$ and $m\in M_{\alpha\beta}$,

$$\rho_{\alpha,\beta}(m)=m_{(0,\alpha)}\otimes m_{(1,\beta)}\in M_\alpha\otimes C_\beta.$$

Axiom (2.1) gives that, for any $\alpha,\beta,\gamma\in\pi$ and $m\in M_{\alpha\beta\gamma}$,

$$m_{(0,\alpha\beta)(0,\alpha)}\otimes m_{(0,\alpha\beta)(1,\beta)}\otimes m_{(1,\gamma)}=m_{(0,\alpha)}\otimes m_{(1,\beta\gamma)(1,\beta)}\otimes m_{(1,\beta\gamma)(2,\gamma)}.$$

This element of $M_\alpha\otimes C_\beta\otimes C_\gamma$ is written as $m_{(0,\alpha)}\otimes m_{(1,\beta)}\otimes m_{(2,\gamma)}$. By iterating the procedure, we define inductively $m_{(0,\alpha_0)}\otimes m_{(1,\alpha_1)}\otimes\cdots\otimes m_{(n,\alpha_n)}$ for any $m\in M_{\alpha_0\alpha_1\cdots\alpha_n}$.

Let $N=\{N_\alpha\}_{\alpha\in\pi}$ be a π-subcomodule of a right π-comodule $M=\{M_\alpha\}_{\alpha\in\pi}$ over a π-coalgebra C. One easily checks that $M/N=\{M_\alpha/N_\alpha\}_{\alpha\in\pi}$ is a right π-comodule over C, with structure maps naturally induced from the structure maps of M. Moreover, this is the unique structure of a right π-comodule over C on M/N which makes the canonical projection $p=\{p_\alpha:M_\alpha\to M_\alpha/N_\alpha\}_{\alpha\in\pi}$ a π-comodule morphism.

If $f=\{f_\alpha:M_\alpha\to M'_\alpha\}_{\alpha\in\pi}$ is a π-comodule morphism between two right π-comodules M and M', then $\ker(f)=\{\ker(f_\alpha)\}_{\alpha\in\pi}$ is a π-subcomodule of M, $f(M)=\{f_\alpha(M_\alpha)\}_{\alpha\in\pi}$ is a π-subcomodule of M', and the canonical isomorphism $\bar{f}=\{\bar{f}_\alpha:M_\alpha/\ker(f_\alpha)\to f_\alpha(M_\alpha)\}_{\alpha\in\pi}$ is a π-comodule isomorphism.

Example 2.1. Let H be a Hopf π-coalgebra and $M=\{M_\alpha\}_{\alpha\in\pi}$ be a right π-comodule over H with structure maps $\rho=\{\rho_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$. The *coinvariants of H on M* are the elements of the $\Bbbk$-space

$$\left\{ m=(m_\alpha)_{\alpha\in\pi}\in\prod_{\alpha\in\pi} M_\alpha \mid \rho_{\alpha,\beta}(m_{\alpha\beta})=m_\alpha\otimes 1_\beta \text{ for all } \alpha,\beta\in\pi \right\}.$$

For any $\alpha\in\pi$, let $M_\alpha^{\mathrm{co}\,H}$ be the image of the (canonical) projection of this set onto M_α. It is easy to verify that $M^{\mathrm{co}\,H}=\{M_\alpha^{\mathrm{co}\,H}\}_{\alpha\in\pi}$ is a right π-subcomodule of M, called the *π-subcomodule of coinvariants.*

2.2. Rational π-graded modules

Throughout this subsection, $C=(\{C_\alpha\},\Delta,\varepsilon)$ will denote a π-coalgebra and $C^*=\bigotimes_{\alpha\in\pi} C_\alpha^*$ its dual π-graded algebra (see Section 1.2). In this subsection we explore the relationships between right π-comodules over C and π-graded left C^*-modules.

Let $M=\bigoplus_{\alpha\in\pi} M_\alpha$ be a π-graded left C^*-module with action $\psi: C^*\otimes M\to M$. Set $\bar{M}_\alpha=M_{\alpha^{-1}}$. For any $\alpha,\beta\in\pi$, define

$$\rho_{\alpha,\beta}:\bar{M}_{\alpha\beta}\to \mathrm{Hom}_{\Bbbk}(C_\beta^*,\bar{M}_\alpha) \quad \text{by } \rho_{\alpha,\beta}(m)(f)=\psi(f\otimes m). \tag{2.3}$$

There is a natural embedding

$$\bar{M}_\alpha\otimes C_\beta \hookrightarrow \mathrm{Hom}_{\Bbbk}(C_\beta^*,\bar{M}_\alpha) \quad m\otimes c\mapsto (f\mapsto f(c)m).$$

Regard this embedding as inclusion, so that $\bar{M}_\alpha\otimes C_\beta\subset \mathrm{Hom}_{\Bbbk}(C_\beta^*,\bar{M}_\alpha)$. The π-graded left C^*-module M is said to be *rational* provided $\rho_{\alpha,\beta}(\bar{M}_{\alpha\beta})\subset \bar{M}_\alpha\otimes C_\beta$ for all $\alpha,\beta\in\pi$. In this case, the restriction of $\rho_{\alpha,\beta}$ onto $\bar{M}_\alpha\otimes C_\beta$ will also be denoted by

$$\rho_{\alpha,\beta}:\bar{M}_{\alpha\beta}\to \bar{M}_\alpha\otimes C_\beta. \tag{2.4}$$

The definition given here generalizes that of a rational π-graded left module given in [10] and agrees with it when π is finite.

The next theorem generalizes [10, Theorem 6.3; 17, Theorem 2.1.3].

Theorem 2.2. *Let C be a π-coalgebra. Then*

(a) *There is a one-to-one correspondence between (isomorphic classes of) right π-comodules over C and (isomorphic classes of) rational π-graded left C^*-modules.*

(b) *Every graded submodule of a rational π-graded left C^*-module is rational.*

(c) *Any π-graded left C^*-module $L=\bigoplus_{\alpha\in\pi} L_\alpha$ has a unique maximal rational graded submodule, noted L^{rat}, which is equal to the sum of all rational graded submodules of L. Moreover, if $\rho=\{\rho_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$ is defined as in* (2.3), *then $(L^{\mathrm{rat}})_\gamma=\bigcap_{\substack{\alpha,\beta\in\pi\\ \alpha\beta=\gamma^{-1}}} \rho_{\alpha,\beta}^{-1}(\bar{L}_\alpha\otimes C_\beta)$ for any $\gamma\in\pi$.*

Before proving the theorem, we needs two lemmas. Recall that a left module M over a π-graded algebra $A=\bigotimes_{\alpha\in\pi} A_\alpha$ is *graded* if M admits a decomposition as a direct sum of $\Bbbk$-spaces $M=\bigotimes_{\alpha\in\pi} M_\alpha$ such that $A_\alpha M_\beta \subset M_{\alpha\beta}$ for all $\alpha,\beta\in\pi$. A submodule N of M is *graded* if $N=\bigotimes_{\alpha\in\pi}(N\cap M_\alpha)$. The quotient M/N is then a left π-graded A-module by setting $(M/N)_\alpha=(M_\alpha+N)/N$ for all $\alpha\in\pi$. This is the unique structure of a π-graded A-module on M/N which makes the canonical projection $M\to M/N$ a graded A-morphism.

Let $M=\{M_\alpha\}_{\alpha\in\pi}$ be a family of $\Bbbk$-spaces and $\rho=\{\rho_{\alpha,\beta}:M_{\alpha\beta}\to M_\alpha\otimes C_\beta\}_{\alpha,\beta\in\pi}$ be a family of $\Bbbk$-linear maps. Set $\bar M=\bigoplus_{\alpha\in\pi}\bar M_\alpha$, where $\bar M_\alpha=M_{\alpha^{-1}}$. Let $\psi_\rho: C^*\otimes\bar M\to\bar M$ be the $\Bbbk$-linear map defined on the summands by

$$C_\alpha^*\otimes\bar M_\beta \xrightarrow{\mathrm{id}_{C_\alpha^*}\otimes\rho_{(\alpha\beta)^{-1},\alpha}} C_\alpha^*\otimes\bar M_{\alpha\beta}\otimes C_\alpha \xrightarrow{\sigma_{C_\alpha^*,\bar M_{\alpha\beta}}\otimes\mathrm{id}_{C_\alpha}}$$
$$\longrightarrow \bar M_{\alpha\beta}\otimes C_\alpha^*\otimes C_\alpha \xrightarrow{\mathrm{id}_{\bar M_{\alpha\beta}}\otimes\langle,\rangle} \bar M_{\alpha\beta}\otimes\Bbbk\cong\bar M_{\alpha\beta},$$

where $\langle,\rangle$ denotes the natural pairing between C_α^* and C_α.

Lemma 2.3. *(M,ρ) is a right π-comodule over C if and only if $(\bar M,\psi_\rho)$ is a π-graded left C^*-module.*

Proof. Suppose that (M,ρ) is a right π-comodule over C. Firstly, for any $m\in\bar M_\alpha$, $\psi_\rho(\varepsilon\otimes m)=m_{(0,\alpha^{-1})}\varepsilon(m_{(1,1)})=m$, by (2.2). Secondly, for any $f\in C_\alpha^*$, $g\in C_\beta^*$, and $m\in\bar M_\gamma$,

$$\begin{aligned}\psi_\rho(fg\otimes m)&=m_{(0,(\alpha\beta\gamma)^{-1})}fg(m_{(1,\alpha\beta)})\\&=m_{(0,(\alpha\beta\gamma)^{-1})}f(m_{(1,\alpha)})g(m_{(2,\beta)})\\&=\psi_\rho(f\otimes m_{(0,(\beta\gamma)^{-1})}g(m_{(1,\beta)}))\\&=\psi_\rho(f\otimes\psi_\rho(g\otimes m)).\end{aligned}$$

Moreover, by construction, $\psi_\rho(C_\alpha^*\otimes\bar M_\beta)\subset\bar M_{\alpha\beta}$ for any $\alpha,\beta\in\pi$. Hence $(\bar M,\psi_\rho)$ is a π-graded left C^*-module.

Conversely, suppose that $(\bar M,\psi_\rho)$ is a left π-graded C^*-module. Axiom (2.2) is satisfied since $(\mathrm{id}_{M_\alpha}\otimes\varepsilon)\rho_{\alpha,1}(m)=\psi_\rho(\varepsilon\otimes m)=m$ for all $\alpha\in\pi$ and $m\in M_\alpha=\bar M_{\alpha^{-1}}$. To show that axiom (2.1) is satisfied, let $\alpha,\beta,\gamma\in\pi$ and $m\in M_{\alpha\beta\gamma}$. Set

$$\delta=(\rho_{\alpha,\beta}\otimes\mathrm{id}_{C_\gamma})\rho_{\alpha\beta,\gamma}(m)-(\mathrm{id}_{M_\alpha}\otimes\Delta_{\beta,\gamma})\rho_{\alpha,\beta\gamma}(m)\in M_\alpha\otimes C_\beta\otimes C_\gamma.$$

Suppose that $\delta\neq0$. Then there exists $F\in(M_\alpha\otimes C_\beta\otimes C_\gamma)^*$ such that $F(\delta)\neq0$. Now $M_\alpha^*\otimes C_\beta^*\otimes C_\gamma^*$ is dense in the linear topological space $(M_\alpha\otimes C_\beta\otimes C_\gamma)^*$ endowed with the $(M_\alpha\otimes C_\beta\otimes C_\gamma)$-topology (see [1, p. 70]). Thus $(M_\alpha^*\otimes C_\beta^*\otimes C_\gamma^*)\cap(F+\delta^\perp)\neq\emptyset$, where $\delta^\perp=\{f\in(M_\alpha\otimes C_\beta\otimes C_\gamma)^*\mid f(\delta)=0\}$. Then there exists $G\in M_\alpha^*\otimes C_\beta^*\otimes C_\gamma^*$ such that $G(\delta)\neq0$. Now for all $f\in M_\alpha^*$, $g\in C_\beta^*$, and $h\in C_\gamma^*$,

$$\begin{aligned}(f\otimes g\otimes h)(\rho_{\alpha,\beta}\otimes\mathrm{id}_{C_\gamma})\rho_{\alpha\beta,\gamma}(m)&=f\circ\psi_\rho(g\otimes\psi_p(h\otimes m))\\&=f\circ\psi_p(gh\otimes m)\\&=(f\otimes g\otimes h)(\mathrm{id}_{M_\alpha}\otimes\Delta_{\beta,\gamma})\rho_{\alpha,\beta\gamma}(m),\end{aligned}$$

i.e., $(f\otimes g\otimes h)(\delta)=0$. Therefore $G(\delta)=0$, which is a contradiction. We conclude that $\delta=0$ and then $(\rho_{\alpha,\beta}\otimes \mathrm{id}_{C_\gamma})\rho_{\alpha\beta,\gamma}=(\mathrm{id}_{M_\alpha}\otimes \Delta_{\beta,\gamma})\rho_{\alpha,\beta\gamma}$. Hence (M,ρ) is a right π-comodule over C. □

Lemma 2.4. *Let* $(M=\bigotimes_{\alpha\in\pi} M_\alpha,\psi)$ *be a rational* π*-graded left* C^**-module. Then* $\bar{M}=\{\bar{M}_\alpha\}_{\alpha\in\pi}$, *endowed with the structure maps* $\rho=\{\rho_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$ *defined by* (2.4), *is a right* π*-comodule over* C.

Proof. Let $\psi_\rho : C^*\otimes \bar{\bar{M}} \to \bar{\bar{M}}$ be the map defined as in Lemma 2.3. It is easy to verify that $(\bar{\bar{M}},\psi_\rho)=(M,\psi)$. Thus $(\bar{\bar{M}},\psi_\rho)$ is a π-graded left C^*-module and hence, by Lemma 2.3, $(\bar{M},\rho)$ is a right π-comodule over C. □

Proof of Theorem 2.2. Part (a) follows directly from Lemmas 2.3 and 2.4. To show part (b), let N be a graded submodule of a rational π-graded left C^*-module (M,ψ). Let $\rho_{\alpha,\beta}:\bar{N}_{\alpha\beta}\to \mathrm{Hom}_{\Bbbk}(C^*_\beta,\bar{N}_\alpha)$ defined by $\rho_{\alpha,\beta}(m)(f)=\psi(f\otimes m)$. Suppose that there exist $\alpha,\beta\in\pi$ and $n\in\bar{N}_{\alpha\beta}$ such that $\rho_{\alpha,\beta}(n)\notin \bar{N}_\alpha\otimes C_\beta$. Since M is rational, we can write $\rho_{\alpha,\beta}(n)=\sum_{i=1}^k n_i\otimes c_i\in \bar{M}_\alpha\otimes C_\beta$. Without loss of generality, we can assume that the c_i are $\Bbbk$-linearly independent and $n_1\notin \bar{N}_\alpha$. Let $f\in C^*_\beta$ such that $f(c_1)=1$ and $f(c_i)=0$ for $i\geqslant 2$. Now $\psi(f\otimes n)=\sum_{i=1}^k n_i f(c_i)=n_1\notin \bar{N}_\alpha=N_{\alpha^{-1}}$, contradicting the fact that N is a graded submodule of M. Thus $\rho_{\alpha,\beta}(\bar{N}_{\alpha\beta})\subset \bar{N}_\alpha\otimes C_\beta$ for all $\alpha,\beta\in\pi$. Hence N is rational.

Let us show part (c). Denote by $\cdot$ the left action of C^* on L. Set $\bar{L}_\alpha=L_{\alpha^{-1}}$ and $\rho_{\alpha,\beta}:\bar{L}_{\alpha\beta}\to \mathrm{Hom}_{\Bbbk}(C^*_\beta,\bar{L}_\alpha)$ given by $\rho_{\alpha,\beta}(m)(f)=f\cdot m$. Recall $\bar{L}_\alpha\otimes C_\beta$ can be viewed as a subspace of $\mathrm{Hom}_{\Bbbk}(C^*_\beta,\bar{L}_\alpha)$ via the embedding $\bar{L}_\alpha\otimes C_\beta\hookrightarrow \mathrm{Hom}_{\Bbbk}(C^*_\beta,\bar{L}_\alpha)$ given by $m\otimes c\mapsto (f\mapsto f(c)m)$. Define $M_\gamma=\bigcap_{\alpha\beta=\gamma^{-1}}\rho^{-1}_{\alpha,\beta}(\bar{L}_\alpha\otimes C_\beta)$ for any $\gamma\in\pi$, and set $M=\bigoplus_{\gamma\in\pi}M_\gamma$. Fix $\alpha,\beta\in\pi$, $f\in C^*_\alpha$, and $m\in M_\beta$. Let $u,v\in\pi$ such that $uv=(\alpha\beta)^{-1}$. We can write $\rho_{u,v\alpha}(m)=\sum_{i=1}^k l_i\otimes c_i\in \bar{L}_u\otimes C_{v\alpha}$. Now, for any $g\in C^*_v$,

$$g\cdot(f\cdot m)=(gf)\cdot m=\sum_{i=1}^k gf(c_i)l_i=\sum_{i=1}^k g(f(c_{i(2,\alpha)})c_{i(1,v)})l_i.$$

Then $\rho_{u,v}(f\cdot m)=\sum_{i=1}^k l_i\otimes f(c_{i(2,\alpha)})c_{i(1,v)}\in \bar{L}_u\otimes C_v$ and so $f\cdot m\in\rho^{-1}_{u,v}(\bar{L}_u\otimes C_v)$. Hence $f\cdot m\in\bigcap_{uv=(\alpha\beta)^{-1}}\rho^{-1}_{u,v}(\bar{L}_u\otimes C_v)=M_{\alpha\beta}$. Therefore M is a graded submodule of L. Moreover one easily checks at this point that $\rho_{\alpha,\beta}(\bar{M}_{\alpha\beta})\subset \bar{M}_\alpha\otimes C_\beta$ for any α,β in π. Thus M is rational.

Suppose now that N is another rational graded submodule of L and denoted by $\varrho=\{\varrho_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$ its corresponding π-comodule structure maps (see Lemma 2.4). Let $\gamma\in\pi$ and $\alpha,\beta\in\pi$ such that $\alpha\beta=\gamma^{-1}$. By the definition of $\rho_{\alpha,\beta}$ and $\varrho_{\alpha,\beta}$ and of the embedding $\bar{N}_\alpha\otimes C_\beta\subset \bar{L}_\alpha\otimes C_\beta\subset \mathrm{Hom}_{\Bbbk}(C^*_\beta,\bar{L}_\alpha)$, it follows that $\rho_{\alpha,\beta|N}=\varrho_{\alpha,\beta}:\bar{N}_{\alpha\beta}\to \bar{N}_\alpha\otimes C_\beta$. Thus $\rho_{\alpha,\beta}(N_\gamma)=\varrho_{\alpha,\beta}(\bar{N}_{\alpha\beta})\subset \bar{N}_\alpha\otimes C_\beta\subset \bar{L}_\alpha\otimes C_\beta$, and so $N_\gamma\subset\rho^{-1}_{\alpha,\beta}(\bar{L}_\alpha\otimes C_\beta)$. This holds for all $\alpha,\beta\in\pi$ such that $\alpha\beta=\gamma^{-1}$. Thus $N_\gamma\subset\bigcap_{\alpha\beta=\gamma^{-1}}\rho^{-1}_{\alpha,\beta}(\bar{L}_\alpha\otimes C_\beta)=M_\gamma$ for any $\gamma\in\pi$. Hence $N\subset M$. Therefore M is the unique maximal rational graded submodule of L and is the sum of all rational graded submodules of L. □

$$\begin{array}{ccccc} M_{\alpha\beta} \otimes H_{\alpha\beta} & \xrightarrow{\psi_{\alpha\beta}} & M_{\alpha\beta} & \xrightarrow{\rho_{\alpha,\beta}} & M_\alpha \otimes H_\beta \\ \downarrow {\scriptstyle \rho_{\alpha,\beta}\otimes\Delta_{\alpha,\beta}} & & & & \uparrow {\scriptstyle \psi_\alpha \otimes m_\beta} \\ M_\alpha \otimes H_\beta \otimes H_\alpha \otimes H_\beta & & \xrightarrow[\mathrm{id}_{M_\alpha}\otimes\sigma_{H_\beta,H_\alpha}\otimes\mathrm{id}_{H_\beta}]{} & & M_\alpha \otimes H_\alpha \otimes H_\beta \otimes H_\beta \end{array}$$

Fig. 1. Compatibility of the structure maps of a right Hopf π-comodule.

Remark 2.5. It follows from Lemma 2.4 and Theorem 2.2(c) that a unique "maximal" right π-comodule $\overline{(M^{\mathrm{rat}})}$ over a π-coalgebra C can be associated to any π-graded left C^*-module M.

2.3. *Hopf π-comodules*

In this subsection, we introduce and discuss the notion of a Hopf π-comodule.

Let $H = (\{H_\alpha, m_\alpha, 1_\alpha\}, \Delta, \varepsilon, S)$ be a Hopf π-coalgebra. A *right Hopf π-comodule over H* is a right π-comodule $M = \{M_\alpha\}_{\alpha\in\pi}$ over H such that

$$M_\alpha \text{ is a right } H_\alpha\text{-module for any } \alpha \in \pi. \tag{2.5}$$

Let us denote by $\psi_\alpha : M_\alpha \otimes H_\alpha \to H_\alpha$ the right action of H_α on M_α and by $\rho = \{\rho_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$ the π-comodule maps of M. These structures are required to be compatible in the sense that, for any $\alpha, \beta \in \pi$, the diagram of Fig. 1 is commutative. (2.6)

When $\pi = 1$, one recovers the definition of a Hopf module (see [9]).

Note that axiom (2.6) means that $\rho_{\alpha,\beta} : M_{\alpha\beta} \to M_\alpha \otimes H_\beta$ is $H_{\alpha\beta}$-linear, where $M_\alpha \otimes H_\beta$ is endowed with the right $H_{\alpha\beta}$-module structure given by

$$(m \otimes h) \cdot a = \psi_\alpha(m \otimes a_{(1,\alpha)}) \otimes h a_{(2,\beta)}.$$

A *Hopf π-subcomodule of M* is a π-subcomodule $N = \{N_\alpha\}_{\alpha\in\pi}$ of M such that N_α is a H_α-submodule of M_α for any $\alpha \in \pi$. Then N is a right Hopf π-comodule over H.

A *Hopf π-comodule morphism* between two right Hopf π-comodules M and M' is a π-comodule morphism $f = \{f_\alpha : M_\alpha \to M'_\alpha\}_{\alpha\in\pi}$ between M and M' such that f_α is H_α-linear for any $\alpha \in \pi$.

Example 2.6. Let $H = \{H_\alpha\}_{\alpha\in\pi}$ be a Hopf π-coalgebra and $M = \{M_\alpha\}_{\alpha\in\pi}$ be a right π-comodule over H, with structure maps $\rho = \{\rho_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$. For any $\alpha \in \pi$, set $(M \otimes H)_\alpha = M_\alpha \otimes H_\alpha$. The multiplication in H_α induces a structure of a right H_α-module on $(M \otimes H)_\alpha$ by setting $(m \otimes h) \triangleleft a = m \otimes ha$. Define the π-comodule structure maps $\xi_{\alpha,\beta} : (M \otimes H)_{\alpha\beta} \to (M \otimes H)_\alpha \otimes H_\beta$ by

$$\xi_{\alpha,\beta}(m \otimes h) = m_{(0,\alpha)} \otimes h_{(1,\alpha)} \otimes m_{(1,\beta)} h_{(2,\beta)}.$$

Here we write as usual $\rho_{\alpha,\beta}(m) = m_{(0,\alpha)} \otimes m_{(1,\beta)}$ and $\Delta_{\alpha,\beta}(h) = h_{(1,\alpha)} \otimes h_{(2,\beta)}$. One easily verifies that $M \otimes H = \{(M \otimes H)_\alpha\}_{\alpha\in\pi}$ is a right Hopf π-comodule over H, called *trivial*.

In the next theorem, we show that a Hopf π-comodule can be canonically decomposed. This generalizes the fundamental theorem of Hopf modules (see [9, Proposition 1]).

