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Hopf monads

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Abstract

We introduce and study Hopf monads on autonomous categories (i.e., monoidal categories with duals). Hopf monads generalize Hopf algebras to a non-braided (and non-linear) setting. In particular, any monoidal adjunction between autonomous categories gives rise to a Hopf monad. We extend many fundamental results of the theory of Hopf algebras (such as the decomposition of Hopf modules, the existence of integrals, Maschke's criterium of semisimplicity, etc.) to Hopf monads. We also introduce and study quasitriangular and ribbon Hopf monads (again defined in a non-braided setting).

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Contents

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0. Introduction

In 1991, Reshetikhin and Turaev [12] introduced a new 3-manifold invariant. Its construction consists in representing the 3-manifold by surgery along a framed link and then assigning a scalar to the link by applying a suitable algorithm involving simple representations of a quantum group at a root of unity. Since then, this construction has been re-visited many times. In particular, Turaev [14] introduced the notion of a modular category, which is a semisimple ribbon category satisfying conditions of finiteness and non-degeneracy, and showed that such a category defines a 3-manifold invariant and indeed a TQFT.

A more general setting for constructing quantum invariants of 3-manifolds has been subsequently developed in [8], and more recently in [7,16], where the input data is a (non-necessarily semisimple) ribbon category $\mathcal{C}$ which admits a coend $A = \int^{X \in \mathcal{C}} {}^{\vee}X \otimes X$. This coend A is endowed with a very rich algebraic structure. In particular, it is a Hopf algebra in $\mathcal{C}$. In fact, in this setting, the quantum invariants depend only on certain structural morphisms of the coend A (see [2] for details).

Recall that a Hopf algebra in a braided category $\mathcal{C}$ is an object A of $\mathcal{C}$ which is both an algebra and a coalgebra in $\mathcal{C}$, and has an antipode. The structural morphisms satisfy the traditional axioms of a Hopf algebra, except that one has to replace the usual flip of vector spaces with the braiding τ of $\mathcal{C}$. More precisely, the axiom expressing the compatibility between the product $m : A \otimes A \to A$ and the coproduct $\Delta : A \to A \otimes A$ of A becomes:

$$\Delta m = (m \otimes m)(\mathrm{id}_A \otimes \tau_{A,A} \otimes \mathrm{id}_A)(\Delta \otimes \Delta).$$

Hopf algebras in braided categories have been extensively studied (see [1] and references therein). Many results about Hopf algebras have been extended to this setting.

However, general (non-necessarily braided) monoidal categories also play an important role in quantum topology. Firstly, they are the input data for another class of 3-manifold invariants, the Turaev–Viro invariants (see [4,15]). Also, via the center construction due to Drinfeld, they lead to a large class of braided categories: if $\mathcal{C}$ is a monoidal category, then its center $\mathcal{Z}(\mathcal{C})$ is a braided category. Under mild hypotheses, the category $\mathcal{Z}(\mathcal{C})$ admits a coend A, which is a Hopf algebra in $\mathcal{Z}(\mathcal{C})$. How can one describe this Hopf algebra A in terms of the category $\mathcal{C}$? Note that, if the coend of $\mathcal{C}$ exists, then it is a coalgebra but not a Hopf algebra (and in general not even an algebra), and therefore is not sufficient to describe A. What we need is a suitable generalization of the notion of Hopf algebra to a non-braided setting.

The aim of this paper is to provide such a generalization by introducing the notion of Hopf monad. What is a Hopf monad? Fix a category $\mathcal{C}$. Recall that the category $\mathrm{End}(\mathcal{C})$ of endofunctors of $\mathcal{C}$ is a monoidal category, with composition of endofunctors for monoidal product and trivial endofunctor $1_{\mathcal{C}}$ for unit object. A *monad* on $\mathcal{C}$ is an algebra in the monoidal category $\mathrm{End}(\mathcal{C})$. In other words, it is an endofunctor T of $\mathcal{C}$ endowed with natural transformations $\mu : T^2 \to T$ (the product) and $\eta : 1_{\mathcal{C}} \to \mathcal{C}$ (the unit) such that, for any object X of $\mathcal{C}$,

$$\mu_X \mu_{T(X)} = \mu