Theorem 2.7. *Let H be a Hopf π-coalgebra and M be a right Hopf π-comodule over H. Consider the π-subcomodule of coinvariants $M^{\mathrm{co}\,H}$ of M (see Example 2.1) and the trivial right Hopf π-comodule $M^{\mathrm{co}\,H} \otimes H$ (see Example 2.6). Then there exists a Hopf π-comodule isomorphism $M \cong M^{\mathrm{co}\,H} \otimes H$.*

Proof. We will denote by $\cdot$ (resp. $\triangleleft$) the right action of H_α on M_α (resp. on $(M^{\mathrm{co}\,H} \otimes H)_\alpha$) and by $\rho = \{\rho_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$ (resp. $\xi = \{\xi_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$) the π-comodule structure maps of M (resp. of $M^{\mathrm{co}\,H} \otimes H$). For any $\alpha \in \pi$, define $P_\alpha : M_1 \to M_\alpha$ by $P_\alpha(m) = m_{(0,\alpha)} \cdot S_{\alpha^{-1}}(m_{(1,\alpha^{-1})})$. Remark first that, for any $m \in M_1, (P_\alpha(m))_{\alpha\in\pi}$ is a coinvariant of H on M. Indeed, for all $\alpha, \beta \in \pi$,

$$\begin{aligned}
&\rho_{\alpha,\beta}(P_{\alpha\beta}(m))\\
&\quad = \rho_{\alpha,\beta}(m_{(0,\alpha\beta)} \cdot S_{(\alpha\beta)^{-1}}(m_{(1,(\alpha\beta)^{-1})}))\\
&\quad = \rho_{\alpha,\beta}(m_{(0,\alpha\beta)}) \cdot \Delta_{\alpha,\beta} S_{(\alpha\beta)^{-1}}(m_{(1,(\alpha\beta)^{-1})}) \quad \text{by (2.6)}\\
&\quad = m_{(0,\alpha)} \cdot S_{\alpha^{-1}}(m_{(3,\alpha^{-1})}) \otimes m_{(1,\beta)} S_{\beta^{-1}}(m_{(2,\beta^{-1})}) \quad \text{by Lemma 1.1(c)}\\
&\quad = m_{(0,\alpha)} \cdot S_{\alpha^{-1}}(\varepsilon(m_{(1,1)}) m_{(2,\alpha^{-1})}) \otimes 1_\beta \quad \text{by (1.5)}\\
&\quad = m_{(0,\alpha)} \cdot S_{\alpha^{-1}}(m_{(1,\alpha^{-1})}) \otimes 1_\beta \quad \text{by (1.2)}\\
&\quad = P_\alpha(m) \otimes 1_\beta.
\end{aligned}$$

For any $\alpha \in \pi$, define $f_\alpha : (M^{\mathrm{co}\,H} \otimes H)_\alpha \to M_\alpha$ by $f(m \otimes h) = m \cdot h$. Then f_α is H_α-linear since $f_\alpha(m \otimes h) \cdot a = (m \cdot h) \cdot a = m \cdot ha = f_\alpha((m \otimes h) \triangleleft a)$ for all $m \in M_\alpha^{\mathrm{co}\,H}$ and $h, a \in H_\alpha$. Moreover $(f_\alpha \otimes \mathrm{id}_{H_\beta})\xi_{\alpha,\beta} = \rho_{\alpha,\beta} f_{\alpha\beta}$ for all $\alpha, \beta \in \pi$. Indeed let $m \in M_{\alpha\beta}^{\mathrm{co}\,H}$ and $h \in H_{\alpha\beta}$. By the definition of $M_{\alpha\beta}^{\mathrm{co}\,H}$, there exists a coinvariant $(m_\gamma)_{\gamma\in\pi}$ of H on M such that $m = m_{\alpha\beta}$. In particular $\rho_{\alpha,\beta}(m) = m_\alpha \otimes 1_\beta$. Thus,

$$\begin{aligned}
(f_\alpha \otimes \mathrm{id}_{H_\beta})\xi_{\alpha,\beta}(m \otimes h) &= m_\alpha \cdot h_{(1,\alpha)} \otimes h_{(2,\beta)}\\
&= \rho_{\alpha,\beta}(m) \cdot \Delta_{\alpha,\beta}(h)\\
&= \rho_{\alpha,\beta}(m \cdot h) \quad \text{by (2.6)}\\
&= \rho_{\alpha,\beta}(f_{\alpha\beta}(m \otimes h)).
\end{aligned}$$

Then $f = \{f_\alpha\}_{\alpha\in\pi} : M^{\mathrm{co}\,H} \otimes H \to M$ is a Hopf π-comodule morphism. To show that f is an isomorphism, we construct its inverse. For any $\alpha \in \pi$, define $g_\alpha : M_\alpha \to (M^{\mathrm{co}\,H} \otimes H)_\alpha$ by $g_\alpha = (P_\alpha \otimes \mathrm{id}_{H_\alpha})\rho_{1,\alpha}$. The map g_α is well-defined since $(P_\gamma(m))_{\gamma\in\pi}$ is a coinvariant of H on M for all $m \in M_1$, and is H_α-linear since, for any $x \in M_\alpha$ and $a \in H_\alpha$,

$$\begin{aligned} g_\alpha(x \cdot a) &= (P_\alpha \otimes \mathrm{id}_{H_\alpha})\rho_{1,\alpha}(x \cdot a) \\ &= P_\alpha(x_{(0,1)} \cdot a_{(1,1)}) \otimes x_{(1,\alpha)}a_{(2,\alpha)} \quad \text{by (2.6)} \\ &= (x_{(0,\alpha)} \cdot a_{(1,\alpha)}) \cdot S_{\alpha^{-1}}(x_{(1,\alpha^{-1})}a_{(2,\alpha^{-1})}) \otimes x_{(2,\alpha)}a_{(3,\alpha)} \quad \text{by (2.6)} \\ &= x_{(0,\alpha)} \cdot (a_{(1,\alpha)}S_{\alpha^{-1}}(a_{(2,\alpha^{-1})})S_{\alpha^{-1}}(x_{(1,\alpha^{-1})})) \otimes x_{(2,\alpha)}a_{(3,\alpha)} \\ &= x_{(0,\alpha)} \cdot S_{\alpha^{-1}}(x_{(1,\alpha^{-1})}) \otimes x_{(2,\alpha)}\varepsilon(a_{(1,1)})a_{(2,\alpha)} \quad \text{by (1.5)} \\ &= x_{(0,\alpha)} \cdot S_{\alpha^{-1}}(x_{(1,\alpha^{-1})}) \otimes x_{(2,\alpha)}a \quad \text{by (1.2)} \\ &= g_\alpha(x) \triangleleft a. \end{aligned}$$

Moreover $(g_\alpha \otimes \mathrm{id}_{H_\beta})\rho_{\alpha,\beta} = \xi_{\alpha,\beta}g_{\alpha\beta}$ for all $\alpha, \beta \in \pi$. Indeed, for any $x \in M_{\alpha\beta}$,

$$\begin{aligned} \xi_{\alpha,\beta}(g_{\alpha\beta}(x)) &= \xi_{\alpha,\beta}(P_{\alpha\beta}(x_{(0,1)}) \otimes x_{(1,\alpha\beta)}) \\ &= P_{\alpha\beta}(x_{(0,1)})_{(0,\alpha)} \otimes x_{(1,\alpha\beta)(1,\alpha)} \otimes P_{\alpha\beta}(x_{(1,1)})_{(1,\beta)}x_{(1,\alpha\beta)(2,\beta)}, \end{aligned}$$

and so, since $(P_\gamma(x_{(0,1)}))_{\gamma\in\pi}$ is a π-coinvariant of H on M,

$$\begin{aligned} \xi_{\alpha,\beta}(g_{\alpha\beta}(x)) &= P_\alpha(x_{(0,1)}) \otimes x_{(1,\alpha)} \otimes x_{(2,\beta)} \\ &= g_\alpha(x_{(0,\alpha)}) \otimes x_{(1,\beta)} \\ &= (g_\alpha \otimes \mathrm{id}_{H_\beta})\rho_{\alpha,\beta}(x). \end{aligned}$$

Thus $g = \{g_\alpha\}_{\alpha\in\pi} : M \to M^{\mathrm{co}\,H} \otimes H$ is a Hopf π-comodule morphism. It remains now to verify that $g_\alpha f_\alpha = \mathrm{id}_{(M^{\mathrm{co}\,H}\otimes H)_\alpha}$ and $f_\alpha g_\alpha = \mathrm{id}_{M_\alpha}$ for any $\alpha \in \pi$. Let $m \in M_\alpha^{\mathrm{co}\,H}$ and $h \in H_\alpha$. By the definition of $M_\alpha^{\mathrm{co}\,H}$, there exists a coinvariant $(m_\gamma)_{\gamma\in\pi}$ of H on M such that $m = m_\alpha$. In particular, $\rho_{1,\alpha}(m) = m_1 \otimes 1_\alpha$ and $P_\alpha(m_1) = m_\alpha \cdot S_{\alpha^{-1}}(1_{\alpha^{-1}}) = m \cdot 1_\alpha = m$. Then

$$\begin{aligned} g_\alpha f_\alpha(m \otimes h) &= g_\alpha(m \cdot h) \\ &= g_\alpha(m) \triangleleft h \quad \text{since } g_\alpha \text{ is } H_\alpha\text{-linear} \\ &= (P_\alpha(m_1) \otimes 1_\alpha) \triangleleft h \\ &= m \otimes h. \end{aligned}$$

Finally, for all $x \in M_\alpha$,

$$\begin{aligned} f_\alpha g_\alpha(x) &= (x_{(0,\alpha)} \cdot S_{\alpha^{-1}}(x_{(1,\alpha^{-1})})) \cdot x_{(2,\alpha)} \\ &= x_{(0,\alpha)} \cdot (S_{\alpha^{-1}}(x_{(1,\alpha^{-1})})x_{(2,\alpha)}) \\ &= x_{(0,\alpha)}\varepsilon(x_{(1,1)}) \cdot 1_\alpha \quad \text{by (1.5)} \\ &= x \quad \text{by (2.2)}. \end{aligned}$$

Hence $g = f^{-1}$ and f and g are Hopf π-comodule isomorphisms. □

3. Existence and uniqueness of π-integrals

In this section, we introduce and discuss the notion of a π-integral for a Hopf π-coalgebra. In particular, by generalizing the arguments of [17, Section 5], we show that, in the finite type case, the space of left (resp. right) π-integrals is one dimensional.

3.1. π-integrals

We first recall that a left (resp. right) integral for a Hopf algebra $(A, \Delta, \varepsilon, S)$ is an element $\Lambda \in A$ such that $x\Lambda = \varepsilon(x)\Lambda$ (resp. $\Lambda x = \varepsilon(x)\Lambda$) for all $x \in A$. A left (resp. right) integral for the dual Hopf algebra A^* is a $\Bbbk$-linear form $\lambda \in A^*$ verifying $(f \otimes \lambda)\Delta = f(1_A)\lambda$ (resp. $(\lambda \otimes f)\Delta = f(1_A)\lambda$) for all $f \in A^*$ Let us extend this notion to the setting of a Hopf π-coalgebra.

Let $H = (\{H_\alpha, m_\alpha, 1_\alpha\}, \Delta, \varepsilon, S)$ be a Hopf π-coalgebra. A *left* (*resp. right*) *π-integral for H* is a family of $\Bbbk$-linear forms $\lambda = (\lambda_\alpha)_{\alpha\in\pi} \in \prod_{\alpha\in\pi} H_\alpha^*$ such that, for all $\alpha, \beta \in \pi$,

$$(\mathrm{id}_{H_\alpha} \otimes \lambda_\beta)\Delta_{\alpha,\beta} = \lambda_{\alpha\beta} 1_\alpha \quad (\text{resp. } (\lambda_\alpha \otimes \mathrm{id}_{H_\beta})\Delta_{\alpha,\beta} = \lambda_{\alpha\beta} 1_\beta). \tag{3.1}$$

Note that λ_1 is a usual left (resp. right) integral for the Hopf algebra H_1^*.

If we use the multiplication of the dual π-graded algebra H^* of H (see Section 1.2), we have that $\lambda = (\lambda_\alpha)_{\alpha\in\pi} \in \prod_{\alpha\in\pi} H_\alpha^*$ is a left (resp. right) π-integral for H if and only if, for all $\alpha, \beta \in \pi$ and $f \in H_\alpha^*$ (resp. $g \in H_\beta^*$),

$$f\lambda_\beta = f(1_\alpha)\lambda_{\alpha\beta} \quad (\text{resp. } \lambda_\alpha g = g(1_\alpha)\lambda_{\alpha\beta}).$$

A π-integral $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ for H is said to be *non-zero* if $\lambda_\beta \neq 0$ for some $\beta \in \pi$.

Lemma 3.1. *Let $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ be a non-zero left* (*resp. right*) *π-integral for H. Then $\lambda_\alpha \neq 0$ for all $\alpha \in \pi$ such that $H_\alpha \neq 0$. In particular $\lambda_1 \neq 0$.*

Proof. Let $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ be a left π-integral for H such that $\lambda_\beta \neq 0$ for some $\beta \in \pi$ and let $\alpha \in \pi$ with $H_\alpha \neq 0$. Then $H_{\beta\alpha^{-1}} \neq 0$ (by Corollary 1.2) and so $1_{\beta\alpha^{-1}} \neq 0$. Using (3.1), we have that $(\mathrm{id}_{H_{\beta\alpha^{-1}}} \otimes \lambda_\alpha)\Delta_{\beta\alpha^{-1},\alpha} = \lambda_\beta 1_{\beta\alpha^{-1}} \neq 0$. Hence $\lambda_\alpha \neq 0$. The right case can be done similarly. □

Remark 3.2. Let H be a finite type Hopf π-coalgebra. Consider the Hopf algebra H^* dual to H (see Section 1.3.4). If $H_\alpha = 0$ for all but a finite number of $\alpha \in \pi$, then $\lambda = (\lambda_\alpha)_{\alpha\in\pi} \in \prod_{\alpha\in\pi} H_\alpha^*$ is a left (resp. right) π-integral for H if and only if $\sum_{\alpha\in\pi} \lambda_\alpha$ is a left (resp. right) integral for H^*. If $H_\alpha \neq 0$ for infinitely many $\alpha \in \pi$, then H^* is infinite dimensional and thus does not have any non-zero left or right integral (see [18]). Nevertheless we show in the next subsection that H always has a non-zero π-integral.

3.2. The space of π-integrals is one dimensional

It is known (see [17, Corollary 5.1.6]) that the space of left (resp. right) integrals for a finite-dimensional Hopf algebra is one dimensional. In this subsection, we generalize this result to finite type Hopf π-coalgebras.

Let $H=\{H_\alpha\}_{\alpha\in\pi}$ be a Hopf π-coalgebra (not necessarily of finite type). The dual π-graded algebra H^* of H (see Section 1.2) is a π-graded left H^*-module via left multiplication. Let $(H^*)^{\mathrm{rat}}$ be its maximal rational π-graded submodule (see Theorem 2.2(c)). Denote by $H^{\square}=\overline{(H^*)^{\mathrm{rat}}}=\{H^{\square}_\alpha\}_{\alpha\in\pi}$ the right π-comodule over H which corresponds to it by Lemma 2.4. Recall that $H^{\square}_\alpha\subset H^*_{\alpha^{-1}}$ for any $\alpha\in\pi$. The π-comodule structure maps of $H^{\square}$ will be denoted by $\rho=\{\rho_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$.

Lemma 3.3. *Let $\lambda=(\lambda_\alpha)_{\alpha\in\pi}\in\prod_{\alpha\in\pi}H^*_\alpha$. Then λ is a left π-integral for H if and only if $(\lambda_{\alpha^{-1}})_{\alpha\in\pi}$ is a coinvariant of H on $H^{\square}$ (see Example 2.1).*

Proof. Suppose that λ is a left π-integral for H. Fix $\gamma\in\pi$. Let $\alpha,\beta\in\pi$ such that $\alpha\beta=\gamma$. We have that $\rho_{\alpha,\beta}(\lambda_{\gamma^{-1}})=\lambda_{\alpha^{-1}}\otimes 1_\beta\in\overline{H^*_\alpha}\otimes H_\beta$ since $f\lambda_{\gamma^{-1}}=f(1_\beta)\lambda_{\alpha^{-1}}$ for all $f\in H^*_\beta$. Therefore $\lambda_{\gamma^{-1}}\in\bigcap_{\alpha\beta=\gamma}\rho^{-1}_{\alpha,\beta}(\overline{H^*_\alpha}\otimes H_\beta)=(H^*)^{\mathrm{rat}}_{\gamma^{-1}}=H^{\square}_\gamma$, see Theorem 2.2(c). Hence, since $\rho_{\alpha,\beta}(\lambda_{(\alpha\beta)^{-1}})=\lambda_{\alpha^{-1}}\otimes 1_\beta$ for all $\alpha,\beta\in\pi$, $(\lambda_{\alpha^{-1}})_{\alpha\in\pi}$ is a coinvariant of H on $H^{\square}$. Conversely, suppose that $(\lambda_{\alpha^{-1}})_{\alpha\in\pi}$ is a coinvariant of H on $H^{\square}$. Let $\alpha,\beta\in\pi$. Then $\rho_{(\alpha\beta)^{-1},\alpha}(\lambda_\beta)=\lambda_{\alpha\beta}\otimes 1_\alpha$, i.e., $f\lambda_\beta=f(1_\alpha)\lambda_{\alpha\beta}$ for all $f\in H^*_\alpha$. Hence λ is a left π-integral for H. $\square$

For all $\alpha\in\pi$, we define a right H_α-module structure on $H^{\square}_\alpha$ by setting

$$(f\leftharpoonup a)(x)=f(xS_\alpha(a))$$

for any $f\in H^{\square}_\alpha$, $a\in H_\alpha$, and $x\in H_{\alpha^{-1}}$.

Lemma 3.4. *$H^{\square}$ is a right Hopf π-comodule over H.*

Proof. Let us first show that for any $\alpha,\beta\in\pi$, $f\in H^{\square}_{\alpha\beta}$, $a\in H_{\alpha\beta}$, and $g\in H^*_\beta$,

$$g(f\leftharpoonup a)=f_{(0,\alpha)}\leftharpoonup a_{(1,\alpha)}\langle g,f_{(1,\beta)}a_{(2,\beta)}\rangle, \tag{3.2}$$

where $\langle\ ,\ \rangle$ denotes the natural pairing between H^*_β and H_β. Remark first that

$$\begin{aligned}1_\beta\otimes S_{\alpha\beta}(a)&=\varepsilon(a_{(2,1)})1_\beta\otimes S_{\alpha\beta}(a_{(1,\alpha\beta)}) \quad \text{by (1.2)}\\ &=S_{\beta^{-1}}(a_{(2,\beta^{-1})})a_{(3,\beta)}\otimes S_{\alpha\beta}(a_{(1,\alpha\beta)}) \quad \text{by (1.5)}\\ &=S_\alpha(a_{(1,\alpha)})_{(1,\beta)}a_{(2,\beta)}\otimes S_\alpha(a_{(1,\alpha)})_{(2,(\alpha\beta)^{-1})} \quad \text{by Lemma 1.1(c)}\end{aligned}$$

Then, for all $x\in H_{\alpha^{-1}}$,

$$\begin{aligned}&x_{(1,\beta)}\otimes x_{(2,(\alpha\beta)^{-1})}S_{\alpha\beta}(a)\\ &\quad=x_{(1,\beta)}S_\alpha(a_{(1,\alpha)})_{(1,\beta)}a_{(2,\beta)}\otimes x_{(2,(\alpha\beta)^{-1})}S_\alpha(a_{(1,\alpha)})_{(2,(\alpha\beta)^{-1})}\\ &\quad=(xS_\alpha(a_{(1,\alpha)}))_{(1,\beta)}a_{(2,\beta)}\otimes(xS_\alpha(a_{(1,\alpha)}))_{(2,(\alpha\beta)^{-1})} \quad \text{by (1.4)}\end{aligned}$$

and so

$$g(f \leftharpoondown a)(x) = \langle g, x_{(1,\beta)} \rangle \langle f \leftharpoondown a, x_{(2,(\alpha\beta)^{-1})} \rangle$$
$$= \langle g, x_{(1,\beta)} \rangle \langle f, x_{(2,(\alpha\beta)^{-1})} S_{\alpha\beta}(a) \rangle$$
$$= \langle g, (xS_\alpha(a_{(1,\alpha)}))_{(1,\beta)} a_{(2,\beta)} \rangle \langle f, (xS_\alpha(a_{(1,\alpha)}))_{(2,(\alpha\beta)^{-1})} \rangle$$
$$= ((a_{(2,\beta)} \rightharpoonup g) f) \leftharpoondown a_{(1,\alpha)}(x),$$

where $\rightharpoonup$ is the left H_β-action on H_β^* defined by $(b \rightharpoonup l)(y) = l(yb)$ for any $l \in H_\beta^*$ and $b, y \in H_\beta$. Then

$$g(f \leftharpoondown a) = ((a_{(2,\beta)} \rightharpoonup g) f) \leftharpoondown a_{(1,\alpha)}$$
$$= (f_{(0,\alpha)} \langle a_{(2,\beta)} \rightharpoonup g, f_{(1,\beta)} \rangle) \leftharpoondown a_{(1,\alpha)} \quad \text{by definition of } \rho_{\alpha,\beta}$$
$$= f_{(0,\alpha)} \leftharpoondown a_{(1,\alpha)} \langle g, f_{(1,\beta)} a_{(2,\beta)} \rangle$$

and hence (3.2) is proved.

Recall that the π-comodule structure map $\rho_{\alpha,\beta}$ of $H^\square$ is, via the natural embedding $H_\alpha^\square \otimes H_\beta \subset \overline{H_\alpha^*} \otimes H_\beta \hookrightarrow \mathrm{Hom}_\Bbbk(H_\beta^*, \overline{H_\alpha^*})$, the restriction onto $H_\alpha^\square \otimes H_\beta$ of the map $\xi_{\alpha,\beta} : H_{\alpha\beta}^\square \to \mathrm{Hom}_\Bbbk(H_\beta^*, \overline{H_\alpha^*})$ defined by $\xi_{\alpha,\beta}(f)(g) = gf$. Let $\gamma \in \pi$. By (3.2), we have that, for any $\alpha, \beta \in \pi$ such that $\alpha\beta = \gamma$, $f \in H_\gamma^\square$, and $a \in H_\gamma$,

$$\xi_{\alpha,\beta}(f \leftharpoondown a) = f_{(0,\alpha)} \leftharpoondown a_{(1,\alpha)} \otimes f_{(1,\beta)} a_{(2,\beta)} \in (H_\alpha^\square \leftharpoondown a_{(1,\alpha)}) \otimes H_\beta \subset \overline{H_\alpha^*} \otimes H_\beta.$$

Therefore, by Theorem 2.2(c), $f \leftharpoondown a \in \bigcap_{\alpha\beta=\gamma} \xi_{\alpha,\beta}^{-1}(\overline{H_\alpha^*} \otimes C_\beta) = H_\gamma^\square$. Hence the action of H_γ on $H_\gamma^\square$ is well-defined. This is a right action because S_γ is unitary and anti-multiplicative (see Lemma 1.1). Finally, axiom (2.6) is satisfied since (3.2) says that $\rho_{\alpha,\beta}(f \leftharpoondown a) = f_{(0,\alpha)} \leftharpoondown a_{(1,\alpha)} \otimes f_{(1,\beta)} a_{(2,\beta)}$ for any $\alpha, \beta \in \pi$, $f \in H_{\alpha\beta}^\square$, and $a \in H_{\alpha\beta}$. Thus $H^\square$ is a right Hopf π-comodule over H. □

By Theorem 2.7, the Hopf π-comodule $H^\square$ is isomorphic to the Hopf π-comodule $(H^\square)^{\mathrm{co}\,H} \otimes H$. Let $f = \{f_\alpha : (H^\square)_\alpha^{\mathrm{co}\,H} \otimes H_\alpha \to H_\alpha^\square\}_{\alpha\in\pi}$ be the right Hopf π-comodule isomorphism between them as in the proof Theorem 2.7. Recall that $f_\alpha(m \otimes h) = m \leftharpoondown h$ for any $\alpha \in \pi$, $m \in (H^\square)_\alpha^{\mathrm{co}\,H}$, and $h \in H_\alpha$.

Lemma 3.5. *If there exists a non-zero left π-integral for H, then S_α is injective for all $\alpha \in \pi$.*

Proof. Suppose that $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ is a non-zero left π-integral for H. Let $\alpha \in \pi$. If $H_\alpha = 0$, then the result is obvious. Let us suppose that $H_\alpha \neq 0$. Then $H_{\alpha^{-1}} \neq 0$ by Corollary 1.2 and so $\lambda_{\alpha^{-1}} \neq 0$ (by Lemma 3.1). Let $h \in H_\alpha$ such that $S_\alpha(h) = 0$. By Lemma 3.3, $\lambda_{\alpha^{-1}} \in H_\alpha^{\square \mathrm{co}\,H}$. Now $f_\alpha(\lambda_{\alpha^{-1}} \otimes h) = \lambda_{\alpha^{-1}} \leftharpoondown h = 0$ (since $S_\alpha(h) = 0$). Thus $\lambda_{\alpha^{-1}} \otimes h = 0$ (since f_α is an isomorphism) and so $h = 0$ (since $\lambda_{\alpha^{-1}} \neq 0$). □

Theorem 3.6. *Let H be a finite type Hopf π-coalgebra. Then the space of left* (*resp. right*) *π-integrals for H is one dimensional.*

Proof. For any $\alpha, \beta \in \pi$, since H is of finite type and $\overline{H_\alpha^*} = H_{\alpha^{-1}}^*$, we have that $\dim \overline{H_\alpha^*} \otimes H_\beta = \dim \mathrm{Hom}_{\Bbbk}(H_\beta^*, \overline{H_\alpha^*}) < +\infty$. Therefore, the natural embedding $\overline{H_\alpha^*} \otimes H_\beta \hookrightarrow \mathrm{Hom}_{\Bbbk}(H_\beta^*, \overline{H_\alpha^*})$ is an isomorphism. Thus H^* is a rational π-graded H^*-module (see Section 2.2) and so $H_\alpha^\square = H_{\alpha^{-1}}^*$ for all $\alpha \in \pi$. Now $\dim (H^\square)_1^{\mathrm{co}\,H} = 1$ since $(H^\square)_1^{\mathrm{co}\,H} \otimes H_1 \cong H_1^\square$, $\dim H_1 = \dim H_1^\square < +\infty$, and $\dim H_1 \neq 0$ (by Corollary 1.2). Hence there exists a π-coinvariant $(\psi_\alpha)_{\alpha\in\pi}$ of H on $H^\square$ such that $\psi_1 \neq 0$. Set $\lambda_\alpha = \psi_{\alpha^{-1}}$ for any $\alpha \in \pi$. By Lemma 3.3, $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ is a left π-integral for H. Moreover $\lambda_1 = \psi_1 \neq 0$ and so λ is non-zero.

Suppose now that $\delta = (\delta_\alpha)_{\alpha\in\pi}$ is another left π-integral for H. Let $\alpha \in \pi$ such that $H_\alpha \neq 0$. By Lemma 3.5, S_α and $S_{\alpha^{-1}}$ are injective (since there exists a non-zero left integral for H) and so $\dim H_\alpha = \dim H_{\alpha^{-1}}$. Therefore, $\dim (H^\square)_\alpha^{\mathrm{co}\,H} = 1$ since $(H^\square)_\alpha^{\mathrm{co}\,H} \otimes H_\alpha \cong H_\alpha^\square$ and $0 \neq \dim H_\alpha = \dim H_\alpha^\square < +\infty$. Now $\lambda_{\alpha^{-1}}, \delta_{\alpha^{-1}} \in (H^\square)_\alpha^{\mathrm{co}\,H}$ by Lemma 3.3 and $\lambda_{\alpha^{-1}} \neq 0$ (by Lemma 3.1). Hence there exists $k_\alpha \in \Bbbk$ such that $\delta_{\alpha^{-1}} = k_\alpha \lambda_{\alpha^{-1}}$. If $\alpha \in \pi$ is such that $H_\alpha \neq 0$, then

$$k_1 \lambda_1 1_\alpha = \delta_1 1_\alpha = (\mathrm{id}_{H_\alpha} \otimes \delta_{\alpha^{-1}}) \Delta_{\alpha,\alpha^{-1}} = k_\alpha (\mathrm{id}_{H_\alpha} \otimes \lambda_{\alpha^{-1}}) \Delta_{\alpha,\alpha^{-1}} = k_\alpha \lambda_1 1_\alpha,$$

and thus $k_\alpha = k_1$ (since $\lambda_1 \neq 0$ and $1_\alpha \neq 0$). If $\alpha \in \pi$ is such that $H_\alpha = 0$, then $\delta_\alpha = 0 = \lambda_\alpha$ and so $\delta_\alpha = k_1 \lambda_\alpha$. Hence we can conclude that δ is a scalar multiple of λ.

To show the existence and the uniqueness of right π-integrals for H, it suffices to consider the opposite and coopposite Hopf π-coalgebra $H^{\mathrm{op,cop}}$ to H (see Section 1.3.3). Indeed $\lambda = (\lambda_\alpha)_{\alpha\in\pi} \in \prod_{\alpha\in\pi} H_\alpha^*$ is a right π-integral for H if and only if $(\lambda_{\alpha^{-1}})_{\alpha\in\pi}$ is a left π-integral for $H^{\mathrm{op,cop}}$. This completes the proof of the theorem. □

Corollary 3.7. *Let $H = \{H_\alpha\}_{\alpha\in\pi}$ be a finite type Hopf π-coalgebra. Then*
(a) *The antipode $S = \{S_\alpha\}_{\alpha\in\pi}$ of H is bijective.*
(b) *Let $\alpha \in \pi$. Then H_α^* is a free left (resp. right) H_α-module for the action defined, for any $f \in H_\alpha^*$ and $a, x \in H_\alpha$, by*

$$(a \rightharpoonup f)(x) = f(xa) \quad (\textit{resp. } (f \leftharpoonup a)(x) = f(ax)).$$

Its rank is 1 if $H_\alpha \neq 0$ and 0 otherwise. Moreover, if $\lambda = (\lambda_\beta)_{\beta\in\pi}$ is a non-zero left (resp. right) π-integral for H, then λ_α is a basis vector for H_α^*.

Proof. To show part (a), let $\alpha \in \pi$. By Lemma 3.5 and Theorem 3.6, $S_\alpha : H_\alpha \to H_{\alpha^{-1}}$ and $S_{\alpha^{-1}} : H_{\alpha^{-1}} \to H_\alpha$ are injective. Thus $\dim H_\alpha = \dim H_{\alpha^{-1}}$ and so S_α is bijective. To show part (b), let $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ be a non-zero left π-integral for H and fix $\alpha \in \pi$. If $H_\alpha = 0$, then the result is obvious. Let us suppose that $H_\alpha \neq 0$. Recall that $H_{\alpha^{-1}}^\square = H_\alpha^*$ and $f_{\alpha^{-1}} : (H^*)_\alpha^{\mathrm{co}\,H} \otimes H_{\alpha^{-1}} \to H_\alpha^*$ defined by $f \otimes h \mapsto S_{\alpha^{-1}}(h) \rightharpoonup f$ is an isomorphism. Since $0 \neq \lambda_\alpha \in (H^*)_\alpha^{\mathrm{co}\,H}$, $\dim (H^*)_\alpha^{\mathrm{co}\,H} = 1$, and $S_{\alpha^{-1}}$ is bijective, the map $H_\alpha \to H_\alpha^*$ defined by $h \mapsto h \rightharpoonup \lambda_\alpha$ is an isomorphism. Thus $(H_\alpha^*, \rightharpoonup)$ is a free left H_α-module of rank 1 with vector basis λ_α. Using $H^{\mathrm{op,cop}}$ (see Section 1.3.3), one easily deduces the right case. □

4. The distinguished π-grouplike element

In this section, we extend the notion of a grouplike element of a Hopf algebra to the setting of a Hopf π-coalgebra. We show that a π-grouplike element is distinguished in a finite type Hopf π-coalgebra and we study its relations with the π-integrals. As a corollary, for any $\alpha \in \pi$ of finite order, we give an upper bound for the (finite) order of $S_{\alpha^{-1}}S_\alpha$.

4.1. π-grouplike elements

A *π-grouplike element* of a Hopf π-coalgebra H is a family $g=(g_\alpha)_{\alpha\in\pi} \in \prod_{\alpha\in\pi} H_\alpha$ such that $\Delta_{\alpha,\beta}(g_{\alpha\beta}) = g_\alpha \otimes g_\beta$ for any $\alpha,\beta\in\pi$ and $\varepsilon(g_1)=1_\Bbbk$ (or equivalently $g_1 \neq 0$). Note that g_1 is then a (usual) grouplike element of the Hopf algebra H_1.

One easily checks that the set $G(H)$ of π-grouplike elements of H is a group (with respect to the multiplication and unit of the product monoid $\prod_{\alpha\in\pi} H_\alpha$) and if $g=(g_\alpha)_{\alpha\in\pi}\in G(H)$, then $g^{-1} = (S_{\alpha^{-1}}(g_{\alpha^{-1}}))_{\alpha\in\pi}$.

We remark that the group $\mathrm{Hom}(\pi,\Bbbk^*)$ acts on $G(H)$ by $\phi g = (\phi(\alpha)g_\alpha)_{\alpha\in\pi}$ for any $g=(g_\alpha)_{\alpha\in\pi}\in G(H)$ and $\phi\in\mathrm{Hom}(\pi,\Bbbk^*)$.

Lemma 4.1. *Let H be a finite type Hopf π-coalgebra. Then there exists a unique π-grouplike element $g=(g_\alpha)_{\alpha\in\pi}$ of H such that $(\mathrm{id}_{H_\alpha}\otimes\lambda_\beta)\Delta_{\alpha,\beta} = \lambda_{\alpha\beta}g_\alpha$ for any right π-integral $\lambda=(\lambda_\alpha)_{\alpha\in\pi}$ and all $\alpha,\beta\in\pi$.*

The π-grouplike element $g=(g_\alpha)_{\alpha\in\pi}$ of the previous lemma is called the *distinguished π-grouplike element of H*. Note that g_1 is the (usual) distinguished grouplike element of the Hopf algebra H_1.

Proof. Let $\lambda=(\lambda_\alpha)_{\alpha\in\pi}$ be a non-zero right π-integral for H. Let $\gamma\in\pi$. For any $\varphi\in H_\gamma^*$, $(\varphi\lambda_{\gamma^{-1}\alpha})_{\alpha\in\pi}$ is a right π-integral for H and thus, by Theorem 3.6, there exists a unique $k_\varphi\in\Bbbk$ such that $\varphi\lambda_{\gamma^{-1}\alpha} = k_\varphi\lambda_\alpha$ for all $\alpha\in\pi$. Now $(\varphi\mapsto k_\varphi)\in H_\gamma^{**}\cong H_\gamma$ ($\dim H_\gamma < +\infty$). Therefore, there exists a unique $g_\gamma\in H_\gamma$ such that $\varphi\lambda_{\gamma^{-1}\alpha} = \varphi(g_\gamma)\lambda_\alpha$ for any $\alpha\in\pi$ and $\varphi\in H_\gamma^*$. Then $\varphi\lambda_\beta = \varphi(g_\alpha)\lambda_{\alpha\beta}$ for any $\alpha,\beta\in\pi$ and $\varphi\in H_\alpha^*$ and hence $(\mathrm{id}_{H_\alpha}\otimes\lambda_\beta)\Delta_{\alpha,\beta} = \lambda_{\alpha\beta}g_\alpha$ for all $\alpha,\beta\in\pi$. Let $\alpha,\beta\in\pi$. If $H_{\alpha\beta}=0$, then either $H_\alpha=0$ or $H_\beta=0$ (by Corollary 1.2) and so $\Delta_{\alpha,\beta}(g_{\alpha\beta}) = 0 = g_\alpha\otimes g_\beta$. If $H_{\alpha\beta}\neq 0$, then, for any $\varphi\in H_\alpha^*$ and $\psi\in H_\beta^*$, $k_{\varphi\psi}\lambda_{\alpha\beta} = (\varphi\psi)\lambda_1 = \varphi(\psi\lambda_1) = k_\psi\varphi\lambda_\beta = k_\varphi k_\psi\lambda_{\alpha\beta}$ and thus $k_{\varphi\psi} = k_\varphi k_\psi$ (since $\lambda_{\alpha\beta}\neq 0$ by Lemma 3.1), that is $\Delta_{\alpha,\beta}(g_{\alpha\beta}) = g_\alpha\otimes g_\beta$. Moreover $\varepsilon(g_1)\lambda_1 = (\varepsilon\otimes\lambda_1)\Delta_{1,1} = \lambda_1$ and so $\varepsilon(g_1)=1$ (since $\lambda_1\neq 0$ by Lemma 3.1). Then $g=(g_\alpha)_{\alpha\in\pi}$ is a π-grouplike element of H. Since all the right π-integrals for H are scalar multiple of λ, the "existence" part of the lemma is demonstrated. Let us now show the uniqueness of g. Suppose that $h=(h_\alpha)_{\alpha\in\pi}$ is another such π-grouplike element of H. Let $\lambda=(\lambda_\alpha)_{\alpha\in\pi}$ be a non-zero right π-integral for H. Fix $\alpha\in\pi$. If $H_\alpha=0$, then $h_\alpha=0=g_\alpha$. If $H_\alpha\neq 0$, then $\lambda_\alpha\neq 0$ (by Lemma 3.1) and so there exists $a\in H_\alpha$ such that $\lambda_\alpha(a)=1$.

Therefore $g_\alpha = \lambda_\alpha(a) g_\alpha = (\mathrm{id}_{H_\alpha} \otimes \lambda_1)\Delta_{\alpha,1}(a) = \lambda_\alpha(a) h_\alpha = h_\alpha$. This completes the proof of the lemma. □

4.2. The distinguished π-grouplike element and π-integrals

Throughout this subsection, $H = \{H_\alpha\}_{\alpha\in\pi}$ will denote a finite type Hopf π-coalgebra.

Since H_1 is a finite-dimensional Hopf algebra, there exists (e.g., see [13]) a unique algebra morphism $v : H_1 \to \Bbbk$ such that if Λ is a left integral for H_1, then $\Lambda x = v(x)\Lambda$ for all $x \in H_1$. This morphism is a grouplike element of the Hopf algebra H_1^*, called the *distinguished grouplike element of* H_1^*. In particular, it is invertible in H_1^* and its inverse v^{-1} is also an algebra morphism and verifies that if Λ is a right integral for H_1, then $x\Lambda = v^{-1}(x)\Lambda$ for all $x \in H_1$.

For all $\alpha \in \pi$, we define a left and a right H_1^*-action on H_α by setting, for any $f \in H_1^*$ and $a \in H_\alpha$,

$$f \rightharpoonup a = a_{(1,\alpha)} f(a_{(2,1)}) \quad \text{and} \quad a \leftharpoonup f = f(a_{(1,1)}) a_{(2,\alpha)}.$$

The next theorem generalizes [15, Theorem 3]. It is used in Section 7 to show the existence of traces.

Theorem 4.2. *Let $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ be a right π-integral for H, $g = (g_\alpha)_{\alpha\in\pi}$ be the distinguished π-grouplike element of H, and v be the distinguished grouplike element of H_1^*. Then, for any $\alpha \in \pi$ and $x, y \in H_\alpha$,*
(a) $\lambda_\alpha(xy) = \lambda_\alpha(S_{\alpha^{-1}} S_\alpha(y \leftharpoonup v)x)$;
(b) $\lambda_\alpha(xy) = \lambda_\alpha(y S_{\alpha^{-1}} S_\alpha(v^{-1} \rightharpoonup g_\alpha^{-1} x g_\alpha))$;
(c) $\lambda_{\alpha^{-1}}(S_\alpha(x)) = \lambda_\alpha(g_\alpha x)$.

Before proving Theorem 4.2, we establish the following lemma.

Lemma 4.3. *Let $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ be a right π-integral for H, $\alpha \in \pi$, and $a \in H_\alpha$.*
(a) *If Λ is a right integral for H_1 such that $\lambda_1(\Lambda) = 1$, then*

$$S_\alpha(a) = \lambda_\alpha(\Lambda_{(1,\alpha)} a)\Lambda_{(2,\alpha^{-1})}.$$

(b) *If Λ is a left integral for H_1 such that $\lambda_1(\Lambda) = 1$, then*

$$S_{\alpha^{-1}}^{-1}(a) = \lambda_\alpha(a\Lambda_{(1,\alpha)})\Lambda_{(2,\alpha^{-1})}.$$

Proof. To show part (a), let $\alpha \in \pi$. Define $f \in H_\alpha^*$ by $f(x) = \lambda_\alpha(\Lambda_{(1,\alpha)} x)\Lambda_{(2,\alpha^{-1})}$ for any $x \in H_\alpha$. If $*$ denotes the product in the convolution algebra $\mathrm{Conv}(H, H_{\alpha^{-1}})$ (see Section 1.2), then, for any $x \in H_1$,

$$\begin{aligned}(f * \mathrm{id}_{H_{\alpha^{-1}}})(x) &= \lambda_\alpha(\Lambda_{(1,\alpha)} x_{(1,\alpha)})\Lambda_{(2,\alpha^{-1})} x_{(2,\alpha^{-1})} \\ &= \lambda_\alpha((\Lambda x)_{(1,\alpha)})(\Lambda x)_{(2,\alpha^{-1})} \quad \text{by (1.4)} \\ &= \varepsilon(x)\lambda_\alpha(\Lambda_{(1,\alpha)})\Lambda_{(2,\alpha^{-1})} \quad \text{since } \Lambda \text{ is a right integral for } H_1\end{aligned}$$

$$= \varepsilon(x)\lambda_1(\Lambda)1_{\alpha^{-1}} \quad \text{by (3.1)}$$

$$= \varepsilon(x)1_{\alpha^{-1}} \quad \text{since } \lambda_1(\Lambda) = 1.$$

Therefore, since $\mathrm{id}_{H_{\alpha^{-1}}}$ is invertible in $\mathrm{Conv}(H, H_{\alpha^{-1}})$ with inverse S_α, we have that $f = S_\alpha$, that is $S_\alpha(a) = \lambda_\alpha(\Lambda_{(1,\alpha)}a)\Lambda_{(2,\alpha^{-1})}$ for all $a \in H_\alpha$. Part (b) can be deduced from part (a) by using the Hopf π-coalgebra H^{op} (see Section 1.3.1). □

Proof of Theorem 4.2. We use the same arguments as in the proof of [15, Theorem 3], even if we cannot use the duality (since the notion a Hopf π-coalgebra is not self-dual). We can assume that λ is a non-zero right π-integral (otherwise the result is obvious). To show part (a), let $\alpha \in \pi$ and $x, y \in H_\alpha$. Since λ_1 is a non-zero right integral for the Hopf algebra H_1^*, there exists a left integral Λ for H_1 such that $\lambda_1(\Lambda) = \lambda_1(S_1(\Lambda)) = 1$ (cf. [15, Proposition 1]). By Lemma 4.3(b) for $a = S_{\alpha^{-1}}S_\alpha(y \leftharpoonup v)$, we have that

$$S_\alpha(y \leftharpoonup v) = \lambda_\alpha(S_{\alpha^{-1}}S_\alpha(y \leftharpoonup v)\Lambda_{(1,\alpha)})\Lambda_{(2,\alpha^{-1})}. \tag{4.1}$$

It is easy to verify that $(v^{-1}\lambda_\gamma)_{\gamma\in\pi}$ is a right π-integral for H and $\Lambda \leftharpoonup v$ is a right integral for H_1 such that $(v^{-1}\lambda_1)(\Lambda \leftharpoonup v) = 1$. Thus Lemma 4.3(a) for $a = y \leftharpoonup v$ gives that

$$\begin{aligned} S_\alpha(y \leftharpoonup v) &= (v^{-1}\lambda_\alpha)((\Lambda \leftharpoonup v)_{(1,\alpha)}(y \leftharpoonup v))\,(\Lambda \leftharpoonup v)_{(2,\alpha^{-1})} \\ &= (v^{-1}\lambda_\alpha)((\Lambda_{(1,\alpha)}y) \leftharpoonup v)\Lambda_{(2,\alpha^{-1})} \quad \text{by (1.4)} \\ &= \lambda_\alpha(((\Lambda_{(1,\alpha)}y) \leftharpoonup v) \leftharpoonup v^{-1})\Lambda_{(2,\alpha^{-1})} \\ &= \lambda_\alpha((\Lambda_{(1,\alpha)}y) \leftharpoonup \varepsilon)\Lambda_{(2,\alpha^{-1})} \\ &= \lambda_\alpha(\Lambda_{(1,\alpha)}y)\Lambda_{(2,\alpha^{-1})} \quad \text{by (1.2).} \end{aligned}$$

Hence, by comparing with (4.1), we obtain that

$$\lambda_\alpha(\Lambda_{(1,\alpha)}y)\Lambda_{(2,\alpha^{-1})} = \lambda_\alpha(S_{\alpha^{-1}}S_\alpha(y \leftharpoonup v)\Lambda_{(1,\alpha)})\Lambda_{(2,\alpha^{-1})}. \tag{4.2}$$

Now $(\lambda_\gamma S_{\gamma^{-1}})_{\gamma\in\pi}$ is a right π-integral for H^{cop} and Λ is a left integral for H_1^{cop} such that $(\lambda_1 S_1)(\Lambda) = 1$. Thus, applying Lemma 4.3(b) for $a = S_{\alpha^{-1}}^{-1}(x) \in H_\alpha^{\mathrm{cop}}$, we have $(S_{\alpha^{-1}}^{\mathrm{cop}})^{-1}(S_{\alpha^{-1}}^{-1}(x)) = \lambda_\alpha S_{\alpha^{-1}}(S_{\alpha^{-1}}^{-1}(x)\Lambda_{(2,\alpha^{-1})})\Lambda_{(1,\alpha)}$, that is

$$x = \Lambda_{(1,\alpha)}\lambda_\alpha(S_{\alpha^{-1}}(\Lambda_{(2,\alpha^{-1})})x). \tag{4.3}$$

Finally, evaluating (4.2) with $\lambda_\alpha(S_{\alpha^{-1}}(\cdot)x)$ and using (4.3) gives that $\lambda_\alpha(xy) = \lambda_\alpha(S_{\alpha^{-1}}S_\alpha(y \leftharpoonup v)x)$.

To show part (b), let $\alpha \in \pi$ and $a, b \in H_\alpha$. For any $\gamma \in \pi$, let us define $\phi_\gamma \in (H_\gamma^{\mathrm{op,cop}})^*$ by $\phi_\gamma(x) = \lambda_{\gamma^{-1}}(g_{\gamma^{-1}}x)$ for all $x \in H_\gamma^{\mathrm{op,cop}}$. Using Lemma 4.1, one easily checks that $\phi = (\phi_\gamma)_{\gamma\in\pi}$ is a right π-integral for $H^{\mathrm{op,cop}}$. Let us denote by $\times^{\mathrm{op}}$ the multiplication of $H_{\alpha^{-1}}^{\mathrm{op,cop}}$ and by $\leftharpoonup^{\mathrm{cop}}$ the right action of $(H_1^{\mathrm{op,cop}})^*$ on $H_{\alpha^{-1}}^{\mathrm{op,cop}}$ defined by $h \leftharpoonup^{\mathrm{cop}} f = (f \otimes \mathrm{id})\Delta_{1,\alpha^{-1}}^{\mathrm{cop}}(h)$. Then, since v^{-1} is the distinguished grouplike element of $(H_1^{\mathrm{op,cop}})^*$, part (a) with $x = g_\alpha^{-1}b$ and $y = g_\alpha^{-1}ag_\alpha$ gives that $\phi_{\alpha^{-1}}(x \times^{\mathrm{op}} y) = \phi_{\alpha^{-1}}(S_\alpha^{\mathrm{op,cop}}S_{\alpha^{-1}}^{\mathrm{op,cop}}(y \leftharpoonup^{\mathrm{cop}} v^{-1}) \times^{\mathrm{op}} x)$, that is $\lambda_\alpha(ab) = \lambda_\alpha(bS_{\alpha^{-1}}S_\alpha(v^{-1} \rightharpoonup g_\alpha^{-1}ag_\alpha))$.

Let us show part (c). For any $\alpha\in\pi$, define $\phi_\alpha\in H_\alpha^*$ by $\phi_\alpha(x)=\lambda_\alpha(g_\alpha x)$ for all $x\in H_\alpha$. Since $(\phi_\alpha)_{\alpha\in\pi}$ and $(\lambda_{\alpha^{-1}}S_\alpha)_{\alpha\in\pi}$ are left π-integrals for H which are non-zero (because λ is non-zero, g is invertible and S is bijective), there exists $k\in\Bbbk$ such that $\phi_\alpha=k\lambda_{\alpha^{-1}}S_\alpha$ for all $\alpha\in\pi$ (by Theorem 3.6). As above, let Λ be a left integral for H_1 such that $\lambda_1(\Lambda)=\lambda_1(S_1(\Lambda))=1$. Recall that $\varepsilon(g_1)=1$. Then $1=\lambda_1(\Lambda)=\lambda_1(\varepsilon(g_1)\Lambda)=\lambda_1(g_1\Lambda)=k\lambda_1(S_1(\Lambda))=k$. Hence $\lambda_{\alpha^{-1}}S_\alpha=\phi_\alpha$ for all $\alpha\in\pi$, that is $\lambda_{\alpha^{-1}}(S_\alpha(x))=\lambda_\alpha(g_\alpha x)$ for all $\alpha\in\pi$ and $x\in H_\alpha$. This completes the proof of the theorem. □

The following corollary will be used later to relate the distinguished grouplike element of a finite type quasitriangular Hopf π-coalgebra to the R-matrix.

Corollary 4.4. *Let Λ be a left integral for H_1 and $g=(g_\alpha)_{\alpha\in\pi}$ be the distinguished π-grouplike element of H. Then, for all $\alpha\in\pi$,*

$$\Lambda_{(1,\alpha)}\otimes\Lambda_{(2,\alpha^{-1})}=S_{\alpha^{-1}}S_\alpha(\Lambda_{(2,\alpha)})g_\alpha\otimes\Lambda_{(1,\alpha^{-1})}.$$

Proof. We can suppose that $\Lambda\neq 0$. Let $\alpha\in\pi$. Remark that it suffices to show that, for all $f\in H_{\alpha^{-1}}^*$,

$$f(\Lambda_{(2,\alpha^{-1})})\Lambda_{(1,\alpha)}=f(\Lambda_{(1,\alpha^{-1})})S_{\alpha^{-1}}S_\alpha(\Lambda_{(2,\alpha)})g_\alpha. \tag{4.4}$$

Fix $f\in H_{\alpha^{-1}}^*$. Let $\lambda=(\lambda_\gamma)_{\gamma\in\pi}$ be a non-zero right π-integral for H (see Theorem 3.6). By multiplying λ by some (non-zero) scalar, we can assume that $\lambda_1(\Lambda)=\lambda_1(S_1(\Lambda))=1$. By Corollary 3.7(b), there exists $a\in H_{\alpha^{-1}}$ such that $f(x)=\lambda_{\alpha^{-1}}(ax)$ for all $x\in H_{\alpha^{-1}}$. By Lemma 4.3(b), $S_{\alpha^{-1}}(a)=\lambda_{\alpha^{-1}}(a\Lambda_{(1,\alpha^{-1})})S_{\alpha^{-1}}S_\alpha(\Lambda_{(2,\alpha)})$. Thus

$$S_{\alpha^{-1}}(a)g_\alpha=f(\Lambda_{(1,\alpha^{-1})})S_{\alpha^{-1}}S_\alpha(\Lambda_{(2,\alpha)})g_\alpha. \tag{4.5}$$

Since $(\lambda_\gamma S_{\gamma^{-1}})_{\gamma\in\pi}$ is a right π-integral for $H^{\mathrm{op,cop}}$ and Λ is a right integral for $H_1^{\mathrm{op,cop}}$ such that $(\lambda_1S_1)(\Lambda)=1$, Lemma 4.3(a) applied to $H^{\mathrm{op,cop}}$ gives that

$$S_{\alpha^{-1}}(a)=\lambda_\alpha S_{\alpha^{-1}}(a\Lambda_{(2,\alpha^{-1})})\Lambda_{(1,\alpha)}.$$

Then, by using Theorem 4.2(c), we get

$$\begin{aligned}S_{\alpha^{-1}}(a)g_\alpha&=\lambda_\alpha S_{\alpha^{-1}}(a\Lambda_{(2,\alpha^{-1})})\Lambda_{(1,\alpha)}g_\alpha\\&=\lambda_{\alpha^{-1}}(g_{\alpha^{-1}}a\Lambda_{(2,\alpha^{-1})})\Lambda_{(1,\alpha)}g_\alpha.\end{aligned} \tag{4.6}$$

Now, since Λ is left integral for H_1,

$$\Lambda_{(1,\alpha)}g_\alpha\otimes\Lambda_{(2,\alpha^{-1})}g_{\alpha^{-1}}=\Delta_{\alpha,\alpha^{-1}}(\Lambda g_1)=v(g_1)\Lambda_{(1,\alpha)}\otimes\Lambda_{(2,\alpha^{-1})}.$$

Therefore,

$$\Lambda_{(1,\alpha)}g_\alpha\otimes g_{\alpha^{-1}}a\Lambda_{(2,\alpha^{-1})}=\Lambda_{(1,\alpha)}\otimes v(g_1)g_{\alpha^{-1}}a\Lambda_{(2,\alpha^{-1})}g_{\alpha^{-1}}^{-1}$$

and so, using (4.6) and then Theorem 4.2(a),

$$\begin{aligned}S_{\alpha^{-1}}(a)g_\alpha&=\lambda_{\alpha^{-1}}(v(g_1)g_{\alpha^{-1}}a\Lambda_{(2,\alpha^{-1})}g_{\alpha^{-1}}^{-1})\Lambda_{(1,\alpha)}\\&=\lambda_{\alpha^{-1}}(v(g_1)S_\alpha S_{\alpha^{-1}}(g_{\alpha^{-1}}^{-1}\leftharpoonup v)g_{\alpha^{-1}}a\Lambda_{(2,\alpha^{-1})})\Lambda_{(1,\alpha)}.\end{aligned}$$

Now $S_\alpha S_{\alpha^{-1}}(g_{\alpha^{-1}}^{-1} \leftharpoonup v) = v(g_1)^{-1} g_{\alpha^{-1}}^{-1}$ since $g^{-1} = (g_\beta^{-1} = S_{\beta^{-1}}(g_{\beta^{-1}}))_{\beta\in\pi}$ is a π-grouplike element and v is an algebra morphism. Thus

$$S_{\alpha^{-1}}(a)g_\alpha = \lambda_{\alpha^{-1}}(a\Lambda_{(2,\alpha^{-1})})\Lambda_{(1,\alpha)} = f(\Lambda_{(2,\alpha^{-1})})\Lambda_{(1,\alpha)}.$$

Finally, by comparing the last equation with (4.5), we get (4.4). □

4.3. The order of the antipode

It is known that the order of the antipode of a finite-dimensional Hopf algebra A is finite (by Radford [13, Theorem 1]) and divides $4\dim A$ (by Nichols and Zoeller [11, Proposition 3.1]). Let us apply this result to the setting of a Hopf π-coalgebra.

Let $H = \{H_\alpha\}_{\alpha\in\pi}$ be a finite-type Hopf π-coalgebra with antipode $S = \{S_\alpha\}_{\alpha\in\pi}$. Let $\alpha\in\pi$ of finite order d and denote by $\langle\alpha\rangle$ the subgroup of π generated by α. By considering the (finite-dimensional) Hopf algebra $\oplus_{\beta\in\langle\alpha\rangle} H_\beta$ (coming from the Hopf $\langle\alpha\rangle$-coalgebra $\{H_\beta\}_{\beta\in\langle\alpha\rangle}$, as in Section 1.3.5), we obtain that the order of $S_{\alpha^{-1}}S_\alpha \in \mathrm{Aut}_{\mathrm{Alg}}(H_\alpha)$ is finite and divides $2\sum_{\beta\in\langle\alpha\rangle}\dim H_\beta$. As a corollary of Theorem 4.2, we give another upper bound for the order of $S_{\alpha^{-1}}S_\alpha$.

Corollary 4.5. *Let $H = \{H_\alpha\}_{\alpha\in\pi}$ be a finite type Hopf π-coalgebra with antipode $S = \{S_\alpha\}_{\alpha\in\pi}$. Then*
(a) *If $\alpha\in\pi$ has a finite order d, then $(S_{\alpha^{-1}}S_\alpha)^{2d\dim H_1} = \mathrm{id}_{H_\alpha}$.*
(b) *If $\alpha\in\pi$ has order 2, then $S_\alpha^{8\dim H_1} = \mathrm{id}_{H_\alpha}$.*

Before proving Corollary 4.5, we establish the following lemma.

Lemma 4.6. *Let H be a finite-type Hopf π-coalgebra, $g = (g_\alpha)_{\alpha\in\pi}$ be the distinguished π-grouplike element of H, and v be the distinguished grouplike element of H_1^*. Then $(S_{\alpha^{-1}}S_\alpha)^2(x) = g_\alpha(v \rightharpoonup x \leftharpoonup v^{-1})g_\alpha^{-1}$ for all $\alpha\in\pi$ and $x\in H_\alpha$.*

Proof. Let $\alpha\in\pi$ and $x, y\in H_\alpha$. If $H_\alpha = 0$, then the result is obvious. Let us suppose that $H_\alpha \neq 0$. Let $\lambda = (\lambda_\gamma)_{\gamma\in\pi}$ be a non-zero right π-integral for H. Then

$$\begin{aligned}
&\lambda_\alpha(g_\alpha(v \rightharpoonup x \leftharpoonup v^{-1})g_\alpha^{-1} y)\\
&\quad = \lambda_\alpha(yS_{\alpha^{-1}}S_\alpha(v^{-1} \rightharpoonup g_\alpha^{-1} g_\alpha(v \rightharpoonup x \leftharpoonup v^{-1})g_\alpha^{-1} g_\alpha)) \quad \text{by Theorem 4.2(b)}\\
&\quad = \lambda_\alpha(yS_{\alpha^{-1}}S_\alpha(x \leftharpoonup v^{-1}))\\
&\quad = \lambda_\alpha(S_{\alpha^{-1}}S_\alpha(S_{\alpha^{-1}}S_\alpha(x \leftharpoonup v^{-1}) \leftharpoonup v)y) \quad \text{by Theorem 4.2(a)}\\
&\quad = \lambda_\alpha((S_{\alpha^{-1}}S_\alpha)^2(x \leftharpoonup v^{-1} \leftharpoonup v)y) \quad \text{since } S_{\alpha^{-1}}S_\alpha \text{ is comultiplicative}\\
&\quad = \lambda_\alpha((S_{\alpha^{-1}}S_\alpha)^2(x \leftharpoonup \varepsilon)y)\\
&\quad = \lambda_\alpha((S_{\alpha^{-1}}S_\alpha)^2(x)y).
\end{aligned}$$

Now, by Corollary 3.7(b), H_α^* is a free right H_α-module of rank 1 for the action defined by $(f \triangleleft a)(x) = f(ax)$ for any $f \in H_\alpha^*$ and $a, x \in H_\alpha$, and λ_α is a basis vector of $(H_\alpha^*, \triangleleft)$. Thus, since the above computation says that

$$\lambda_\alpha \triangleleft g_\alpha(v \rightharpoonup x \leftharpoonup v^{-1})g_\alpha^{-1} = \lambda_\alpha \triangleleft (S_{\alpha^{-1}}S_\alpha)^2(x),$$

we conclude that $(S_{\alpha^{-1}}S_\alpha)^2(x) = g_\alpha(v \rightharpoonup x \leftharpoonup v^{-1})g_\alpha^{-1}$. □

Proof of Corollary 4.5. To show part (a), let $\alpha \in \pi$ of finite order d. Consider the distinguished π-grouplike element $g = (g_\alpha)_{\alpha\in\pi}$ of H and the distinguished grouplike element v of H_1^*. Using Lemma 4.6, one easily shows by induction that, for all $x \in H_\alpha$ and $l \in \mathbb{N}$,

$$(S_{\alpha^{-1}}S_\alpha)^{2l}(x) = g_\alpha^l(v^l \rightharpoonup x \leftharpoonup v^{-l})g_\alpha^{-l}. \tag{4.7}$$

Recall that the order of a grouplike element of a finite-dimensional Hopf algebra A is finite and divides $\dim A$ (see [11, Theorem 2.2]). Therefore g_1 has a finite order which divides $\dim H_1$ and v has a finite order which divides $\dim H_1^* = \dim H_1$. Now, since $\alpha^d = 1$ and $(g_\beta^{\dim H_1})_{\beta\in\pi} \in G(H)$,

$$g_\alpha^{d \dim H_1} = (g_1^{\dim H_1})_{(1,\alpha)} \cdots (g_1^{\dim H_1})_{(d,\alpha)} = 1_{1(1,\alpha)} \cdots 1_{1(d,\alpha)} = 1_\alpha^d = 1_\alpha.$$

Then, for all $x \in H_\alpha$, by (4.7),

$$\begin{aligned}(S_{\alpha^{-1}}S_\alpha)^{2d \dim H_1}(x) &= g_\alpha^{d \dim H_1}(v^{d \dim H_1} \rightharpoonup x \leftharpoonup v^{-d \dim H_1})g_\alpha^{-d \dim H_1}\\ &= 1_\alpha(\varepsilon \rightharpoonup x \leftharpoonup \varepsilon)1_\alpha = x.\end{aligned}$$

Hence $(S_{\alpha^{-1}}S_\alpha)^{2d \dim H_1} = \mathrm{id}_{H_\alpha}$. Part (b) is part (a) for $d = 2$, since in this case S_α is an endomorphism of H_α. □

5. Semisimplicity and cosemisimplicity

In this section, we define the semisimplicity and the cosemisimplicity for Hopf π-coalgebras, and we give criteria for a Hopf π-coalgebra to be semisimple (resp. cosemisimple).

5.1. Semisimple Hopf π-coalgebras

A Hopf π-coalgebra $H = \{H_\alpha\}_{\alpha\in\pi}$ is said to be *semisimple* if each algebra H_α is semisimple.

Note that, since any infinite-dimensional Hopf algebra (over a field) is never semisimple (see [18, Corollary 2.7]), a necessary condition for H to be semisimple is that H_1 is finite dimensional.

Lemma 5.1. *Let $H = \{H_\alpha\}_{\alpha\in\pi}$ be a finite type Hopf π-coalgebra. Then H is semisimple if and only if H_1 is semisimple.*

Proof. We have to show that if H_1 is semisimple then H is semisimple. Suppose that H_1 is semisimple and fix $\alpha \in \pi$. Since H_α is a finite-dimensional algebra, it suffices to show that all left H_α-modules are completely reducible. Thus let M be a left H_α-module and N be a submodule of M. Since H_1 is a finite-dimensional semisimple Hopf algebra, there exists a left integral Λ for H_1 such that $\varepsilon(\Lambda)=1$ (cf. [17, Theorem 5.1.8]). Let $p: M \to N$ by any $\Bbbk$-linear projection which is the identity on N. Let $P: M \to N$ be the $\Bbbk$-linear map defined, for any $m \in M$, by

$$P(m) = \Lambda_{(1,\alpha)} \cdot p(S_{\alpha^{-1}}(\Lambda_{(2,\alpha^{-1})}) \cdot m),$$

where $\cdot$ denotes the action of H_α on M. The map P is the identify on N since, for any $n \in N$,

$$\begin{aligned} P(n) &= \Lambda_{(1,\alpha)} \cdot p(S_{\alpha^{-1}}(\Lambda_{(2,\alpha^{-1})}) \cdot n) = \Lambda_{(1,\alpha)} \cdot (S_{\alpha^{-1}}(\Lambda_{(2,\alpha^{-1})}) \cdot n) \\ &= (\Lambda_{(1,\alpha)} S_{\alpha^{-1}}(\Lambda_{(2,\alpha^{-1})})) \cdot n = \varepsilon(\Lambda) 1_\alpha \cdot n = n. \end{aligned}$$

Let $h \in H_\alpha$. Using (1.2) and the fact that Λ is a left integral for H_1, we have

$$\begin{aligned} \Lambda_{(1,\alpha)} \otimes \Lambda_{(2,\alpha^{-1})} \otimes h &= \Delta_{\alpha,\alpha^{-1}}(\varepsilon(h_{(1,1)})\Lambda) \otimes h_{(2,\alpha)} \\ &= \Delta_{\alpha,\alpha^{-1}}(h_{(1,1)}\Lambda) \otimes h_{(2,\alpha)} \\ &= h_{(1,\alpha)}\Lambda_{(1,\alpha)} \otimes h_{(2,\alpha^{-1})}\Lambda_{(2,\alpha^{-1})} \otimes h_{(3,\alpha)} \end{aligned}$$

and so

$$\begin{aligned} &\Lambda_{(1,\alpha)} \otimes S_{\alpha^{-1}}(\Lambda_{(2,\alpha^{-1})})h \\ &\quad = h_{(1,\alpha)}\Lambda_{(1,\alpha)} \otimes S_{\alpha^{-1}}(h_{(2,\alpha^{-1})}\Lambda_{(2,\alpha^{-1})})h_{(3,\alpha)} \\ &\quad = h_{(1,\alpha)}\Lambda_{(1,\alpha)} \otimes S_{\alpha^{-1}}(\Lambda_{(2,\alpha^{-1})})S_{\alpha^{-1}}(h_{(2,\alpha^{-1})})h_{(3,\alpha)} \quad \text{by Lemma 1.1(c)} \\ &\quad = h_{(1,\alpha)}\varepsilon(h_{(2,1)})\Lambda_{(1,\alpha)} \otimes S_{\alpha^{-1}}(\Lambda_{(2,\alpha^{-1})})1_\alpha \quad \text{by (1.5)} \\ &\quad = h\Lambda_{(1,\alpha)} \otimes S_{\alpha^{-1}}(\Lambda_{(2,\alpha^{-1})}) \quad \text{by (1.2).} \end{aligned}$$

Therefore, for all $h \in H_\alpha$ and $m \in M$,

$$\begin{aligned} P(h \cdot m) &= \Lambda_{(1,\alpha)} \cdot p(S_{\alpha^{-1}}(\Lambda_{(2,\alpha^{-1})})h \cdot m) \\ &= h\Lambda_{(1,\alpha)} \cdot p(S_{\alpha^{-1}}(\Lambda_{(2,\alpha^{-1})}) \cdot m) = h \cdot P(m). \end{aligned}$$

Hence P is H_α-linear and $\ker P$ is a H_α-supplement of N in M. □

5.2. *Cosemisimple π-comodules and π-coalgebras*

Let C be a π-coalgebra and M be a right π-comodule over C. If $\{N^i = \{N^i_\alpha\}_{\alpha\in\pi}\}_{i\in I}$ is a family of π-subcomodules of M, we define their *sum* by $\{\sum_{i\in I} N^i_\alpha\}_{\alpha\in\pi}$. It is easy to see that it is a π-subcomodule of M. We denote it by $\sum_{i\in I} N^i$. This sum is said to be *direct* provided $\sum_{i\in I} N^i_\alpha$ is a direct sum for all $\alpha \in \pi$. In this case $\sum_{i\in I} N^i$ will be denoted by $\bigoplus_{i\in I} N^i$.

A right π-comodule $M=\{M_\alpha\}_{\alpha\in\pi}$ is said to be *simple* if it is *non-zero* (i.e., $M_\alpha \neq 0$ for some $\alpha\in\pi$) and if it has no π-subcomodules other than $0=\{0\}_{\alpha\in\pi}$ and itself.

Lemma 5.2. *Let M be a right π-comodule over a π-coalgebra C. The following conditions are equivalent*:
(a) *M is a sum of a family of simple π-subcomodules*;
(b) *M is a direct sum of a family of simple π-subcomodules*;
(c) *every π-subcomodule N of M is a direct summand, i.e., there exists a π-subcomodule N' of M such that $M=N\oplus N'$.*

Proof. Let us show condition (a) $\Rightarrow$ condition (b). Suppose that $M=\sum_{i\in I} M^i$ is a sum of simple π-submodules. Let J be a maximal subset of I such that $\sum_{j\in J} M^j$ is direct. Let us show that this sum is in fact equal to M. It suffices to prove that each M^i $(i\in I)$ is contained in this sum. The intersection of our sum with M^i is a π-subcomodule of M^i, thus equal to 0 or M^i. If it is equal to 0, then J is not maximal since we can adjoin i to it. Hence M^i is contained in the sum.

To show condition (b) $\Rightarrow$ condition (c), suppose that $M=\bigoplus_{i\in I} M^i$ and let N be a π-subcomodule of M. Let J be a maximal subset of I such that the sum $N+\bigoplus_{j\in J} M^j$ is direct. The same reasoning as before shows this sum is equal to M.

Let us show condition (c) $\Rightarrow$ condition (a). Let N be the π-subcomodule of M defined as the sum of all simple π-subcomodules of M. Suppose that $M\neq N$. Then $M=N\oplus F$ where F is a non-zero π-subcomodule of M. Let us show that there exists a simple π-subcomodule of F, contradicting the definition of N. By Theorem 2.2(a), $\bar{F}=\bigoplus_{\alpha\in\pi} \bar{F}_\alpha$ (where $\bar{F}_\alpha = F_{\alpha^{-1}}$) is a rational π-graded left C^*-module which is non-zero. Let $v\in\bar{F}$, $v\neq 0$. The kernel of the morphism of π-graded left C^*-modules $C^*\to C^*v$ is a π-graded left ideal $J\neq C^*$. Therefore, J is contained in a maximal π-graded left ideal $I\neq C^*$ (by Zorn's lemma). Then I/J is a maximal π-graded left C^*-submodule of C^*/J (not equal to C^*/J), and hence Iv is a maximal π-graded C^*-submodule of C^*v, not equal to C^*v (corresponding to I/J under the π-graded isomorphism $C^*/J\to C^*v$). Moreover, it is rational since it is a submodule of the rational module $\bar{F}$ (see Theorem 2.2(b)). So we can consider the π-subcomodule $\overline{Iv}$ of M (see Lemma 2.4). Write then $M=\overline{Iv}\oplus L$ where L is π-subcomodule of M. Therefore $\bar{M}=Iv\oplus\bar{L}$ and so $C^*v=Iv\oplus(\bar{L}\cap C^*v)$. Now, since Iv is a maximal π-graded C^*-submodule of C^*v (not equal to C^*v), we have that $\bar{L}\cap C^*v$ is a non-zero π-graded C^*-submodule of $\bar{F}$ which does not contain any π-graded submodule other than 0 and itself. Moreover $\bar{L}\cap C^*v$ is rational since it is a π-graded C^*-submodule of the rational π-graded C^*-module $\bar{F}$ (see Theorem 2.2(b)). Finally $\overline{\bar{L}\cap C^*v}$ is a simple π-subcomodule of F. $\square$

A right π-comodule satisfying the equivalent conditions of Lemma 5.2 is said to be *cosemisimple*. A π-coalgebra is called *cosemisimple* if it is cosemisimple as a right π-comodule over itself (with comultiplication as structure maps).

When $\pi=1$, one recovers the usual notions of cosemisimple comodules and coalgebras.

When π is finite, a π-coalgebra $C=\{C_\alpha\}_{\alpha\in\pi}$ is cosemisimple if and only if the π-graded coalgebra $\tilde{C}=\bigoplus_{\alpha\in\pi} C_\alpha$ (defined as in Section 3.5) is *graded-cosemisimple* (i.e., is a direct sum of simple π-graded right comodules).

Lemma 5.3. *Every π-subcomodule or quotient of a cosemisimple right π-comodule is cosemisimple.*

Proof. Let N be a π-subcomodule of a cosemisimple right π-comodule M. Let F be the sum of all simple π-subcomodules of N and write $M=F\oplus F'$. Therefore $N=F\oplus(F'\cap N)$. If $F'\cap N\neq 0$, it contains a simple π-subcomodule (see the demonstration of Lemma 5.2). Thus $F'\cap N=0$ and $N=F$, which is cosemisimple. Now write $M=N\oplus N'$. N' is a sum of simple π-subcomodules (it is a π-subcomodule of M and thus cosemisimple) and the canonical projection $M\to M/N$ induces a π-comodule isomorphism between N' onto M/N. Hence M/N is cosemisimple. $\square$

5.3. Cosemisimple Hopf π-coalgebras

A Hopf π-coalgebra $H=\{H_\alpha\}_{\alpha\in\pi}$ is said to be *cosemisimple* if it is cosemisimple as a π-coalgebra. A right π-comodule $M=\{M_\alpha\}_{\alpha\in\pi}$ over H is said to be *reduced* if, for all $\alpha\in\pi$, $M_\alpha=0$ whenever $H_\alpha=0$.

The next theorem is the Hopf π-coalgebra version of the dual Maschke theorem (see [17, Section 14.0.3]).

Theorem 5.4. *Let H be a Hopf π-coalgebra. The following conditions are equivalent*:
(a) *every reduced right π-comodule over H is cosemisimple*;
(b) *H is cosemisimple*;
(c) *there exists a right π-integral $\lambda=(\lambda_\alpha)_{\alpha\in\pi}$ for H such that $\lambda_\alpha(1_\alpha)=1$ for some $\alpha\in\pi$*;
(d) *there exists a right π-integral $\lambda=(\lambda_\alpha)_{\alpha\in\pi}$ for H such that $\lambda_\alpha(1_\alpha)=1$ for all $\alpha\in\pi$ with $H_\alpha\neq 0$.*

Proof. Condition (a) implies trivially condition (b). Moreover condition (c) is equivalent to condition (d). Indeed condition (d) implies condition (c) since $H_1\neq 0$ (by Corollary 1.2). Conversely, suppose that $\beta\in\pi$ is such that $\lambda_\beta(1_\beta)=1$. Let $\alpha\in\pi$ such that $H_\alpha\neq 0$. Then $\lambda_\alpha(1_\alpha)1_{\beta^{-1}\alpha}=(\lambda_\beta\otimes \mathrm{id}_{H_{\beta^{-1}\alpha}})\Delta_{\beta,\beta^{-1}\alpha}(1_\alpha)=\lambda_\beta(1_\beta)1_{\beta^{-1}\alpha}=1_{\beta^{-1}\alpha}$. Now $1_{\beta^{-1}\alpha}\neq 0$ by Corollary 1.2. Hence $\lambda_\alpha(1_\alpha)=1$.

Let us show that condition (b) implies condition (d). Consider H as a right π-comodule over itself (with comultiplication as structure maps). For any $\alpha\in\pi$, set $N_\alpha=\mathbb{k}1_\alpha$. Since the comultiplication is unitary, N is a π-subcomodule of H. Therefore N is a direct summand of H (since H is cosemisimple), that is there exists a π-comodule morphism $p=\{p_\alpha\}_{\alpha\in\pi}:H\to N$ such that $p_\alpha|_{N_\alpha}=\mathrm{id}_{N_\alpha}$ for all $\alpha\in\pi$. For any $\alpha\in\pi$, since $N_\alpha=\mathbb{k}1_\alpha$, there exists a (unique) $\mathbb{k}$-form $\lambda_\alpha\in H_\alpha^*$ such that $p_\alpha(h)=\lambda_\alpha(h)1_\alpha$ for all $h\in H_\alpha$. Let us verify that $\lambda=(\lambda_\alpha)_{\alpha\in\pi}$ is a right π-integral for H. Let $\alpha,\beta\in\pi$. Since

p is a π-comodule morphism, we have that

$$\lambda_{\alpha\beta}1_\alpha \otimes 1_\beta = \Delta_{\alpha,\beta}\, p_{\alpha\beta} = (\lambda_\alpha 1_\alpha \otimes \mathrm{id}_{H_\beta})\Delta_{\alpha,\beta}. \tag{5.1}$$

If $H_\alpha = 0$, then either $H_\beta = 0$ or $H_{\alpha\beta} = 0$ (by Corollary 1.2) and so $\lambda_{\alpha\beta}1_\beta = 0 = (\lambda_\alpha \otimes \mathrm{id}_{H_\beta})\Delta_{\alpha\beta}$. If $H_\alpha \neq 0$, then there exists $f \in H_\alpha^*$ such that $f(1_\alpha) = 1$ and, by applying $(f \otimes \mathrm{id}_{H_\beta})$ to both sides of (5.1), we get that $\lambda_{\alpha\beta}1_\beta = (\lambda_\alpha \otimes \mathrm{id}_{H_\beta})\Delta_{\alpha,\beta}$. Therefore λ is a right π-integral for H. Finally, let $\alpha \in \pi$ such that $H_\alpha \neq 0$. Then $\lambda_\alpha(1_\alpha)1_\alpha = p_\alpha(1_\alpha) = 1_\alpha$ (since $1_\alpha \in N_\alpha$) and so $\lambda_\alpha(1_\alpha) = 1$ (since $1_\alpha \neq 0$).

To show that condition (d) implies condition (a), let $M = \{M_\alpha\}_{\alpha\in\pi}$ be a reduced right π-comodule over H with structure maps by $\rho = \{\rho_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$ and $N = \{N_\alpha\}_{\alpha\in\pi}$ be a π-subcomodule of M. We have to show that N is a direct summand of M (see Lemma 5.2). Define $\delta_\alpha : H_{\alpha^{-1}} \otimes H_\alpha \to \Bbbk$ by $\delta_\alpha(x \otimes y) = \lambda_\alpha(S_{\alpha^{-1}}(x)y)$ for all $\alpha \in \pi$. We first prove that, for any $\alpha, \beta, \gamma \in \pi$,

$$(\mathrm{id}_{H_\beta} \otimes \delta_{\alpha\beta})(\Delta_{\beta,(\alpha\beta)^{-1}} \otimes \mathrm{id}_{H_{\alpha\beta}}) = (\delta_\alpha \otimes \mathrm{id}_{H_\beta})(\mathrm{id}_{H_{\alpha^{-1}}} \otimes \Delta_{\alpha,\beta}). \tag{5.2}$$

Indeed, for any $x \in H_{\alpha^{-1}}$ and $y \in H_{\alpha\beta}$,

$$\begin{aligned}
&(\mathrm{id}_{H_\beta} \otimes \delta_{\alpha\beta})(\Delta_{\beta,(\alpha\beta)^{-1}} \otimes \mathrm{id}_{H_{\alpha\beta}})(x \otimes y)\\
&\quad = x_{(1,\beta)}\lambda_{\alpha\beta}(S_{(\alpha\beta)^{-1}}(x_{(2,(\alpha\beta)^{-1})})y)\\
&\quad = x_{(1,\beta)}(\lambda_\alpha \otimes \mathrm{id}_{H_\beta})\Delta_{\alpha,\beta}(S_{(\alpha\beta)^{-1}}(x_{(2,(\alpha\beta)^{-1})})y) \quad \text{by (3.1)}\\
&\quad = x_{(1,\beta)}S_{\beta^{-1}}(x_{(2,\beta^{-1})})y_{(2,\beta)}\lambda_\alpha(S_{\alpha^{-1}}(x_{(3,\alpha^{-1})})y_{(1,\alpha)}) \quad \text{by Lemma 1.1(c)}\\
&\quad = y_{(2,\beta)}\lambda_\alpha(S_{\alpha^{-1}}(\varepsilon(x_{(1,1)})x_{(2,\alpha^{-1})})y_{(1,\alpha)}) \quad \text{by (1.5)}\\
&\quad = \lambda_\alpha(S_{\alpha^{-1}}(x)y_{(1,\alpha)})y_{(2,\beta)} \quad \text{by (1.2)}\\
&\quad = (\delta_\alpha \otimes \mathrm{id}_{H_\beta})(\mathrm{id}_{H_{\alpha^{-1}}} \otimes \Delta_{\alpha,\beta})(x \otimes y).
\end{aligned}$$

Let $q : M_1 \to N_1$ be any $\Bbbk$-linear projection which is the identity on N_1. Define, for all $\alpha \in \pi$,

$$p_\alpha = (\mathrm{id}_{N_\alpha} \otimes \delta_\alpha)(\rho_{\alpha,\alpha^{-1}} \circ q \otimes \mathrm{id}_{H_\alpha})\rho_{1,\alpha} : M_\alpha \to N_\alpha.$$

For any $\alpha, \beta \in \pi$, using (2.1) and (5.2), we have

$$\begin{aligned}
&\rho_{\alpha,\beta}\, p_{\alpha\beta}\\
&\quad = \rho_{\alpha,\beta}(\mathrm{id}_{N_{\alpha\beta}} \otimes \delta_{\alpha\beta})(\rho_{\alpha\beta,(\alpha\beta)^{-1}} \circ q \otimes \mathrm{id}_{H_{\alpha\beta}})\rho_{1,\alpha\beta}\\
&\quad = (\mathrm{id}_{N_\alpha} \otimes \mathrm{id}_{H_\beta} \otimes \delta_{\alpha\beta})((\rho_{\alpha,\beta} \otimes \mathrm{id}_{H_{(\alpha\beta)^{-1}}})\rho_{\alpha\beta,(\alpha\beta)^{-1}} \circ q \otimes \mathrm{id}_{H_{\alpha\beta}})\rho_{1,\alpha\beta}\\
&\quad = (\mathrm{id}_{N_\alpha} \otimes \mathrm{id}_{H_\beta} \otimes \delta_{\alpha\beta})((\mathrm{id}_{N_\alpha} \otimes \Delta_{\beta,(\alpha\beta)^{-1}})\rho_{\alpha,\alpha^{-1}} \circ q \otimes \mathrm{id}_{H_{\alpha\beta}})\rho_{1,\alpha\beta}\\
&\quad = (\mathrm{id}_{N_\alpha} \otimes (\mathrm{id}_{H_\beta} \otimes \delta_{\alpha\beta})(\Delta_{\beta,(\alpha\beta)^{-1}} \otimes \mathrm{id}_{H_{\alpha\beta}}))(\rho_{\alpha,\alpha^{-1}} \circ q \otimes \mathrm{id}_{H_{\alpha\beta}})\rho_{1,\alpha\beta}\\
&\quad = (\mathrm{id}_{N_\alpha} \otimes (\delta_\alpha \otimes \mathrm{id}_{H_\beta})(\mathrm{id}_{H_{\alpha^{-1}}} \otimes \Delta_{\alpha,\beta}))(\rho_{\alpha,\alpha^{-1}} \circ q \otimes \mathrm{id}_{H_{\alpha\beta}})\rho_{1,\alpha\beta}\\
&\quad = (\mathrm{id}_{N_\alpha} \otimes \delta_\alpha \otimes \mathrm{id}_{H_\beta})(\rho_{\alpha,\alpha^{-1}} \circ q \otimes \mathrm{id}_{H_\alpha} \otimes \mathrm{id}_{H_\beta})(\mathrm{id}_{M_1} \otimes \Delta_{\alpha,\beta})\rho_{1,\alpha\beta}
\end{aligned}$$

$$= (\mathrm{id}_{N_\alpha} \otimes \delta_\alpha \otimes \mathrm{id}_{H_\beta})(\rho_{\alpha,\alpha^{-1}} \circ q \otimes \mathrm{id}_{H_\alpha} \otimes \mathrm{id}_{H_\beta})(\rho_{1,\alpha} \otimes \mathrm{id}_{H_\beta})\rho_{\alpha,\beta}$$

$$= (p_\alpha \otimes \mathrm{id}_{H_\beta})\rho_{\alpha,\beta}.$$

Thus $p = \{p_\alpha\}_{\alpha\in\pi}$ is a π-comodule morphism between M and N. Let $\alpha \in \pi$ and $n \in N_\alpha$. If $H_\alpha = 0$, then $N_\alpha = 0$ (since M and thus N is reduced) and so $p_\alpha(n) = 0 = n$. If $H_\alpha \neq 0$, then

$$p_\alpha(n) = n_{(0,\alpha)}\lambda_\alpha(S_{\alpha^{-1}}(n_{(1,\alpha^{-1})})n_{(2,\alpha)}) \quad \text{since } q|_{N_1} = \mathrm{id}_{N_1}$$

$$= n_{(0,\alpha)}\varepsilon(n_{(1,1)})\lambda_\alpha(1_\alpha) \quad \text{by (1.5)}$$

$$= n \quad \text{by (2.2) and since } \lambda_\alpha(1_\alpha) = 1.$$

Therefore, q is a π-comodule projection of M onto N and consequently N is a direct summand of M (namely $M = N \oplus \ker q$). This finishes the proof of the theorem. □

Corollary 5.5. *Let H be a Hopf π-coalgebra. Then*
(a) *if H is cosemisimple, then the Hopf algebra H_1 is cosemisimple*;
(b) *if H is of finite type, then H is cosemisimple if and only if H_1 is cosemisimple.*

Proof. To show part (a), suppose that H is cosemisimple. By Theorem 5.4 and Corollary 1.2, there exists a right π-integral $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ for H such that $\lambda_1(1_1) = 1$. Since λ_1 is a right integral for H_1^* such that $\lambda_1(1_1) \neq 0$, H_1 is cosemisimple (by Sweedler [17, Theorem 14.0.3]). Let us show part (b). Suppose that H is of finite type and H_1 is cosemisimple. By Sweedler [17, Theorem 14.0.3], there exists a right integral ϕ for H_1^* such that $\phi(1_1) = 1$. By Theorem 3.6, there exists a non-zero right π-integral $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ for H. In particular, λ_1 is a non-zero right integral for H_1^*. Therefore, since H_1 is a finite dimensional, there exists $k \in \Bbbk$ such that $\phi = k\lambda_1$ (by Sweedler [17, Theorem 5.1.6]). Thus $(k\lambda_\alpha)_{\alpha\in\pi}$ is a right π-integral for H such that $k\lambda_1(1_1) = 1$. Hence H is cosemisimple by Theorem 5.4. □

Corollary 5.6. *Let H be a finite-type Hopf π-coalgebra over a field $\Bbbk$ of characteristic 0. Then H is semisimple if and only if it is cosemisimple.*

Proof. By Lemma 5.1, H is semisimple if and only if H_1 is semisimple, and by Corollary 5.5(b), H is cosemisimple if and only if H_1 is cosemisimple. It is then easy to conclude using the fact that, in characteristic 0, a finite-dimensional Hopf algebra is semisimple if and only if it is cosemisimple (see [8, Theorem 3.3]). □

Corollary 5.7. *Let H be a finite type cosemisimple Hopf π-coalgebra. If $g = (g_\alpha)_{\alpha\in\pi}$ is the distinguished π-grouplike element of H, then $g = 1$ in $G(H)$, i.e., $g_\alpha = 1_\alpha$ for all $\alpha \in \pi$. Consequently, the spaces of left and right π-integrals for H coincide.*

Proof. Let $\alpha \in \pi$. If $H_\alpha = 0$, then $g_\alpha = 0 = 1_\alpha$. Suppose that $H_\alpha \neq 0$. By Theorem 5.4, there exists a right π-integral $\lambda = (\lambda_\gamma)_{\gamma\in\pi}$ for H such that $\lambda_\alpha(1_\alpha) = 1$ and $\lambda_1(1_1) = 1$.

Then $g_\alpha = \lambda_\alpha(1_\alpha) g_\alpha = (\mathrm{id}_{H_\alpha} \otimes \lambda_1)\Delta_{\alpha,1}(1_\alpha) = \lambda_1(1_1)1_\alpha = 1_\alpha$. Moreover, by Theorem 3.6 and Lemma 4.1, the spaces of left and right π-integrals for H coincide. □

6. Quasitriangular Hopf π-coalgebras

In this section, we recall the definitions of crossed, quasitriangular, and ribbon Hopf π-coalgebras given by Turaev [19], and we generalize the main properties of quasitriangular Hopf algebras to the setting of Hopf π-coalgebras.

6.1. Crossed Hopf π-coalgebras

Following [19, Section 11.2], a Hopf π-coalgebra $H = (\{H_\alpha\}, \Delta, \varepsilon, S)$ is said to be *crossed* provided it is endowed with a family $\varphi = \{\varphi_\beta : H_\alpha \to H_{\beta\alpha\beta^{-1}}\}_{\alpha,\beta\in\pi}$ of $\Bbbk$-linear maps (the *crossing*) such that

each $\varphi_\beta : H_\alpha \to H_{\beta\alpha\beta^{-1}}$ is an algebra isomorphism, (6.1)

each φ_β preserves the comultiplication, i.e., for all $\alpha, \beta, \gamma \in \pi$,

$$(\varphi_\beta \otimes \varphi_\beta)\Delta_{\alpha,\gamma} = \Delta_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}}\varphi_\beta, \tag{6.2}$$

each φ_β preserves the counit, i.e., $\varepsilon\varphi_\beta = \varepsilon$, (6.3)

φ is *multiplicative* in the sense that $\varphi_{\beta\beta'} = \varphi_\beta\varphi_{\beta'}$ for all $\beta, \beta' \in \pi$. (6.4)

Lemma 6.1. *Let H be a crossed Hopf π-coalgebra with crossing φ. Then*
(a) $\varphi_{1|H_\alpha} = \mathrm{id}_{H_\alpha}$ *for all* $\alpha \in \pi$;
(b) $\varphi_\beta^{-1} = \varphi_{\beta^{-1}}$ *for all* $\beta \in \pi$;
(c) φ *preserves the antipode, i.e.,* $\varphi_\beta S_\alpha = S_{\beta\alpha\beta^{-1}}\varphi_\beta$ *for all* $\alpha, \beta \in \pi$;
(d) *if* $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ *is a left (resp. right) π-integral for H and $\beta \in \pi$, then* $(\lambda_{\beta\alpha\beta^{-1}}\varphi_\beta)_{\alpha\in\pi}$ *is also a left (resp. right) π-integral for H;*
(e) *if* $g = (g_\alpha)_{\alpha\in\pi}$ *is a π-grouplike element of H and $\beta \in \pi$, then* $(\varphi_\beta(g_{\beta^{-1}\alpha\beta}))_{\alpha\in\pi}$ *is also a π-grouplike element of H.*

Proof. Parts (a), (b), (d) and (e) follow directly from the axioms of a crossing. To show part (c), let $\alpha, \beta \in \pi$. Using the axioms, it is easy to verify that $\varphi_\beta^{-1}S_{\beta\alpha\beta^{-1}}\varphi_\beta * \mathrm{id}_{H_{\alpha^{-1}}} = \varepsilon 1_{\alpha^{-1}}$ in the convolution algebra $\mathrm{Conv}(H, H_{\alpha^{-1}})$ (see Section 1.2). Thus, since S_α is the inverse of $\mathrm{id}_{H_{\alpha^{-1}}}$ in $\mathrm{Conv}(H, H_{\alpha^{-1}})$, we have that $\varphi_\beta^{-1}S_{\beta\alpha\beta^{-1}}\varphi_\beta = S_\alpha$ and so $S_{\beta\alpha\beta^{-1}}\varphi_\beta = \varphi_\beta S_\alpha$. □

Corollary 6.2. *Let H be a finite type crossed Hopf π-coalgebra with crossing φ. Then there exists a unique group homomorphism $\hat{\varphi} : \pi \to \Bbbk^*$ such that if $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ is a left or right π-integral for H, then $\lambda_{\beta\alpha\beta^{-1}}\varphi_\beta = \hat{\varphi}(\beta)\lambda_\alpha$ for all $\alpha, \beta \in \pi$.*

Proof. Let $\lambda=(\lambda_\alpha)_{\alpha\in\pi}$ be a non-zero left π-integral for H. For any $\beta\in\pi$, since $(\lambda_{\beta\alpha\beta^{-1}}\varphi_\beta)_{\alpha\in\pi}$ is a non-zero left π-integral for H (see Lemma 6.1(d)) and by the uniqueness (within scalar multiple) of a left π-integral in the finite-type case (see Theorem 3.6), there exists a unique $\hat{\varphi}(\beta)\in\Bbbk^*$ such that $\lambda_{\beta\alpha\beta^{-1}}\varphi_\beta=\hat{\varphi}(\beta)\lambda_\alpha$ for all $\alpha\in\pi$. Using (6.4) and Lemma 6.1, one verifies that $\hat{\varphi}:\pi\to\Bbbk^*$ is a group homomorphism. Since any left π-integral for H is a scalar multiple of λ, the result holds for any left π-integral. Finally, let $\lambda=(\lambda_\alpha)_{\alpha\in\pi}$ be a right π-integral for H. Since the antipode is bijective (H is of finite type), and using Lemma 6.1(d) and the fact that $(\lambda_{\alpha^{-1}}S_\alpha)_{\alpha\in\pi}$ is a left π-integral for H, we have that, for all $\alpha,\beta\in\pi$, $\lambda_{\beta\alpha\beta^{-1}}\varphi_\beta=\lambda_{\beta\alpha\beta^{-1}}S_{\beta\alpha^{-1}\beta^{-1}}\varphi_\beta S^{-1}_{\alpha^{-1}}=\hat{\varphi}(\beta)\lambda_\alpha S_{\alpha^{-1}}S^{-1}_{\alpha^{-1}}=\hat{\varphi}(\beta)\lambda_\alpha$. □

Lemma 6.3. *Let H be a finite type crossed Hopf π-coalgebra with crossing φ. Let $\hat{\varphi}$ be as in Corollary 6.2. Then, for any $\beta\in\pi$,*

(a) *if Λ is a left or right integral for H_1, then $\varphi_\beta(\Lambda)=\hat{\varphi}(\beta)\Lambda$;*
(b) *if v is the distinguished grouplike element of H_1^*, then $v\varphi_\beta=v$;*
(c) *if $g=(g_\alpha)_{\alpha\in\pi}$ is the distinguished π-grouplike element of H, then $\varphi_\beta(g_\alpha)=g_{\beta\alpha\beta^{-1}}$ for all $\alpha\in\pi$.*

Proof. Let us show part (a). Let Λ be a left integral for H_1. We can assume that $\Lambda\neq0$ (if $\Lambda=0$, then the result is obvious). By Lemma 6.1 and (6.3), $x\varphi_\beta(\Lambda)=\varphi_\beta(\varphi_{\beta^{-1}}(x)\Lambda)=\varphi_\beta(\varepsilon\varphi_{\beta^{-1}}(x)\Lambda)=\varepsilon(x)\varphi_\beta(\Lambda)$ for any $x\in H_1$. Thus $\varphi_\beta(\Lambda)$ is a left integral for H_1. Therefore, since H_1 is finite dimensional and $\Lambda\neq0$, there exists $k\in\Bbbk$ such that $\varphi_\beta(\Lambda)=k\Lambda$. Let $\lambda=(\lambda_\alpha)_{\alpha\in\pi}$ be a non-zero right π-integral for H. We have that $\hat{\varphi}(\beta)\lambda_1(\Lambda)=\lambda_1(\varphi_\beta(\Lambda))=\lambda_1(k\Lambda)=k\lambda_1(\Lambda)$. Now $\lambda_1(\Lambda)\neq0$ (because Λ is a non-zero left-integral for H_1 and λ_1 is a non-zero right integral for H_1^*). Hence $k=\hat{\varphi}(\beta)$ and so $\varphi_\beta(\Lambda)=\hat{\varphi}(\beta)\Lambda$. It can be shown similarly that the result holds if Λ is a right integral for H_1.

Let us show part (b). If Λ is a left integral for H_1, then, for all $x\in H_1$, $\Lambda x=\varphi_{\beta^{-1}}(\varphi_\beta(\Lambda)\varphi_\beta(x))=\varphi_{\beta^{-1}}(v(\varphi_\beta(x))\varphi_\beta(\Lambda))=v\varphi_\beta(x)\Lambda$ (since $\varphi_\beta(\Lambda)$ is a left integral for H_1). Thus, by the uniqueness of the distinguished grouplike element of the Hopf algebra H_1^*, we have that $v\varphi_\beta=v$.

To show part (c), let $\lambda=(\lambda_\alpha)_{\alpha\in\pi}$ be a right π-integral for H. By Lemma 6.1(d), $(\lambda_{\beta^{-1}\alpha\beta}\varphi_{\beta^{-1}})_{\alpha\in\pi}$ is also a right π-integral for H. Then, for any $\alpha,\gamma\in\pi$, using (6.2) and Lemmas 4.1 and 6.1,

$$\begin{aligned}(\mathrm{id}_{H_\alpha}\otimes\lambda_\gamma)\Delta_{\alpha,\gamma}&=\varphi_{\beta^{-1}}(\mathrm{id}_{H_{\beta\alpha\beta^{-1}}}\otimes\lambda_\gamma\varphi_{\beta^{-1}})\Delta_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}}\varphi_\beta\\&=\varphi_{\beta^{-1}}(\lambda_{\alpha\gamma}\varphi_{\beta^{-1}}\varphi_\beta g_{\beta\alpha\beta^{-1}})\\&=\lambda_{\alpha\gamma}\varphi_{\beta^{-1}}(g_{\beta\alpha\beta^{-1}}).\end{aligned}$$

Hence, by the uniqueness of the distinguished π-grouplike element (see Lemma 4.1), we have that $\varphi_{\beta^{-1}}(g_{\beta\alpha\beta^{-1}})=g_\alpha$ and so $\varphi_\beta(g_\alpha)=g_{\beta\alpha\beta^{-1}}$ for all $\alpha\in\pi$. □

6.1.1. The opposite (resp. coopposite) Hopf π-coalgebra

Let H be a crossed Hopf π-coalgebra with crossing φ. If the antipode of H is bijective, then the opposite (resp. coopposite) Hopf π-coalgebra to H (see Sections 1.3.1 and 1.3.2) is crossed with crossing given by $\varphi_\beta^{\mathrm{op}}|_{H_\alpha^{\mathrm{op}}}=\varphi_\beta|_{H_\alpha}$ (resp. $\varphi_\beta^{\mathrm{cop}}|_{H_\alpha^{\mathrm{cop}}}=\varphi_\beta|_{H_{\alpha}^{-1}}$) for all $\alpha,\beta\in\pi$.

6.1.2. The mirror Hopf π-coalgebra

Let $H=(\{H_\alpha\},\Delta,\varepsilon,S,\varphi)$ be a crossed Hopf π-coalgebra. Following [19, Section 11.6], its *mirror* $\bar{H}$ is defined by the following procedure: set $\bar{H}_\alpha=H_{\alpha^{-1}}$ as an algebra, $\bar{\Delta}_{\alpha,\beta}=(\varphi_\beta\otimes \mathrm{id}_{H_{\beta^{-1}}})\Delta_{\beta^{-1}\alpha^{-1}\beta,\beta^{-1}}$, $\bar{\varepsilon}=\varepsilon$, $\bar{S}_\alpha=\varphi_\alpha S_{\alpha^{-1}}$ and $\bar{\varphi}_\beta|_{\bar{H}_\alpha}=\varphi_\beta|_{H_{\alpha^{-1}}}$. It is also a crossed Hopf π-coalgebra.

6.2. Quasitriangular Hopf π-coalgebras

Following [19, Section 11.3], a *quasitriangular* Hopf π-coalgebra is a crossed Hopf π-coalgebra $H=(\{H_\alpha\},\Delta,\varepsilon,S,\varphi)$ endowed with a family $R=\{R_{\alpha,\beta}\in H_\alpha\otimes H_\beta\}_{\alpha,\beta\in\pi}$ of invertible elements (the *R-matrix*) such that

for any $\alpha,\beta\in\pi$ and $x\in H_{\alpha\beta}$,

$$R_{\alpha,\beta}\cdot\Delta_{\alpha,\beta}(x)=\sigma_{\beta,\alpha}(\varphi_{\alpha^{-1}}\otimes \mathrm{id}_{H_\alpha})\Delta_{\alpha\beta\alpha^{-1},\alpha}(x)\cdot R_{\alpha,\beta}, \tag{6.5}$$

where $\sigma_{\beta,\alpha}$ denotes the flip map $H_\beta\otimes H_\alpha\to H_\alpha\otimes H_\beta$;

for any $\alpha,\beta\in\pi$,

$$(\mathrm{id}_{H_\alpha}\otimes\Delta_{\beta,\gamma})(R_{\alpha,\beta\gamma})=(R_{\alpha,\gamma})_{1\beta 3}\cdot(R_{\alpha,\beta})_{12\gamma},$$

$$(\Delta_{\alpha,\beta}\otimes \mathrm{id}_{H_\gamma})(R_{\alpha\beta,\gamma})=[(\mathrm{id}_{H_\alpha}\otimes\varphi_{\beta^{-1}})(R_{\alpha,\beta\gamma\beta^{-1}})]_{1\beta 3}\cdot(R_{\beta,\gamma})_{\alpha 23}, \tag{6.6}$$

where, for $\Bbbk$-spaces P,Q and $r=\sum_j p_j\otimes q_j\in P\otimes Q$, we set $r_{12\gamma}=r\otimes 1_\gamma\in P\otimes Q\otimes H_\gamma$, $r_{\alpha 23}=1_\alpha\otimes r\in H_\alpha\otimes P\otimes Q$ and $r_{1\beta 3}=\sum_j p_j\otimes 1_\beta\otimes q_j\in P\otimes H_\beta\otimes Q$;

the family R is invariant under the crossing, i.e., for any $\alpha,\beta,\gamma\in\pi$,

$$(\varphi_\beta\otimes\varphi_\beta)(R_{\alpha,\gamma})=R_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}}. \tag{6.7}$$

Note that $R_{1,1}$ is a (classical) R-matrix for the Hopf algebra H_1.

When π is abelian and φ is *trivial* (that is $\varphi_\beta|H_\alpha=\mathrm{id}_{H_\alpha}$ for all $\alpha,\beta\in\pi$), one recovers the definition of quasitriangular π-colored Hopf algebra given by Ohtsuki [12].

If π is finite, then an R-matrix for H does not necessarily give rise to a (usual) R-matrix for the Hopf algebra $\tilde{H}=\bigoplus_{\alpha\in\pi}H_\alpha$ (see Section 1.3.5). However, if the group π is finite abelian and if φ is trivial, then $\tilde{R}=\sum_{\alpha,\beta\in\pi}R_{\alpha,\beta}$ is an R-matrix for $\tilde{H}$.

Notation. In the proofs, when we write a component $R_{\alpha,\beta}$ of an R-matrix as $R_{\alpha,\beta}=a_\alpha\otimes b_\beta$, it is to signify that $R_{\alpha,\beta}=\sum_j a_j\otimes b_j$ for some $a_j\in H_\alpha$ and $b_j\in H_\beta$, where j runs over a finite set of indices.

We now generalize the main properties of quasitriangular Hopf algebras (see [3]) to the setting of quasitriangular Hopf π-coalgebras.

Lemma 6.4. *Let $H=(\{H_\alpha\},\Delta,\varepsilon,S,\varphi,R)$ be a quasitriangular Hopf π-coalgebra. Then, for any $\alpha,\beta,\gamma\in\pi$,*
(a) $(\varepsilon\otimes \mathrm{id}_{H_\alpha})(R_{1,\alpha})=1_\alpha=(\mathrm{id}_{H_\alpha}\otimes\varepsilon)(R_{\alpha,1})$;
(b) $(S_{\alpha^{-1}}\varphi_\alpha\otimes \mathrm{id}_{H_\beta})(R_{\alpha^{-1},\beta})=R^{-1}_{\alpha,\beta}$ *and* $(\mathrm{id}_{H_\alpha}\otimes S_\beta)(R^{-1}_{\alpha,\beta})=R_{\alpha,\beta^{-1}}$;
(c) $(S_\alpha\otimes S_\beta)(R_{\alpha,\beta})=(\varphi_\alpha\otimes \mathrm{id}_{H_{\beta^{-1}}})(R_{\alpha^{-1},\beta^{-1}})$;
(d) $(R_{\beta,\gamma})_{\alpha 23}\cdot(R_{\alpha,\gamma})_{1\beta 3}\cdot(R_{\alpha\beta})_{12\gamma}=(R_{\alpha,\beta})_{12\gamma}\cdot[(\mathrm{id}_{H_\alpha}\otimes\varphi_{\beta^{-1}})(R_{\alpha,\beta\gamma\beta^{-1}})]_{1\beta 3}\cdot(R_{\beta,\gamma})_{\alpha 23}$.

Part (d) of Lemma 6.4, which is the Yang-Baxter equality for $R=\{R_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$, first appeared in [19, Section 11.3]. We prove it here for completeness sake.

Proof. Let us show part (a). We have

$$\begin{aligned}R_{1,\alpha}&=(\varepsilon\otimes \mathrm{id}_{H_1}\otimes \mathrm{id}_{H_\alpha})(\Delta_{1,1}\otimes \mathrm{id}_{H_\alpha})(R_{1,\alpha}) \quad \text{by (1.2)}\\ &=(\varepsilon\otimes \mathrm{id}_{H_1}\otimes \mathrm{id}_{H_\alpha})([(\mathrm{id}_{H_1}\otimes\varphi_1)(R_{1,\alpha})]_{1_{1_\pi}3}\cdot(R_{1,\alpha})_{1_\pi 23}) \quad \text{by (6.6)}\\ &=(\varepsilon\otimes \mathrm{id}_{H_1}\otimes \mathrm{id}_{H_\alpha})((R_{1,\alpha})_{1_{1_\pi}3}\cdot(R_{1,\alpha})_{1_\pi 23}) \quad \text{by Lemma 6.1(a)}\\ &=(\varepsilon\otimes \mathrm{id}_{H_1}\otimes \mathrm{id}_{H_\alpha})((R_{1,\alpha})_{1_{1_\pi}3})\cdot(\varepsilon\otimes \mathrm{id}_{H_1}\otimes \mathrm{id}_{H_\alpha})((R_{1,\alpha})_{1_\pi 23}) \quad \text{by (1.4)}\\ &=(1_1\otimes(\varepsilon\otimes \mathrm{id}_{H_\alpha})(R_{1,\alpha}))\cdot R_{1,\alpha}.\end{aligned}$$

Thus $1_1\otimes(\varepsilon\otimes \mathrm{id}_{H_\alpha})(R_{1,\alpha})=1_1\otimes 1_\alpha$ (since $R_{1,\alpha}$ is invertible). By applying $(\varepsilon\otimes \mathrm{id}_{H_\alpha})$ on both sides, we get the first equality of part (a). The second one can be obtained similarly.

To show the first equality of part (b), set

$$\mathscr{E}=(m_\alpha\otimes \mathrm{id}_{H_\beta})(S_{\alpha^{-1}}\otimes \mathrm{id}_{H_\alpha}\otimes \mathrm{id}_{H_\beta})(\Delta_{\alpha^{-1},\alpha}\otimes \mathrm{id}_{H_\beta})(R_{1,\beta}).$$

Let us compute $\mathscr{E}$ in two different ways. On one hand,

$$\begin{aligned}\mathscr{E}&=(m_\alpha\otimes \mathrm{id}_{H_\beta})(S_{\alpha^{-1}}\otimes \mathrm{id}_{H_\alpha}\otimes \mathrm{id}_{H_\beta})\\ &\quad([(\mathrm{id}_{H_{\alpha^{-1}}}\otimes\varphi_{\alpha^{-1}})(R_{\alpha^{-1},\alpha\beta\alpha^{-1}})]_{1\alpha 3}\cdot(R_{\alpha,\beta})_{\alpha^{-1}23}) \quad \text{by (6.6)}\\ &=(S_{\alpha^{-1}}\otimes\varphi_{\alpha^{-1}})(R_{\alpha^{-1},\alpha\beta\alpha^{-1}})\cdot R_{\alpha,\beta}\\ &=(S_{\alpha^{-1}}\varphi_\alpha\otimes \mathrm{id}_{H_\beta})(R_{\alpha^{-1},\beta})\cdot R_{\alpha,\beta} \quad \text{by (6.7)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathscr{E}&=(\varepsilon 1_\alpha\otimes \mathrm{id}_{H_\beta})(R_{1,\beta}) \quad \text{by (1.5)}\\ &=1_\alpha\otimes 1_\beta \quad \text{by part (a)}.\end{aligned}$$

Comparing these two calculations and since $R_{\alpha,\beta}$ is invertible, we get the first equality of part (b). The second one can be proved similarly by computing the expression $\mathscr{F}=(\mathrm{id}_{H_\alpha}\otimes m_{\beta^{-1}})(\mathrm{id}_{H_\alpha}\otimes \mathrm{id}_{H_{\beta^{-1}}}\otimes S_\beta)(\mathrm{id}_{H_\alpha}\otimes\Delta_{\beta^{-1},\beta})(R^{-1}_{\alpha,1})$.

Part (c) is a direct consequence of part (b) and Lemma 6.1(a) and (c).

Finally, part (d) follows from axioms (6.5) and (6.6):

$$\begin{aligned}
&(R_{\beta,\gamma})_{\alpha 23} \cdot (R_{\alpha,\gamma})_{1\beta 3} \cdot (R_{\alpha,\beta})_{12\gamma} \\
&\quad = (R_{\beta,\gamma})_{\alpha 23} \cdot (\mathrm{id}_{H_\alpha} \otimes \Delta_{\beta,\gamma})(R_{\alpha,\beta\gamma}) \\
&\quad = (\mathrm{id}_{H_\alpha} \otimes R_{\beta,\gamma} \cdot \Delta_{\beta,\gamma})(R_{\alpha,\beta\gamma}) \\
&\quad = (\mathrm{id}_{H_\alpha} \otimes \sigma_{\gamma,\beta}(\varphi_{\beta^{-1}} \otimes \mathrm{id}_{H_\beta})\Delta_{\beta\gamma\beta^{-1},\beta} \cdot R_{\beta,\gamma})(R_{\alpha,\beta\gamma}) \\
&\quad = (\mathrm{id}_{H_\alpha} \otimes \sigma_{\gamma,\beta}(\varphi_{\beta^{-1}} \otimes \mathrm{id}_{H_\beta}))((R_{\alpha,\beta})_{1\beta\gamma\beta^{-1}3} \cdot (R_{\alpha,\beta\gamma\beta^{-1}})_{12\beta}) \cdot (R_{\beta,\gamma})_{\alpha 23} \\
&\quad = (R_{\alpha,\beta})_{12\gamma} \cdot [(\mathrm{id}_{H_\alpha} \otimes \varphi_{\beta^{-1}})(R_{\alpha,\beta\gamma\beta^{-1}})]_{1\beta 3} \cdot (R_{\beta,\gamma})_{\alpha 23}.
\end{aligned}$$

This completes the proof of the lemma. □

6.3. *The Drinfeld elements*

Let $H = (\{H_\alpha, m_\alpha, 1_\alpha\}, \Delta, \varepsilon, S, \varphi, R)$ be a quasitriangular Hopf π-coalgebra. We define the (*generalized*) *Drinfeld elements* of H, for any $\alpha \in \pi$, by

$$u_\alpha = m_\alpha(S_{\alpha^{-1}}\varphi_\alpha \otimes \mathrm{id}_{H_\alpha})\sigma_{\alpha,\alpha^{-1}}(R_{\alpha,\alpha^{-1}}) \in H_\alpha.$$

Note that u_1 is the Drinfeld element of the quasitriangular Hopf algebra H_1 (see [3]).

Lemma 6.5. *For any* $\alpha, \beta \in \pi$,
(a) u_α *is invertible and* $u_\alpha^{-1} = m_\alpha(\mathrm{id}_{H_\alpha} \otimes S_{\alpha^{-1}}S_\alpha)\sigma_{\alpha,\alpha}(R_{\alpha,\alpha})$;
(b) $S_{\alpha^{-1}}S_\alpha(\varphi_\alpha(x)) = u_\alpha x u_\alpha^{-1}$ *for all* $x \in H_\alpha$;
(c) *the antipode of* H *is bijective*;
(d) $\varphi_\beta(u_\alpha) = u_{\beta\alpha\beta^{-1}}$;
(e) $S_{\alpha^{-1}}(u_{\alpha^{-1}})u_\alpha = u_\alpha S_{\alpha^{-1}}(u_{\alpha^{-1}})$ *and this element*, *noted* c_α, *verifies* $c_\alpha\varphi_{\alpha^{-1}}(x) = \varphi_\alpha(x)c_\alpha$ *for all* $x \in H_\alpha$;
(f) $\Delta_{\alpha,\beta}(u_{\alpha\beta}) = [\sigma_{\beta,\alpha}(\mathrm{id}_{H_\beta} \otimes \varphi_\alpha)(R_{\beta,\alpha}) \cdot R_{\alpha,\beta}]^{-1} \cdot (u_\alpha \otimes u_\beta)$
$= (u_\alpha \otimes u_\beta) \cdot [\sigma_{\beta,\alpha}(\varphi_{\beta^{-1}} \otimes \mathrm{id}_{H_\alpha})(R_{\beta,\alpha}) \cdot (\varphi_{\alpha^{-1}} \otimes \varphi_{\beta^{-1}})(R_{\alpha,\beta})]^{-1}$;
(g) $\varepsilon(u_1) = 1$.

Proof. We adapt the methods used in [3] to our setting. Let us prove parts (a) and (b). We first show that for all $x \in H_\alpha$,

$$S_{\alpha^{-1}}S_\alpha(\varphi_\alpha(x))u_\alpha = u_\alpha x. \tag{6.8}$$

Write $R_{\alpha,\alpha^{-1}} = a_\alpha \otimes b_{\alpha^{-1}}$ so that $u_\alpha = S_{\alpha^{-1}}(\varphi_\alpha(b_{\alpha^{-1}}))a_\alpha$. Let $x \in H_\alpha$. Using (1.1) and (6.5), we have that

$$\begin{aligned}
&(R_{\alpha,\alpha^{-1}})_{12\alpha} \cdot (\mathrm{id}_{H_\alpha} \otimes \Delta_{\alpha^{-1},\alpha})\Delta_{\alpha,1}(x) \\
&\quad = (\sigma_{\alpha^{-1},\alpha}(\varphi_{\alpha^{-1}} \otimes \mathrm{id}_{H_\alpha})\Delta_{\alpha^{-1},\alpha} \otimes \mathrm{id}_{H_\alpha})\Delta_{1,\alpha}(x) \cdot (R_{\alpha,\alpha^{-1}})_{12\alpha},
\end{aligned}$$

that is $a_\alpha x_{(1,\alpha)} \otimes b_{\alpha^{-1}} x_{(2,\alpha^{-1})} \otimes x_{(3,\alpha)} = x_{(2,\alpha)} a_\alpha \otimes \varphi_{\alpha^{-1}}(x_{(1,\alpha^{-1})}) b_{\alpha^{-1}} \otimes x_{(3,\alpha)}$. Evaluate both sides of this equality with $(\mathrm{id}_{H_\alpha} \otimes S_{\alpha^{-1}}\varphi_\alpha \otimes S_{\alpha^{-1}} S_\alpha \varphi_\alpha)$, reverse the order of the tensorands and multiply them to obtain

$$S_{\alpha^{-1}} S_\alpha \varphi_\alpha(x_{(3,\alpha)}) S_{\alpha^{-1}} \varphi_\alpha(b_{\alpha^{-1}} x_{(2,\alpha^{-1})}) a_\alpha x_{(1,\alpha)}$$

$$= S_{\alpha^{-1}} S_\alpha \varphi_\alpha(x_{(3,\alpha)}) S_{\alpha^{-1}} \varphi_\alpha(\varphi_{\alpha^{-1}}(x_{(1,\alpha^{-1})}) b_{\alpha^{-1}}) x_{(2,\alpha)} a_\alpha.$$

Now, by Lemmas 1.1(a) and 6.1(c), the left-hand side is equal to

$$S_{\alpha^{-1}} \varphi_\alpha S_\alpha(x_{(3,\alpha)}) S_{\alpha^{-1}} \varphi_\alpha(x_{(2,\alpha^{-1})}) S_{\alpha^{-1}}(\varphi_\alpha(b_{\alpha^{-1}})) a_\alpha x_{(1,\alpha)}$$

$$= S_{\alpha^{-1}} \varphi_\alpha(x_{(2,\alpha^{-1})} S_\alpha(x_{(3,\alpha)})) u_\alpha x_{(1,\alpha)}$$

$$= S_{\alpha^{-1}} \varphi_\alpha(\varepsilon(x_{(2,1)}) 1_{\alpha^{-1}}) u_\alpha x_{(1,\alpha)} \quad \text{by (1.5)}$$

$$= u_\alpha \varepsilon(x_{(2,1)}) x_{(1,\alpha)} \quad \text{since } S_{\alpha^{-1}} \varphi_\alpha(1_{\alpha^{-1}}) = 1_\alpha$$

$$= u_\alpha x \quad \text{by (1.2)},$$

and, by Lemma 1.1(a), the right-hand side is equal to

$$S_{\alpha^{-1}} S_\alpha \varphi_\alpha(x_{(3,\alpha)}) S_{\alpha^{-1}}(\varphi_\alpha(b_{\alpha^{-1}})) S_{\alpha^{-1}}(x_{(1,\alpha^{-1})}) x_{(2,\alpha)} a_\alpha$$

$$= S_{\alpha^{-1}} S_\alpha \varphi_\alpha(\varepsilon(x_{(1,1)}) x_{(2,\alpha)}) S_{\alpha^{-1}}(\varphi_\alpha(b_{\alpha^{-1}})) a_\alpha \quad \text{by (1.5)}$$

$$= S_{\alpha^{-1}} S_\alpha \varphi_\alpha(x) u_\alpha \quad \text{by (1.2)}.$$

Thus (6.8) is proven. Let us show that u_α is invertible. Set

$$\tilde{u}_\alpha = m_\alpha(\mathrm{id}_{H_\alpha} \otimes S_{\alpha^{-1}} S_\alpha) \sigma_{\alpha,\alpha}(R_{\alpha,\alpha}) \in H_\alpha.$$

By Lemma 6.4(b) and (6.7), $R_{\alpha,\alpha} = (\mathrm{id}_{H_\alpha} \otimes S_{\alpha^{-1}})(\varphi_\alpha \otimes \varphi_\alpha)(R^{-1}_{\alpha,\alpha^{-1}})$. Write $R^{-1}_{\alpha,\alpha^{-1}} = c_\alpha \otimes d_{\alpha^{-1}}$. Then $\tilde{u}_\alpha = S_{\alpha^{-1}}(\varphi_\alpha(d_{\alpha^{-1}})) S_{\alpha^{-1}} S_\alpha(\varphi_\alpha(c_\alpha))$ and $a_\alpha c_\alpha \otimes b_{\alpha^{-1}} d_{\alpha^{-1}} = 1_\alpha \otimes 1_{\alpha^{-1}}$. Now

$$\tilde{u}_\alpha u_\alpha = S_{\alpha^{-1}}(\varphi_\alpha(d_{\alpha^{-1}})) S_{\alpha^{-1}} S_\alpha(\varphi_\alpha(c_\alpha)) u_\alpha$$

$$= S_{\alpha^{-1}}(\varphi_\alpha(d_{\alpha^{-1}})) u_\alpha c_\alpha \quad \text{by (6.8) with } x = c_\alpha$$

$$= S_{\alpha^{-1}}(\varphi_\alpha(d_{\alpha^{-1}})) S_{\alpha^{-1}}(\varphi_\alpha(b_{\alpha^{-1}})) a_\alpha c_\alpha$$

$$= S_{\alpha^{-1}}(\varphi_\alpha(b_{\alpha^{-1}} d_{\alpha^{-1}})) a_\alpha c_\alpha \quad \text{by Lemma 1.1(a)}$$

$$= S_{\alpha^{-1}}(\varphi_\alpha(1_{\alpha^{-1}})) 1_\alpha = 1_\alpha.$$

It can be shown similarly that $u_\alpha \tilde{u}_\alpha = 1_\alpha$. Thus u_α is invertible, $u_\alpha^{-1} = \tilde{u}_\alpha$, and so $S_{\alpha^{-1}} S_\alpha(\varphi_\alpha(x)) = u_\alpha x u_\alpha^{-1}$ for any $x \in H_\alpha$.

Part (c) is a direct consequence of part (b). Part (d) follows from (6.1), (6.4), and (6.7). Let us show part (e). For any $x \in H_\alpha$,

$$S_{\alpha^{-1}}(u_{\alpha^{-1}}) u_\alpha \varphi_{\alpha^{-1}}(x)$$

$$= S_{\alpha^{-1}}(u_{\alpha^{-1}}) S_{\alpha^{-1}} S_\alpha(x) u_\alpha \quad \text{by part (b)}$$

$$= S_{\alpha^{-1}}(u_{\alpha^{-1}}) S_{\alpha^{-1}} S_\alpha S_{\alpha^{-1}}(\varphi_{\alpha^{-1}} S^{-1}_{\alpha^{-1}}(\varphi_\alpha(x))) u_\alpha \quad \text{by Lemma 6.1(c)}$$

$$= S_{\alpha^{-1}}(u_{\alpha^{-1}}) S_{\alpha^{-1}}(u_{\alpha^{-1}} S^{-1}_{\alpha^{-1}}(\varphi_\alpha(x)) u^{-1}_{\alpha^{-1}}) u_\alpha \quad \text{by part (b)}$$

$$= \varphi_\alpha(x) S_{\alpha^{-1}}(u_{\alpha^{-1}}) u_\alpha \quad \text{since } S_{\alpha^{-1}} \text{ is anti-multiplicative.}$$

In particular, for $x = u_\alpha$, one gets that $S_{\alpha^{-1}}(u_{\alpha^{-1}}) u_\alpha = u_\alpha S_{\alpha^{-1}}(u_{\alpha^{-1}})$.

For the proof of the first equality of part (f), set $\tilde{R}_{\alpha,\beta} = \sigma_{\beta,\alpha}(\mathrm{id}_{H_\beta} \otimes \varphi_\alpha)(R_{\beta,\alpha})$. By Lemma 6.1 and (6.7), we have also that $\tilde{R}_{\alpha,\beta} = \sigma_{\beta,\alpha}(\varphi_{\alpha^{-1}} \otimes \mathrm{id}_{H_\alpha})(R_{\alpha\beta\alpha^{-1},\alpha})$. We first show that for all $x \in H_{\alpha\beta}$,

$$\tilde{R}_{\alpha,\beta} \cdot R_{\alpha,\beta} \cdot \Delta_{\alpha,\beta}(x) = (\varphi_\alpha \otimes \varphi_\beta) \Delta_{\alpha,\beta}(\varphi_{(\alpha\beta)^{-1}}(x)) \cdot \tilde{R}_{\alpha,\beta} \cdot R_{\alpha,\beta}. \tag{6.9}$$

By (6.5), $R_{\beta,\alpha} \cdot \Delta_{\beta,\alpha}(\varphi_{\alpha^{-1}}(x)) = \sigma_{\alpha,\beta}((\varphi_{\beta^{-1}} \otimes \mathrm{id}_{H_\beta}) \Delta_{\beta\alpha\beta^{-1},\beta}(\varphi_{\alpha^{-1}}(x))) \cdot R_{\beta,\alpha}$. Evaluate both sides of this equality with the algebra homomorphism $\sigma_{\beta,\alpha}(\mathrm{id}_{H_\beta} \otimes \varphi_\alpha)$ and multiply them on the right by $R_{\alpha,\beta}$ to obtain

$$\sigma_{\beta,\alpha}(\mathrm{id}_{H_\beta} \otimes \varphi_\alpha)(R_{\beta,\alpha}) \cdot \sigma_{\beta,\alpha}(\mathrm{id}_{H_\beta} \otimes \varphi_\alpha) \Delta_{\beta,\alpha}(\varphi_{\alpha^{-1}}(x)) \cdot R_{\alpha,\beta}$$

$$= (\varphi_\alpha \varphi_{\beta^{-1}} \otimes \mathrm{id}_{H_\beta}) \Delta_{\beta\alpha\beta^{-1},\beta}(\varphi_{\alpha^{-1}}(x)) \cdot \sigma_{\beta,\alpha}(\mathrm{id}_{H_\beta} \otimes \varphi_\alpha)(R_{\beta,\alpha}) \cdot R_{\alpha,\beta}.$$

Then, using (6.5) and (6.2), one gets equality (6.9). Set now

$$\mathscr{E} = \tilde{R}_{\alpha,\beta} \cdot R_{\alpha,\beta} \cdot \Delta_{\alpha,\beta}(u_{\alpha\beta}).$$

We have to show that $\mathscr{E} = u_\alpha \otimes u_\beta$. Write $R_{\alpha\beta,(\alpha\beta)^{-1}} = r \otimes s$, $R_{\alpha,\beta} = a_\alpha \otimes b_\beta$, and $\tilde{R}_{\alpha,\beta} = c_\alpha \otimes d_\beta$. Then $u_{\alpha\beta} = S_{(\alpha\beta)^{-1}}(\varphi_{\alpha\beta}(s)) r = \varphi_{\alpha\beta} S_{(\alpha\beta)^{-1}}(s) r$. We have that

$$\mathscr{E} = \tilde{R}_{\alpha,\beta} \cdot R_{\alpha,\beta} \cdot \Delta_{\alpha,\beta}(\varphi_{\alpha\beta} S_{(\alpha\beta)^{-1}}(s) r)$$

$$= \tilde{R}_{\alpha,\beta} \cdot R_{\alpha,\beta} \cdot \Delta_{\alpha,\beta}(\varphi_{\alpha\beta} S_{(\alpha\beta)^{-1}}(s)) \cdot \Delta_{\alpha,\beta}(r) \quad \text{by (1.4).}$$

Therefore, using (6.9) for $x = \varphi_{\alpha\beta} S_{(\alpha\beta)^{-1}}(s)$ and then Lemmas 1.1(c) and 6.1(c), we obtain that

$$\mathscr{E} = (\varphi_\alpha \otimes \varphi_\beta) \cdot \Delta_{\alpha,\beta}(S_{(\alpha\beta)^{-1}}(s)) \cdot \tilde{R}_{\alpha,\beta} \cdot R_{\alpha,\beta} \cdot \Delta_{\alpha,\beta}(r)$$

$$= (\varphi_\alpha \otimes \varphi_\beta) \sigma_{\beta,\alpha}(S_{\beta^{-1}} \otimes S_{\alpha^{-1}}) \Delta_{\beta^{-1},\alpha^{-1}}(s) \cdot \tilde{R}_{\alpha,\beta} \cdot R_{\alpha,\beta} \cdot \Delta_{\alpha,\beta}(r)$$

$$= \varphi_\alpha S_{\alpha^{-1}}(s_{(2,\alpha^{-1})}) c_\alpha a_\alpha r_{(1,\alpha)} \otimes \varphi_\beta S_{\beta^{-1}}(s_{(1,\beta^{-1})}) d_\beta b_\beta r_{(2,\beta)}$$

$$= S_{\alpha^{-1}}(\varphi_\alpha(s_{(2,\alpha^{-1})})) c_\alpha a_\alpha r_{(1,\alpha)} \otimes S_{\beta^{-1}}(\varphi_\beta(s_{(1,\beta^{-1})})) d_\beta b_\beta r_{(2,\beta)}.$$

Now $H_\alpha \otimes H_\beta$ is a right $H_\alpha \otimes H_\beta \otimes H_{\alpha^{-1}} \otimes H_{\beta^{-1}}$-module under the action

$$(x \otimes y) \twoheadleftarrow (h_1 \otimes h_2 \otimes h_3 \otimes h_4) = S_{\alpha^{-1}}(\varphi_\alpha(h_3)) x h_1 \otimes S_{\beta^{-1}}(\varphi_\beta(h_4)) y h_2.$$

For any $\Bbbk$-spaces P, Q and any $x = \sum_j p_j \otimes q_j \in P \otimes Q$, we set $x_{12\alpha\beta} = x \otimes 1_\alpha \otimes 1_\beta \in P \otimes Q \otimes H_\alpha \otimes H_\beta$, $x_{\alpha 2 \beta 4} = \sum_j 1_\alpha \otimes p_j \otimes 1_\beta \otimes q_j \in H_\alpha \otimes P \otimes H_\beta \otimes Q$, etc. Then

$$\mathscr{E} = c_\alpha \otimes d_\beta \twoheadleftarrow a_\alpha r_{(1,\alpha)} \otimes b_\beta r_{(2,\alpha^{-1})} \otimes s_{(2,\alpha^{-1})} \otimes s_{(1,\beta^{-1})}$$

$$= \tilde{R}_{\alpha,\beta} \twoheadleftarrow (R_{\alpha,\beta})_{12\alpha^{-1}\beta^{-1}} \cdot (\Delta_{\alpha,\beta} \otimes \sigma_{\beta^{-1},\alpha^{-1}} \Delta_{\beta^{-1},\alpha^{-1}})(R_{\alpha\beta,(\alpha\beta)^{-1}})$$

$$= \tilde{R}_{\alpha,\beta} \twoheadleftarrow (R_{\alpha,\beta})_{12\alpha^{-1}\beta^{-1}} \cdot (\Delta_{\alpha,\beta} \otimes \mathrm{id}_{H_{\alpha^{-1}}} \otimes \mathrm{id}_{H_{\beta^{-1}}})$$

$$((R_{\alpha\beta,\alpha^{-1}})_{12\beta^{-1}} \cdot (R_{\alpha\beta,\beta^{-1}})_{1\alpha^{-1}3}) \quad \text{by (6.6)}$$

$$= \tilde{R}_{\alpha,\beta} \twoheadleftarrow (R_{\alpha,\beta})_{12\alpha^{-1}\beta^{-1}} \cdot (\Delta_{\alpha,\beta} \otimes \mathrm{id}_{H_{\alpha^{-1}}} \otimes \mathrm{id}_{H_{\beta^{-1}}})((R_{\alpha\beta,\alpha^{-1}})_{12\beta^{-1}})$$

$$\cdot (\Delta_{\alpha,\beta} \otimes \mathrm{id}_{H_{\alpha^{-1}}} \otimes \mathrm{id}_{H_{\beta^{-1}}})((R_{\alpha\beta,\beta^{-1}})_{1\alpha^{-1}3}) \quad \text{by (1.4).}$$

Therefore, by (6.6) and Lemma 6.4(d),

$$\mathscr{E} = \tilde{R}_{\alpha,\beta} \twoheadleftarrow (R_{\alpha,\beta})_{12\alpha^{-1}\beta^{-1}} \cdot [(\mathrm{id}_{H_\alpha} \otimes \varphi_{\beta^{-1}})(R_{\alpha,\beta\alpha^{-1}\beta^{-1}})]_{1\beta 3\beta^{-1}}$$

$$\cdot (R_{\beta,\alpha^{-1}})_{\alpha 23\beta^{-1}} \cdot [(\mathrm{id}_{H_\alpha} \otimes \varphi_{\beta^{-1}})(R_{\alpha,\beta^{-1}})]_{1\beta\alpha^{-1}4} \cdot (R_{\beta,\beta^{-1}})_{a2\alpha^{-1}4}$$

$$= \tilde{R}_{\alpha,\beta} \twoheadleftarrow (R_{\beta,\alpha^{-1}})_{\alpha 23\beta^{-1}} \cdot (R_{\alpha,\alpha^{-1}})_{1\beta 3\beta^{-1}} \cdot (R_{\alpha,\beta})_{12\alpha^{-1}\beta^{-1}}$$

$$\cdot [(\mathrm{id}_{H_\alpha} \otimes \varphi_{\beta^{-1}})(R_{\alpha,\beta^{-1}})]_{1\beta\alpha^{-1}4} \cdot (R_{\beta,\beta^{-1}})_{\alpha 2\alpha^{-1}4}.$$

Write $R_{\beta,\alpha} = e_\beta \otimes f_\alpha$ and $R_{\beta,\alpha^{-1}} = h_\beta \otimes k_{\alpha^{-1}}$. Then $\tilde{R}_{\alpha,\beta} = \varphi_\alpha(f_\alpha) \otimes e_\beta$ and so

$$\tilde{R}_{\alpha,\beta} \twoheadleftarrow (R_{\beta,\alpha^{-1}})_{\alpha 23\beta^{-1}}$$

$$= S_{\alpha^{-1}}(\varphi_\alpha(k_{\alpha^{-1}}))\varphi_\alpha(f_\alpha) \otimes e_\beta h_\beta$$

$$= \sigma_{\beta,\alpha}(\mathrm{id}_{H_\beta} \otimes \varphi_\alpha S_{\alpha^{-1}})((\mathrm{id}_{H_\beta} \otimes S_{\alpha^{-1}}^{-1})(R_{\beta,\alpha}) \cdot R_{\beta,\alpha^{-1}}) \quad \text{by Lemma 6.1(c)}$$

$$= \sigma_{\beta,\alpha}(\mathrm{id}_{H_\beta} \otimes \varphi_\alpha S_{\alpha^{-1}})(R_{\beta,\alpha^{-1}}^{-1} \cdot R_{\beta,\alpha^{-1}}) \quad \text{by Lemma 6.4(b)}$$

$$= 1_\alpha \otimes 1_\beta.$$

If we write $R_{\alpha,\alpha^{-1}} = m_\alpha \otimes n_{\alpha^{-1}}$, then

$$1_\alpha \otimes 1_\beta \twoheadleftarrow (R_{\alpha,\alpha^{-1}})_{1\alpha^{-1}3\beta^{-1}} = S_{\alpha^{-1}}\varphi_\alpha(n_{\alpha^{-1}})m_\alpha \otimes 1_\beta = u_\alpha \otimes 1_\beta.$$

Therefore

$$\mathscr{E} = u_\alpha \otimes 1_\beta \twoheadleftarrow (R_{\alpha,\beta})_{12\alpha^{-1}\beta^{-1}} \cdot [(\mathrm{id}_{H_\alpha} \otimes \varphi_{\beta^{-1}})(R_{\alpha,\beta^{-1}})]_{1\beta\alpha^{-1}4} \cdot (R_{\beta,\beta^{-1}})_{\alpha 2\alpha^{-1}4}.$$

Write now $R_{\alpha,\beta^{-1}} = p_\alpha \otimes q_{\beta^{-1}}$. Then

$$u_\alpha \otimes 1_\beta \twoheadleftarrow (R_{\alpha,\beta})_{12\alpha^{-1}\beta^{-1}} \cdot [(\mathrm{id}_{H_\alpha} \otimes \varphi_{\beta^{-1}})(R_{\alpha,\beta^{-1}})]_{1\beta\alpha^{-1}4}$$

$$= u_\alpha a_\alpha p_\alpha \otimes S_{\beta^{-1}}(q_{\beta^{-1}})b_\beta$$

$$= (u_\alpha \otimes 1_\beta) \cdot (\mathrm{id}_{H_\alpha} \otimes S_{\beta^{-1}})((\mathrm{id}_{H_\alpha} \otimes S_{\beta^{-1}}^{-1})(R_{\alpha,\beta}) \cdot R_{\alpha,\beta^{-1}})$$

$$= (u_\alpha \otimes 1_\beta) \cdot (\mathrm{id}_{H_\alpha} \otimes S_{\beta^{-1}})(R_{\alpha,\beta^{-1}}^{-1} \cdot R_{\alpha,\beta^{-1}}) \quad \text{by Lemma 6.4(b)}$$

$$= u_\alpha \otimes 1_\beta.$$

Hence $\mathscr{E} = u_\alpha \otimes 1_\beta \twoheadleftarrow (R_{\beta,\beta^{-1}})_{\alpha 2\alpha^{-1}4}$. Finally, write $R_{\beta,\beta^{-1}} = x_\beta \otimes y_{\beta^{-1}}$. Then $\mathscr{E} = u_\alpha \otimes S_{\beta^{-1}}(\varphi_\beta(y_{\beta^{-1}}))x_\beta = u_\alpha \otimes u_\beta$. This completes the proof of the first equality of part (f). Let us show the second one. Using the first equality of part (f) and then part (b), we have that

$$\begin{aligned}\Delta_{\alpha,\beta}(u_{\alpha\beta}) &= [\sigma_{\beta,\alpha}(\mathrm{id}_{H_\beta}\otimes\varphi_\alpha)(R_{\beta,\alpha})\cdot R_{\alpha,\beta}]^{-1}\cdot(u_\alpha\otimes u_\beta)\\ &= (u_\alpha\otimes u_\beta)\cdot(\varphi_{\alpha^{-1}}(S_{\alpha^{-1}}S_\alpha)^{-1}\otimes\varphi_{\beta^{-1}}(S_{\beta^{-1}}S_\beta)^{-1})\\ &\quad([\sigma_{\beta,\alpha}(\mathrm{id}_{H_\beta}\otimes\varphi_\alpha)(R_{\beta,\alpha})\cdot R_{\alpha,\beta}]^{-1})\end{aligned}$$

and so, by Lemmas 6.1 and 6.4(c),

$$\Delta_{\alpha,\beta}(u_{\alpha\beta}) = (u_\alpha\otimes u_\beta)\cdot[\sigma_{\beta,\alpha}(\varphi_{\beta^{-1}}\otimes\mathrm{id}_{H_\alpha})(R_{\beta,\alpha})\cdot(\varphi_{\alpha^{-1}}\otimes\varphi_{\beta^{-1}})(R_{\alpha,\beta})]^{-1}.$$

It remains to show part (g). We have

$$\begin{aligned}u_1 &= (\varepsilon\otimes\mathrm{id}_{H_1})\Delta_{1,1}(u_1)\quad\text{by (1.2)}\\ &= (\varepsilon\otimes\mathrm{id}_{H_1})((\sigma_{1,1}(R_{1,1})\cdot R_{1,1})^{-1}\cdot(u_1\otimes u_1))\quad\text{by Part (f)}\\ &= (\varepsilon\otimes\mathrm{id}_{H_1})(R_{1,1})^{-1}\cdot(\mathrm{id}_{H_1}\otimes\varepsilon)(R_{1,1})^{-1}\cdot\varepsilon(u_1)u_1\quad\text{by (1.4)}\\ &= \varepsilon(u_1)u_1\quad\text{by Lemma 6.4(a)}.\end{aligned}$$

Now $u_1\neq 0$ since u_1 is invertible (by part (a)) and $H_1\neq 0$ (by Corollary 1.2). Hence $\varepsilon(u_1)=1$. This finishes the proof of the lemma. □

6.3.1. The coopposite Hopf π-coalgebra

Let H be a quasitriangular Hopf π-coalgebra with R-matrix $R=\{R_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$. By Lemma 6.5(c), the antipode of H is bijective. Thus, we can consider the coopposite crossed Hopf π-coalgebra H^{cop} to H (see Section 6.1.1). It is quasitriangular by setting $R^{\mathrm{cop}}_{\alpha,\beta}=(\varphi_\alpha\otimes\mathrm{id}_{H_{\beta^{-1}}})(R^{-1}_{\alpha^{-1},\beta^{-1}})=(S_\alpha\otimes\mathrm{id}_{H_{\beta^{-1}}})(R_{\alpha,\beta^{-1}})$. The Drinfeld elements of H and H^{cop} are related by $u^{\mathrm{cop}}_\alpha=u^{-1}_{\alpha^{-1}}$.

6.3.2. The mirror Hopf π-coalgebra

Let H be a quasitriangular Hopf π-coalgebra with R-matrix $R=\{R_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$. Following [19, Section 11.6], the mirror crossed Hopf π-coalgebra $\bar H$ to H (see Section 6.1.2) is quasitriangular with R-matrix given by $\bar R_{\alpha,\beta}=\sigma_{\beta^{-1},\alpha^{-1}}(R^{-1}_{\beta^{-1},\alpha^{-1}})$. The Drinfeld elements associated to H and $\bar H$ verify $\bar u_\alpha=S_\alpha(u_\alpha)^{-1}$.

The following corollary of Lemma 6.5 will be used in Section 6.6 to compute the distinguished π-grouplike element from the R-matrix.

Corollary 6.6. *Let H be a quasitriangular Hopf π-coalgebra. For all $\alpha\in\pi$, set $\ell_\alpha= S_{\alpha^{-1}}(u_{\alpha^{-1}})^{-1}\,u_\alpha=u_\alpha S_{\alpha^{-1}}(u_{\alpha^{-1}})^{-1}\in H_\alpha$. Then*
(a) *$\ell=(\ell_\alpha)_{\alpha\in\pi}$ is a π-grouplike element of H;*
(b) *$(S_{\alpha^{-1}}S_\alpha)^2(x)=\ell_\alpha x\ell_\alpha^{-1}$ for all $\alpha\in\pi$ and $x\in H_\alpha$.*

Proof. Let us show part (a). Denote by $\bar u_\alpha$ the Drinfeld elements of the mirror Hopf π-coalgebra $\bar H$ to H (see Section 6.3.2). Since $\bar u_\alpha=S_\alpha(u_\alpha)^{-1}$, Lemma 6.5(f) applied to $\bar{H}$ gives that, for any $\alpha, \beta \in \pi$,

$$\Delta_{\alpha,\beta}(S_{(\alpha\beta)^{-1}}(u_{(\alpha\beta)^{-1}})^{-1}) = \sigma_{\beta,\alpha}(\varphi_{\beta^{-1}} \otimes \mathrm{id}_{H_\alpha})(R_{\beta,\alpha}) \cdot (\varphi_{\alpha^{-1}} \otimes \varphi_{\beta^{-1}})(R_{\alpha,\beta}) \cdot (S_{\alpha^{-1}}(u_{\alpha^{-1}})^{-1} \otimes S_{\beta^{-1}}(u_{\beta^{-1}})^{-1}).$$

Now, by Lemma 6.5(f),

$$\Delta_{\alpha,\beta}(u_{\alpha\beta}) = (u_\alpha \otimes u_\beta) \cdot [\sigma_{\beta,\alpha}(\varphi_{\beta^{-1}} \otimes \mathrm{id}_{H_\alpha})(R_{\beta,\alpha}) \cdot (\varphi_{\alpha^{-1}} \otimes \varphi_{\beta^{-1}})(R_{\alpha,\beta})]^{-1}.$$

Thus we obtain that $\Delta_{\alpha,\beta}(\ell_{\alpha\beta}) = \Delta_{\alpha,\beta}(u_{\alpha\beta}) \cdot \Delta_{\alpha,\beta}(S_{(\alpha\beta)^{-1}}(u_{(\alpha\beta)^{-1}})^{-1}) = \ell_\alpha \otimes \ell_\beta$. Moreover $\varepsilon(\ell_1) = \varepsilon(u_1 S_1(u_1)^{-1}) = \varepsilon(u_1)\varepsilon(S_1(u_1))^{-1} = \varepsilon(u_1)\varepsilon(u_1)^{-1} = 1$ by (1.4) and Lemma 1.1(d). Hence $\ell = (\ell_\alpha)_{\alpha\in\pi} \in G(H)$.

To show part (b), let $\alpha \in \pi$ and $x \in H_\alpha$. Applying Lemma 6.5(b) to $\bar{H}$ and then to H gives that

$$\begin{aligned}(S_{\alpha^{-1}} S_\alpha)^2(x) &= S_{\alpha^{-1}} S_\alpha(S_{\alpha^{-1}}(u_{\alpha^{-1}})^{-1} \varphi_\alpha(x) S_{\alpha^{-1}}(u_{\alpha^{-1}}))\\ &= S_{\alpha^{-1}} S_\alpha(S_{\alpha^{-1}}(u_{\alpha^{-1}})^{-1}) S_{\alpha^{-1}} S_\alpha(\varphi_\alpha(x)) S_{\alpha^{-1}} S_\alpha(S_{\alpha^{-1}}(u_{\alpha^{-1}}))\\ &= u_\alpha S_{\alpha^{-1}}(u_{\alpha^{-1}})^{-1} x S_{\alpha^{-1}}(u_{\alpha^{-1}}) u_\alpha^{-1}\\ &= \ell_\alpha x \ell_\alpha^{-1}.\end{aligned}$$

This completes the proof of the corollary. □

6.3.3. The double of a crossed Hopf π-coalgebra

The Drinfeld double construction for Hopf algebras can be extended to the setting of crossed Hopf π-coalgebras, see [21]. This yields examples of quasitriangular Hopf π-coalgebras.

6.4. Ribbon Hopf π-coalgebras

Following [19, Section 11.4], a quasitriangular Hopf π-coalgebra $H = (\{H_\alpha\}, \Delta, \varepsilon, S, \varphi, R)$ is said to be *ribbon* if it is endowed with a family $\theta = \{\theta_\alpha \in H_\alpha\}_{\alpha\in\pi}$ of invertible elements (the *twist*) such that

$$\varphi_\alpha(x) = \theta_\alpha^{-1} x \theta_\alpha \text{ for all } \alpha \in \pi \text{ and } x \in H_\alpha, \tag{6.10}$$

$$S_\alpha(\theta_\alpha) = \theta_{\alpha^{-1}} \text{ for all } \alpha \in \pi, \tag{6.11}$$

$$\varphi_\beta(\theta_\alpha) = \theta_{\beta\alpha\beta^{-1}} \text{ for all } \alpha, \beta \in \pi, \tag{6.12}$$

for all $\alpha, \beta \in \pi$,

$$\Delta_{\alpha,\beta}(\theta_{\alpha\beta}) = (\theta_\alpha \otimes \theta_\beta) \cdot \sigma_{\beta,\alpha}((\varphi_{\alpha^{-1}} \otimes \mathrm{id}_{H_\alpha})(R_{\alpha\beta\alpha^{-1},\alpha})) \cdot R_{\alpha,\beta}. \tag{6.13}$$

Note that θ_1 is a (classical) twist of the quasitriangular Hopf algebra H_1.

Lemma 6.7. *Let $H=(\{H_\alpha\},\Delta,\varepsilon,S,\varphi,R,\theta)$ be a ribbon Hopf π-coalgebra. Then*
(a) $\varphi_{\alpha^{-1}}(x)=\theta_\alpha x\theta_\alpha^{-1}$ *for all* $\alpha\in\pi$ *and* $x\in H_\alpha$;
(b) $\varepsilon(\theta_1)=1$;
(c) *if* $\alpha\in\pi$ *has a finite order d, then* θ_α^d *is a central element of* H_α. *In particular* θ_1 *is central*;
(d) $\theta_\alpha u_\alpha=u_\alpha\theta_\alpha$ *for all* $\alpha\in\pi$, *where the* u_α *are the Drinfeld elements of H.*

Proof. Part (a) is a direct consequence of (6.10), (6.12), and Lemma 6.1. Let us show part (b). We have

$$\begin{aligned}\theta_1&=(\varepsilon\otimes\mathrm{id}_{H_1})\Delta_{1,1}(\theta_1)\quad\text{by (1.2)}\\&=(\varepsilon\otimes\mathrm{id}_{H_1})((\theta_1\otimes\theta_1)\cdot\sigma_{1,1}(R_{1,1})\cdot R_{1,1})\quad\text{by (6.13) and Lemma 6.1(a)}\\&=(\varepsilon\otimes\mathrm{id}_{H_1})(\theta_1\otimes\theta_1)\cdot(\mathrm{id}_{H_1}\otimes\varepsilon)(R_{1,1})\cdot(\varepsilon\otimes\mathrm{id}_{H_1})(R_{1,1})\quad\text{by (1.4)}\\&=\varepsilon(\theta_1)\theta_1\quad\text{by Lemma 6.4(a)}\end{aligned}$$

Now $\theta_1\neq 0$ since it is invertible and $H_1\neq 0$ (by Corollary 1.2). Hence $\varepsilon(\theta_1)=1$. To show part (c), let $\alpha\in\pi$ of finite order d. For any $x\in H_\alpha$, using (6.4), Lemma 6.1 and (6.10), we have that $x=\varphi_1(x)=\varphi_{\alpha^d}(x)=\varphi_\alpha^d(x)=\theta_\alpha^{-d}x\theta_\alpha^d$ and so $\theta_\alpha^d x=x\theta_\alpha^d$. Hence θ_α^d is central in H_α. Finally, let us show part (d). Using Lemma 6.5(d) and (6.10), we have that $u_\alpha=\varphi_\alpha(u_\alpha)=\theta_\alpha^{-1}u_\alpha\theta_\alpha$, and so $\theta_\alpha u_\alpha=u_\alpha\theta_\alpha$. □

6.4.1. The coopposite Hopf π-coalgebra

Let H be a ribbon Hopf π-coalgebra with twist $\theta=\{\theta_\alpha\}_{\alpha\in\pi}$. The coopposite quasitriangular Hopf π-coalgebra H^{cop} (see Section 6.3.1) is ribbon with twist $\theta_\alpha^{\mathrm{cop}}=\theta_{\alpha^{-1}}^{-1}$.

6.4.2. The mirror Hopf π-coalgebra

Let H be a ribbon Hopf π-coalgebra with twist $\theta=\{\theta_\alpha\}_{\alpha\in\pi}$. Following [19, Section 11.6], the mirror quasitriangular Hopf π-coalgebra $\bar{H}$ (see Section 6.3.2) is ribbon with twist $\bar{\theta}_\alpha=\theta_{\alpha^{-1}}^{-1}$.

6.5. The spherical π-grouplike element

Let $H=(\{H_\alpha\},\Delta,\varepsilon,S,\varphi,R,\theta)$ be a ribbon Hopf π-coalgebra. For any $\alpha\in\pi$, we set (see Lemma 6.7(d))

$$G_\alpha=\theta_\alpha u_\alpha=u_\alpha\theta_\alpha\in H_\alpha.$$

Lemma 6.8. (a) $G=(G_\alpha)_{\alpha\in\pi}$ *is a π-grouplike element of H*;
(b) $\varphi_\beta(G_\alpha)=G_{\beta\alpha\beta^{-1}}$ *for all* $\alpha,\beta\in\pi$;
(c) $S_\alpha(G_\alpha)=G_{\alpha^{-1}}^{-1}$ *for all* $\alpha\in\pi$;
(d) $\theta_\alpha^{-2}=c_\alpha$ *for all* $\alpha\in\pi$, *where* $c_\alpha=S_{\alpha^{-1}}(u_{\alpha^{-1}})u_\alpha=u_\alpha S_{\alpha^{-1}}(u_{\alpha^{-1}})$ *as in Lemma* 6.5(e);
(e) $S_\alpha(u_\alpha)=G_{\alpha^{-1}}^{-1}u_{\alpha^{-1}}G_{\alpha^{-1}}^{-1}$ *for all* $\alpha\in\pi$;
(f) $S_{\alpha^{-1}}S_\alpha(x)=G_\alpha xG_\alpha^{-1}$ *for all* $\alpha\in\pi$ *and* $x\in H_\alpha$.

The π-grouplike element $G=(G_\alpha)_{\alpha\in\pi}$ of the previous lemma is called the *spherical* π*-grouplike element of* H.

Proof. Let us show part (a). Firstly $\varepsilon(G_1)=\varepsilon(\theta_1 u_1)=\varepsilon(\theta_1)\varepsilon(u_1)=1$ by Lemmas 6.5(g) and 6.7(b). Secondly, for any $\alpha,\beta\in\pi$, using (6.13) and Lemma 6.5(f),

$$\begin{aligned}\Delta_{\alpha,\beta}(G_{\alpha\beta}) &= \Delta_{\alpha,\beta}(\theta_{\alpha\beta}u_{\alpha\beta})\\ &= \Delta_{\alpha,\beta}(\theta_{\alpha\beta})\cdot\Delta_{\alpha,\beta}(u_{\alpha\beta})\\ &= (\theta_\alpha\otimes\theta_\beta)\cdot[\sigma_{\beta,\alpha}((\varphi_{\alpha^{-1}}\otimes \mathrm{id}_{H_\alpha})(R_{\alpha\beta\alpha^{-1},\alpha}))\cdot R_{\alpha,\beta}]\\ &\quad\cdot[\sigma_{\beta,\alpha}((\varphi_{\alpha^{-1}}\otimes \mathrm{id}_{H_\alpha})(R_{\alpha\beta\alpha^{-1},\alpha}))\cdot R_{\alpha,\beta}]^{-1}\cdot(u_\alpha\otimes u_\beta)\\ &= G_\alpha\otimes G_\beta.\end{aligned}$$

Thus $G=(G_\alpha)_{\alpha\in\pi}\in G(H)$. Part (b) follows directly from Lemma 6.5(d) and (6.12), and part (c) from the fact that G is a π-grouplike element. By part (c) and (6.11), $\theta_\alpha^{-2}=u_\alpha G_\alpha^{-1}\theta_\alpha^{-1}=u_\alpha S_{\alpha^{-1}}(G_{\alpha^{-1}})\theta_\alpha^{-1}=u_\alpha S_{\alpha^{-1}}(\theta_{\alpha^{-1}}u_{\alpha^{-1}})\theta_\alpha^{-1}=c_\alpha$ and so part (d) is established. Let us show part (e). By (6.11) and part (c), $G_{\alpha^{-1}}^{-1}u_{\alpha^{-1}}=\theta_{\alpha^{-1}}^{-1}=S_\alpha(\theta_\alpha^{-1})=S_\alpha(G_\alpha^{-1}u_\alpha)=S_\alpha(u_\alpha)\ S_\alpha(G_\alpha)^{-1}=S_\alpha(u_\alpha)G_{\alpha^{-1}}$. Therefore $S_\alpha(u_\alpha)=G_{\alpha^{-1}}^{-1}u_{\alpha^{-1}}G_{\alpha^{-1}}^{-1}$. Finally, to show part (f), let $x\in H_\alpha$. Then, using Lemmas 6.5(b) and 6.7(a),

$$S_{\alpha^{-1}}S_\alpha(x)=u_\alpha\varphi_{\alpha^{-1}}(x)u_\alpha^{-1}=u_\alpha\theta_\alpha x\theta_\alpha^{-1}u_\alpha^{-1}=G_\alpha x G_\alpha^{-1}.$$

This completes the proof of the lemma. □

6.6. The distinguished π-grouplike element from the R-matrix

In this subsection, we show that the distinguished π-grouplike element of a finite type quasitriangular Hopf π-coalgebra can be computed by using the R-matrix. This generalizes [14, Theorem 2].

Theorem 6.9. *Let H be a finite type quasitriangular Hopf π-coalgebra. Let $g=(g_\alpha)_{\alpha\in\pi}$ be the distinguished π-grouplike element of H, v be the distinguished grouplike element of H_1^*, $\ell=(\ell_\alpha)_{\alpha\in\pi}\in G(H)$ be as in Corollary* 6.6, *and $\hat\varphi$ be as in Corollary* 6.2. *We define $h_\alpha=(\mathrm{id}_{H_\alpha}\otimes v)(R_{\alpha,1})$ for any $\alpha\in\pi$. Then*
(a) *$h=(h_\alpha)_{\alpha\in\pi}$ is a π-grouplike element of H;*
(b) *$g=\hat\varphi^{-1}\ell h$ in $G(H)$, i.e., $g_\alpha=\hat\varphi(\alpha)^{-1}\ell_\alpha h_\alpha$ for all $\alpha\in\pi$.*

Proof. We adapt the technique used in the proof of [14, Theorem 2]. Let us first show part (a). For any $\alpha,\beta\in\pi$, using (6.6), the multiplicativity of v, and Lemma 6.3(b), we have that

$$\begin{aligned}\Delta_{\alpha,\beta}(h_{\alpha\beta}) &= \Delta_{\alpha,\beta}(\mathrm{id}_{H_{\alpha\beta}}\otimes v)(R_{\alpha\beta,1})\\ &= (\mathrm{id}_{H_\alpha}\otimes \mathrm{id}_{H_\beta}\otimes v)([(\mathrm{id}_{H_\alpha}\otimes\varphi_{\beta^{-1}})(R_{\alpha,1})]_{1\beta 3}\cdot(R_{\beta,1})_{\alpha 23})\end{aligned}$$

$$= ((\mathrm{id}_{H_\alpha} \otimes \nu\varphi_{\beta^{-1}})(R_{\alpha,1}) \otimes 1_\beta) \cdot (1_\alpha \otimes (\mathrm{id}_{H_\beta} \otimes \nu)(R_{\beta,1}))$$

$$= ((\mathrm{id}_{H_\alpha} \otimes \nu)(R_{\alpha,1}) \otimes 1_\beta) \cdot (1_\alpha \otimes h_\beta)$$

$$= h_\alpha \otimes h_\beta.$$

Moreover, using Lemma 6.4(a), $\varepsilon(h_1) = (\varepsilon \otimes \nu)(R_{1,1}) = \nu(1_1) = 1$. Thus $h \in G(H)$.

To show part (b), let $\alpha \in \pi$ and Λ be a non-zero left integral for H_1. We first show that, for any $x \in H_{\alpha^{-1}}$,

$$\Lambda_{(1,\alpha)} \otimes x\Lambda_{(2,\alpha^{-1})} = S_{\alpha^{-1}}(x)\Lambda_{(1,\alpha)} \otimes \Lambda_{(2,\alpha^{-1})} \tag{6.14}$$

and

$$\Lambda_{(1,\alpha^{-1})}x \otimes \Lambda_{(2,\alpha)} = \Lambda_{(1,\alpha^{-1})} \otimes \Lambda_{(2,\alpha)}S_{\alpha^{-1}}(x \leftharpoonup \nu). \tag{6.15}$$

Indeed

$$\Lambda_{(1,\alpha)} \otimes x\Lambda_{(2,\alpha^{-1})}$$

$$= \varepsilon(x_{(1,1)})\Lambda_{(1,\alpha)} \otimes x_{(2,\alpha^{-1})}\Lambda_{(2,\alpha^{-1})} \quad \text{by (1.2)}$$

$$= S_{\alpha^{-1}}(x_{(1,\alpha^{-1})})x_{(2,\alpha)}\Lambda_{(1,\alpha)} \otimes x_{(3,\alpha^{-1})}\Lambda_{(2,\alpha^{-1})} \quad \text{by (1.5)}$$

$$= S_{\alpha^{-1}}(x_{(1,\alpha^{-1})})(x_{(2,1)}\Lambda)_{(1,\alpha)} \otimes (x_{(2,1)}\Lambda)_{(2,\alpha^{-1})} \quad \text{by (1.4)},$$

and so, since Λ is a left integral for H_1,

$$\Lambda_{(1,\alpha)} \otimes x\Lambda_{(2,\alpha^{-1})} = S_{\alpha^{-1}}(x_{(1,\alpha^{-1})}\varepsilon(x_{(2,1)}))\Lambda_{(1,\alpha)} \otimes \Lambda_{(2,\alpha^{-1})}$$

$$= S_{\alpha^{-1}}(x)\Lambda_{(1,\alpha)} \otimes \Lambda_{(2,\alpha^{-1})} \quad \text{by (1.2)}.$$

Similarly,

$$\Lambda_{(1,\alpha^{-1})}x \otimes \Lambda_{(2,\alpha^{-1})}$$

$$= \Lambda_{(1,\alpha^{-1})}x_{(1,\alpha^{-1})} \otimes \Lambda_{(2,\alpha)}\varepsilon(x_{(2,1)}) \quad \text{by (1.2)}$$

$$= \Lambda_{(1,\alpha^{-1})}x_{(1,\alpha^{-1})} \otimes \Lambda_{(2,\alpha)}x_{(2,\alpha)}S_{\alpha^{-1}}(x_{(3,\alpha^{-1})}) \quad \text{by (1.5)}$$

$$= (\Lambda x_{(1,1)})_{(1,\alpha^{-1})} \otimes (\Lambda x_{(1,1)})_{(2,\alpha)}S_{\alpha^{-1}}(x_{(2,\alpha^{-1})}) \quad \text{by (1.4)},$$

and so, since Λ is a left integral for H_1,

$$\Lambda_{(1,\alpha^{-1})}x \otimes \Lambda_{(2,\alpha^{-1})} = \Lambda_{(1,\alpha^{-1})} \otimes \Lambda_{(2,\alpha)}S_{\alpha^{-1}}(\nu(x_{(1,1)})x_{(2,\alpha^{-1})})$$

$$= \Lambda_{(1,\alpha^{-1})} \otimes \Lambda_{(2,\alpha)}S_{\alpha^{-1}}(x \leftharpoonup \nu).$$

Write $R_{\alpha,\alpha^{-1}} = a_\alpha \otimes b_{\alpha^{-1}}$. Recall that $u_\alpha = S_{\alpha^{-1}}\varphi_\alpha(b_{\alpha^{-1}})a_\alpha$. By Lemma 6.4(c) and (6.7), $R_{\alpha^{-1},\alpha} = S_\alpha(a_\alpha) \otimes \varphi_\alpha S_{\alpha^{-1}}(b_{\alpha^{-1}})$. Thus $u_{\alpha^{-1}} = S_\alpha S_{\alpha^{-1}}(b_{\alpha^{-1}})S_\alpha(a_\alpha)$ and so, using Lemma 6.5(b) and (d), $S_{\alpha^{-1}}(u_{\alpha^{-1}}) = S_\alpha^{-1}(u_{\alpha^{-1}}) = a_\alpha S_{\alpha^{-1}}(b_{\alpha^{-1}})$. Then

$$\Lambda_{(2,\alpha)}S_{\alpha^{-1}}(\varphi_\alpha(b_{\alpha^{-1}}) \leftharpoonup \nu)a_\alpha \otimes \Lambda_{(1,\alpha^{-1})}$$

$$= \Lambda_{(2,\alpha)}a_\alpha \otimes \Lambda_{(1,\alpha^{-1})}\varphi_\alpha(b_{\alpha^{-1}}) \quad \text{by (6.15) for } x = \varphi_\alpha(b_{\alpha^{-1}})$$

$$= (\mathrm{id}_{H_\alpha} \otimes \varphi_\alpha)(\Lambda_{(2,\alpha)} a_\alpha \otimes \varphi_{\alpha^{-1}}(\Lambda_{(1,\alpha^{-1})}) b_{\alpha^{-1}})$$

$$= (\mathrm{id}_{H_\alpha} \otimes \varphi_\alpha)(a_\alpha \Lambda_{(1,\alpha)} \otimes b_{\alpha^{-1}} \Lambda_{(2,\alpha^{-1})}) \quad \text{by (6.5)}$$

$$= (\mathrm{id}_{H_\alpha} \otimes \varphi_\alpha)(a_\alpha S_{\alpha^{-1}}(b_{\alpha^{-1}}) \Lambda_{(1,\alpha)} \otimes \Lambda_{(2,\alpha^{-1})}) \quad \text{by (6.14) for } x = b_{\alpha^{-1}}$$

$$= S_{\alpha^{-1}}(u_{\alpha^{-1}}) \Lambda_{(1,\alpha)} \otimes \varphi_\alpha(\Lambda_{(2,\alpha^{-1})})$$

$$= (\varphi_{\alpha^{-1}} \otimes \mathrm{id}_{H_{\alpha^{-1}}})(\varphi_\alpha S_{\alpha^{-1}}(u_{\alpha^{-1}}) \varphi_\alpha(\Lambda_{(1,\alpha)}) \otimes \varphi_\alpha(\Lambda_{(2,\alpha^{-1})})) \quad \text{by (6.4)}$$

$$= (\varphi_{\alpha^{-1}} \otimes \mathrm{id}_{H_{\alpha^{-1}}})(\varphi_\alpha S_{\alpha^{-1}}(u_{\alpha^{-1}}) \varphi_\alpha(\Lambda)_{(1,\alpha)} \otimes \varphi_\alpha(\Lambda)_{(2,\alpha^{-1})}) \quad \text{by (6.2).}$$

Now $\varphi_\alpha(\Lambda) = \hat{\varphi}(\alpha)\Lambda$ by Lemma 6.3(a) and

$$\Lambda_{(1,\alpha)} \otimes \Lambda_{(2,\alpha^{-1})} = S_{\alpha^{-1}} S_\alpha(\Lambda_{(2,\alpha)}) g_\alpha \otimes \Lambda_{(1,\alpha^{-1})}$$

by Corollary 4.4. Therefore

$$\Lambda_{(2,\alpha)} S_{\alpha^{-1}}(\varphi_\alpha(b_{\alpha^{-1}}) \leftharpoonup v) a_\alpha \otimes \Lambda_{(1,\alpha^{-1})}$$

$$= \hat{\varphi}(\alpha)(\varphi_{\alpha^{-1}} \otimes \mathrm{id}_{H_{\alpha^{-1}}})(\varphi_\alpha S_{\alpha^{-1}}(u_{\alpha^{-1}}) S_{\alpha^{-1}} S_\alpha(\Lambda_{(2,\alpha)}) g_\alpha \otimes \Lambda_{(1,\alpha^{-1})})$$

$$= \hat{\varphi}(\alpha) S_{\alpha^{-1}}(u_{\alpha^{-1}}) \varphi_{\alpha^{-1}} S_{\alpha^{-1}} S_\alpha(\Lambda_{(2,\alpha)}) \varphi_{\alpha^{-1}}(g_\alpha) \otimes \Lambda_{(1,\alpha^{-1})}$$

$$= \hat{\varphi}(\alpha) S_{\alpha^{-1}}(u_{\alpha^{-1}}) \varphi_{\alpha^{-1}} S_{\alpha^{-1}} S_\alpha(\Lambda_{(2,\alpha)}) g_\alpha \otimes \Lambda_{(1,\alpha^{-1})} \quad \text{by Lemma 6.3(c).}$$

Let $\lambda = (\lambda_\gamma)_{\gamma \in \pi}$ be right π-integral for H such that $\lambda_1(\Lambda) = 1$ (see the proof of Corollary 4.4). Applying $(\mathrm{id}_{H_\alpha} \otimes \lambda_{\alpha^{-1}})$ on both sides of the last equality, we get

$$\lambda_{\alpha^{-1}}(\Lambda_{(1,\alpha^{-1})}) \Lambda_{(2,\alpha)} S_{\alpha^{-1}}(\varphi_\alpha(b_{\alpha^{-1}}) \leftharpoonup v) a_\alpha$$

$$= \hat{\varphi}(\alpha) S_{\alpha^{-1}}(u_{\alpha^{-1}}) \varphi_{\alpha^{-1}} S_{\alpha^{-1}} S_\alpha(\lambda_{\alpha^{-1}}(\Lambda_{(1,\alpha^{-1})}) \Lambda_{(2,\alpha)}) g_\alpha,$$

and so, since $\lambda_{\alpha^{-1}}(\Lambda_{(1,\alpha^{-1})}) \Lambda_{(2,\alpha)} = \lambda_1(\Lambda) 1_\alpha = 1_\alpha$,

$$S_{\alpha^{-1}}(\varphi_\alpha(b_{\alpha^{-1}}) \leftharpoonup v) a_\alpha = \hat{\varphi}(\alpha) S_{\alpha^{-1}}(u_{\alpha^{-1}}) g_\alpha. \tag{6.16}$$

Write $R_{\alpha,1} = c_\alpha \otimes d_1$ so that $h_\alpha = v(d_1) c_\alpha$. Since, by (6.2) and Lemma 6.3(b), $\varphi_\alpha(x) \leftharpoonup v = \varphi_\alpha(x \leftharpoonup v)$ for all $x \in H_{\alpha^{-1}}$, we have that

$$a_\alpha \otimes (\varphi_\alpha(b_{\alpha^{-1}}) \leftharpoonup v) = a_\alpha \otimes \varphi_\alpha(b_{\alpha^{-1}} \leftharpoonup v)$$

$$= (\mathrm{id}_{H_\alpha} \otimes v \otimes \varphi_\alpha)(\mathrm{id}_{H_\alpha} \otimes \Delta_{1,\alpha^{-1}})(R_{\alpha,\alpha^{-1}})$$

$$= (\mathrm{id}_{H_\alpha} \otimes v \otimes \varphi_\alpha)((R_{\alpha,\alpha^{-1}})_{1 1_\pi 3} \cdot (R_{\alpha,1})_{12\alpha^{-1}}) \quad \text{by (6.6)}$$

$$= a_\alpha v(d_1) c_\alpha \otimes \varphi_\alpha(b_{\alpha^{-1}})$$

$$= a_\alpha h_\alpha \otimes \varphi_\alpha(b_{\alpha^{-1}}).$$

Therefore $S_{\alpha^{-1}}(\varphi_\alpha(b_{\alpha^{-1}}) \leftharpoonup v) a_\alpha = S_{\alpha^{-1}}(\varphi_\alpha(b_{\alpha^{-1}})) a_\alpha h_\alpha = u_\alpha h_\alpha$. Finally, comparing with (6.16), we get $\hat{\varphi}(\alpha) S_{\alpha^{-1}}(u_{\alpha^{-1}}) g_\alpha = u_\alpha h_\alpha$. Hence $g_\alpha = \hat{\varphi}(\alpha)^{-1} \ell_\alpha h_\alpha$, since $\ell_\alpha = S_{\alpha^{-1}}(u_{\alpha^{-1}})^{-1} u_\alpha$. This finishes the proof of the theorem. □

Corollary 6.10. *Let H be a finite type ribbon Hopf π-coalgebra. Let $g=(g_\alpha)_{\alpha\in\pi}$ be the distinguished π-grouplike element of H, $G=(G_\alpha)_{\alpha\in\pi}$ be the spherical π-grouplike element of H, $h=(h_\alpha)_{\alpha\in\pi}\in G(H)$ as in Theorem 6.9, and $\hat{\varphi}$ as in Corollary 6.2. Then $\hat{\varphi}g=G^2h$ in $G(H)$, i.e., $\hat{\varphi}(\alpha)g_\alpha=G_\alpha^2h_\alpha$ for all $\alpha\in\pi$.*

Proof. For any $\alpha\in\pi$, $\hat{\varphi}(\alpha)g_\alpha=S_{\alpha^{-1}}(u_{\alpha^{-1}})^{-1}u_\alpha h_\alpha=\theta_\alpha^2u_\alpha^2h_\alpha=G_\alpha^2h_\alpha$ by Theorem 6.9(b) and Lemma 6.8(d). □

7. Existence of π-traces

In this section, we introduce the notion of a π-trace for a crossed Hopf π-coalgebra and we show the existence of π-traces for a finite type unimodular Hopf π-coalgebra whose crossing φ verifies that $\hat{\varphi}=1$. Moreover, we give sufficient conditions for the homomorphism $\hat{\varphi}$ to be trivial.

7.1. Unimodular Hopf π-coalgebras

A Hopf π-coalgebra $H=\{H_\alpha\}_{\alpha\in\pi}$ is said to be *unimodular* if the Hopf algebra H_1 is unimodular (it means that the spaces of left and right integrals for H_1 coincide). If H_1 is finite dimensional, then H is unimodular if and only if $v=\varepsilon$, where v is the distinguished grouplike element of H_1^*.

If π is finite, then a left (resp. right) integral for the Hopf algebra $\tilde{H}=\bigoplus_{\alpha\in\pi}H_\alpha$ (see Section 1.3.5) must belong to H_1, and so the spaces of left (resp. right) integrals for H and H_1 coincide. Hence, when π is finite, H is unimodular if and only if $\tilde{H}$ is unimodular.

One can remark that a semisimple finite type of Hopf π-coalgebra $H=\{H_\alpha\}_{\alpha\in\pi}$ is unimodular (since the finite-dimensional Hopf algebra H_1 is semisimple and so unimodular). Note that a cosemisimple Hopf π-coalgebra is not necessarily unimodular.

7.2. π-traces

Let $H=(\{H_\alpha\},\Delta,\varepsilon,S,\varphi)$ be a crossed Hopf π-coalgebra. A *π-trace* for H is a family of $\Bbbk$-linear forms $\mathrm{tr}=(\mathrm{tr}_\alpha)_{\alpha\in\pi}\in\prod_{\alpha\in\pi}H_\alpha^*$ such that, for any $\alpha,\beta\in\pi$ and $x,y\in H_\alpha$,

$$\mathrm{tr}_\alpha(xy)=\mathrm{tr}_\alpha(yx), \tag{7.1}$$

$$\mathrm{tr}_{\alpha^{-1}}(S_\alpha(x))=\mathrm{tr}_\alpha(x), \tag{7.2}$$

$$\mathrm{tr}_{\beta\alpha\beta^{-1}}(\varphi_\beta(x))=\mathrm{tr}_\alpha(x). \tag{7.3}$$

This notion is motivated mainly by topological purposes: π-traces are used in [20] to construct Hennings-like invariants (see [4,6]) of principal π-bundles over link complements and over 3-mainfolds.

Note that tr_1 is a (usual) trace for the Hopf algebra H_1, invariant under the action φ of π.

In the next lemma, generalizing [4, Proposition 4.2], we give a characterization of the π-traces.

Lemma 7.1. *Let $H=\{H_\alpha\}_{\alpha\in\pi}$ be a finite type unimodular ribbon Hopf π-coalgebra with crossing φ. Let $\lambda=(\lambda_\alpha)_{\alpha\in\pi}$ be a non-zero right π-integral for H, $G=(G_\alpha)_{\alpha\in\pi}$ be the spherical π-grouplike element of H, and $\hat{\varphi}$ be as in Corollary 6.2. Let $\mathrm{tr}=(\mathrm{tr}_\alpha)_{\alpha\in\pi}\in\prod_{\alpha\in\pi}H_\alpha^*$. Then tr is a π-trace for H if and only if there exists a family $z=(z_\alpha)_{\alpha\in\pi}\in\prod_{\alpha\in\pi}H_\alpha$ satisfying, for all $\alpha,\beta\in\pi$,*
(a) $\mathrm{tr}_\alpha(x)=\lambda_\alpha(G_\alpha z_\alpha x)$ *for all* $x\in H_\alpha$;
(b) z_α *is central in* H_α;
(c) $S_\alpha(z_\alpha)=\hat{\varphi}(\alpha)^{-1}z_{\alpha^{-1}}$;
(d) $\varphi_\beta(z_\alpha)=\hat{\varphi}(\beta)z_{\beta\alpha\beta^{-1}}$.

Proof. We first show that, for all $\alpha\in\pi$ and $x,y\in H_\alpha$,

$$\lambda_\alpha(G_\alpha xy)=\lambda_\alpha(G_\alpha yx), \tag{7.4}$$

and

$$\hat{\varphi}(\alpha)\lambda_{\alpha^{-1}}(S_\alpha(x))=\lambda_\alpha(G_\alpha^2 x). \tag{7.5}$$

Indeed, let v be the distinguished grouplike element of H_1^*. Since $v=\varepsilon$ (H is unimodular), Theorem 4.2(a) gives that $\lambda_\alpha(G_\alpha xy)=\lambda_\alpha(S_{\alpha^{-1}}S_\alpha(y)G_\alpha x)$. Now, by Lemma 6.8(f), $S_{\alpha^{-1}}S_\alpha(y)=G_\alpha yG_\alpha^{-1}$. Thus $\lambda_\alpha(G_\alpha xy)=\lambda_\alpha(G_\alpha yx)$ and (7.4) is proven. Moreover, Corollary 6.10 gives that $\hat{\varphi}(\alpha)g_\alpha=G_\alpha^2 h_\alpha$, where $g=(g_\alpha)_{\alpha\in\pi}$ is the distinguished π-grouplike element of H and $h_\alpha=(\mathrm{id}_{H_\alpha}\otimes v)(R_{\alpha,1})$. Since $v=\varepsilon$ and by Lemma 6.4(a), $h_\alpha=(\mathrm{id}_{H_\alpha}\otimes\varepsilon)(R_{\alpha,1})=1_\alpha$. Thus $\hat{\varphi}(\alpha)g_\alpha=G_\alpha^2$. Now $\lambda_{\alpha^{-1}}(S_\alpha(x))=\lambda_\alpha(g_\alpha x)$ by Theorem 4.2(c). Hence $\hat{\varphi}(\alpha)\lambda_{\alpha^{-1}}(S_\alpha(x))=\lambda_\alpha(G_\alpha^2 x)$ and (7.5) is proven.

Let us suppose that there exists $z=(z_\alpha)_{\alpha\in\pi}\in\prod_{\alpha\in\pi}H_\alpha$ verifying conditions (a)–(d). For any $\alpha,\beta\in\pi$ and $x,y\in H_\alpha$,

$$\begin{aligned}\mathrm{tr}_\alpha(xy)&=\lambda_\alpha(G_\alpha z_\alpha xy) \quad \text{by Condition (a)}\\ &=\lambda_\alpha(G_\alpha yz_\alpha x) \quad \text{by(7.4)}\\ &=\lambda_\alpha(G_\alpha z_\alpha yx) \quad \text{since } z_\alpha \text{ is central}\\ &=\mathrm{tr}_\alpha(yx) \quad \text{by Condition (a)},\end{aligned}$$

$$\begin{aligned}&\mathrm{tr}_{\alpha^{-1}}(S_\alpha(x))\\ &\quad=\lambda_{\alpha^{-1}}(G_{\alpha^{-1}}z_{\alpha^{-1}}S_\alpha(x))\\ &\quad=\hat{\varphi}(\alpha)\lambda_{\alpha^{-1}}(S_\alpha(G_\alpha^{-1})S_\alpha(z_\alpha)S_\alpha(x)) \quad \text{by Condition (c) and Lemma 6.8 (c)}\\ &\quad=\hat{\varphi}(\alpha)\lambda_{\alpha^{-1}}(S_\alpha(xz_\alpha G_\alpha^{-1})) \quad \text{by Lemma 1.1(a)}\end{aligned}$$

$$= \lambda_\alpha(G_\alpha^2 x z_\alpha G_\alpha^{-1}) \quad \text{by (7.5)}$$
$$= \lambda_\alpha(G_\alpha z_\alpha G_\alpha x G_\alpha^{-1}) \quad \text{since } z_\alpha \text{ is central}$$
$$= \mathrm{tr}_\alpha(G_\alpha x G_\alpha^{-1})$$
$$= \mathrm{tr}_\alpha(x) \quad \text{since } \mathrm{tr}_\alpha \text{ is symmetric,}$$

and

$$\mathrm{tr}_{\beta\alpha\beta^{-1}}(\varphi_\beta(x))$$
$$= \lambda_{\beta\alpha\beta^{-1}}(G_{\beta\alpha\beta^{-1}} z_{\beta\alpha\beta^{-1}} \varphi_\beta(x))$$
$$= \hat{\varphi}(\beta)^{-1} \lambda_{\beta\alpha\beta^{-1}}(\varphi_\beta(G_\alpha)\varphi_\beta(z_\alpha)\varphi_\beta(x)) \quad \text{by Condition (d) and Lemma 6.8(b)}$$
$$= \hat{\varphi}(\beta)^{-1} \lambda_{\beta\alpha\beta^{-1}}(\varphi_\beta(G_\alpha z_\alpha x))$$
$$= \hat{\varphi}(\beta)^{-1} \hat{\varphi}(\beta) \lambda_\alpha(G_\alpha z_\alpha x) \quad \text{by Corollary 6.2}$$
$$= \mathrm{tr}_\alpha(x).$$

Hence tr is a π-trace.

Conversely, suppose that tr is a π-trace. Recall that H_α^* is a right H_α-module for the action defined, for all $f \in H_\alpha^*$ and $a, x \in H_\alpha$, by

$$(f \leftharpoonup a)(x) = f(ax).$$

By Corollary 3.7(b), $(H_\alpha^*, \leftharpoonup)$ is free, its rank is 1 (resp. 0) if $H_\alpha \neq 0$ (resp. $H_\alpha = 0$), and λ_α is a basis vector for $(H_\alpha^*, \leftharpoonup)$. Thus, for any $\alpha \in \pi$, there exists $w_\alpha \in H_\alpha$ such that $\mathrm{tr}_\alpha = \lambda_\alpha \leftharpoonup w_\alpha$. Set $z_\alpha = G_\alpha^{-1} w_\alpha$. Let us verify that the family $z = (z_\alpha)_{\alpha \in \pi}$ verify conditions (a)–(d). By the definition of z_α, condition (a) is clearly verified. Let $\alpha \in \pi$ and $x \in H_\alpha$. For any $y \in H_\alpha$,

$$(\lambda_\alpha \leftharpoonup G_\alpha z_\alpha x)(y) = \lambda_\alpha(G_\alpha z_\alpha x y)$$
$$= \mathrm{tr}_\alpha(xy)$$
$$= \mathrm{tr}_\alpha(yx) \quad \text{by (7.1)}$$
$$= \lambda_\alpha(G_\alpha z_\alpha y x)$$
$$= \lambda_\alpha(G_\alpha x z_\alpha y) \quad \text{by (7.4)}$$
$$= (\lambda_\alpha \leftharpoonup G_\alpha x z_\alpha)(y).$$

Therefore $\lambda_\alpha \leftharpoonup G_\alpha z_\alpha x = \lambda_\alpha \leftharpoonup G_\alpha x z_\alpha$. Hence $G_\alpha z_\alpha x = G_\alpha x z_\alpha$ (since λ_α is a basis vector for $(H_\alpha^*, \leftharpoonup)$) and so $z_\alpha x = x z_\alpha$. Condition (b) is then verified. Let $\alpha \in \pi$. For any $x \in H_\alpha$,

$$(\lambda_{\alpha^{-1}} \leftharpoonup G_{\alpha^{-1}} S_\alpha(z_\alpha))(x)$$
$$= \lambda_{\alpha^{-1}}(G_{\alpha^{-1}} S_\alpha(z_\alpha) x)$$

$$= \lambda_{\alpha^{-1}}(S_\alpha(S_\alpha^{-1}(x) z_\alpha G_\alpha^{-1})) \quad \text{by Lemmas 1.1(a) and 6.8(c)}$$

$$= \hat{\varphi}(\alpha)^{-1} \lambda_\alpha(G_\alpha^2 S_\alpha^{-1}(x) z_\alpha G_\alpha^{-1}) \quad \text{by (7.5)}$$

$$= \hat{\varphi}(\alpha)^{-1} \lambda_\alpha(G_\alpha z_\alpha S_\alpha^{-1}(x)) \quad \text{by (7.4) and since } z_\alpha \text{ is central}$$

$$= \hat{\varphi}(\alpha)^{-1} \mathrm{tr}_\alpha(S_\alpha^{-1}(x))$$

$$= \hat{\varphi}(\alpha)^{-1} \mathrm{tr}_{\alpha^{-1}}(x) \quad \text{by (7.2)}$$

$$= (\lambda_{\alpha^{-1}} \leftharpoonup G_{\alpha^{-1}} \hat{\varphi}(\alpha)^{-1} z_{\alpha^{-1}})(x).$$

We conclude as above that $S_\alpha(z_\alpha) = \hat{\varphi}(\alpha)^{-1} z_{\alpha^{-1}}$, and so condition (c) is satisfied, Finally, let $\alpha, \beta \in \pi$. For any $x \in H_\alpha$,

$$(\lambda_\alpha \leftharpoonup \hat{\varphi}(\beta) G_\alpha \varphi_{\beta^{-1}}(z_{\beta\alpha\beta^{-1}}))(x)$$

$$= \hat{\varphi}(\beta) \lambda_\alpha(G_\alpha \varphi_{\beta^{-1}}(z_{\beta\alpha\beta^{-1}}) x)$$

$$= \lambda_{\beta\alpha\beta^{-1}}(\varphi_\beta(G_\alpha \varphi_{\beta^{-1}}(z_{\beta\alpha\beta^{-1}}) x)) \quad \text{by Corollary 6.2}$$

$$= \lambda_{\beta\alpha\beta^{-1}}(G_{\beta\alpha\beta^{-1}} z_{\beta\alpha\beta^{-1}} \varphi_\beta(x)) \quad \text{by Lemma 6.8(b)}$$

$$= \mathrm{tr}_{\beta\alpha\beta^{-1}}(\varphi_\beta(x))$$

$$= \mathrm{tr}_\alpha(x) \quad \text{by (7.3)}$$

$$= (\lambda_\alpha \leftharpoonup G_\alpha z_\alpha)(x).$$

Thus $G_\alpha z_\alpha = \hat{\varphi}(\beta) G_\alpha \varphi_{\beta^{-1}}(z_{\beta\alpha\beta^{-1}})$ and so $\varphi_\beta(z_\alpha) = \hat{\varphi}(\beta) z_{\beta\alpha\beta^{-1}}$. Hence condition (d) is verified and the lemma is proven. □

In the setting of Lemma 7.1, constructing a π-trace from a right π-integral $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ reduces to finding a family $z = (z_\alpha)_{\alpha\in\pi}$ which satisfies conditions (b)–(d) of Lemma 7.1. Let us give two possible choices of the family z.

Let Λ be a left integral for H_1 such that $\lambda_1(\Lambda) = 1$. Set $z_1 = \Lambda$ and $z_\alpha = 0$ if $\alpha \neq 1$. This family $z = (z_\alpha)_{\alpha\in\pi}$ verifies conditions (b)–(d) since H is unimodular (and so Λ is central and $S_1(\Lambda) = \Lambda$) and by Lemma 6.3(a). The π-trace obtained is given by $\mathrm{tr}_1 = \varepsilon$ and $\mathrm{tr}_\alpha = 0$ if $\alpha \neq 1$.

If the homomorphism $\hat{\varphi}$ of Corollary 6.2 is trivial (that is $\hat{\varphi}(\alpha) = 1$ for all $\alpha \in \pi$), then another possible choice is $z_\alpha = 1_\alpha$. In the two next lemmas, we give sufficient conditions for the homomorphism $\hat{\varphi}$ to be trivial.

Lemma 7.2. *Let H be a finite type crossed Hopf π-coalgebra with crossing φ. If H is semisimple or cosemisimple or if $\varphi_\beta|_{H_1} = \mathrm{id}_{H_1}$ for all $\beta \in \pi$, then $\hat{\varphi} = 1$.*

Proof. Let $\beta \in \pi$. If H is semisimple, then H_1 is semisimple and thus there exists a left integral Λ for H_1 such that $\varepsilon(\Lambda) = 1$ (by Sweedler [17, Theorem 5.1.8]). Now $\varphi_\beta(\Lambda) = \hat{\varphi}(\beta)\Lambda$ by Lemma 6.3(a). Therefore, using (6.3), $\hat{\varphi}(\beta) = \hat{\varphi}(\beta)\varepsilon(\Lambda) =$

$\varepsilon(\hat{\varphi}(\beta)\Lambda) = \varepsilon\varphi_\beta(\Lambda) = \varepsilon(\Lambda) = 1$. Suppose now that H is cosemisimple. By Theorem 5.4, there exists a right π-integral $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ for H such that $\lambda_1(1_1) = 1$. Then $\hat{\varphi}(\beta) = \hat{\varphi}(\beta)\lambda_1(1_1) = \lambda_1(\varphi_\beta(1_1)) = \lambda_1(1_1) = 1$. Suppose finally that $\varphi_\beta|_{H_1} = \mathrm{id}_{H_1}$. Let $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ be a non-zero right π-integral for H. Then $\hat{\varphi}(\beta)\lambda_1 = \lambda_1\varphi_\beta|_{H_1} = \lambda_1$ and thus $\hat{\varphi}(\beta) = 1$ (since $\lambda_1 \neq 0$ by Lemma 3.1). □

Lemma 7.3. *Let H be a finite type ribbon Hopf π-coalgebra with crossing φ and twist $\theta = \{\theta_\alpha\}_{\alpha\in\pi}$. Let $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ be a right π-integral for H. If $\lambda_1(\theta_1) \neq 0$, then $\hat{\varphi} = 1$.*

Proof. Let $\beta \in \pi$. By (6.4.c) and Corollary 6.2, $\lambda_1(\theta_1) = \lambda_1(\varphi_\beta(\theta_1)) = \hat{\varphi}(\beta)\lambda_1(\theta_1)$. Therefore $\hat{\varphi}(\beta) = 1$ since $\lambda_1(\theta_1) \neq 0$. □

We conclude with the following theorem, which follows directly from Lemma 7.1 (by choosing $z_\alpha = 1_\alpha$ for all $\alpha \in \pi$) and Lemmas 7.2 and 7.3.

Theorem 7.4. *Let H be a finite type unimodular ribbon Hopf π-coalgebra with crossing φ and twist $\theta = \{\theta_\alpha\}_{\alpha\in\pi}$. Let $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ be a right π-integral for H and $G = (G_\alpha)_{\alpha\in\pi}$ be the spherical π-grouplike element of H. Suppose that at least one of the following conditions is verified*:
(a) *H is semisimple*;
(b) *H is cosemisimple*;
(c) $\lambda_1(\theta_1) \neq 0$;
(d) $\varphi_\beta|_{H_1} = \mathrm{id}_{H_1}$ *for all $\beta \in \pi$.*
Then $\mathrm{tr} = (\mathrm{tr}_\alpha)_{\alpha\in\pi}$, defined by $\mathrm{tr}_\alpha(x) = \lambda_\alpha(G_\alpha x)$ for all $\alpha\in\pi$ and $x \in H_\alpha$, is a π-trace for H.

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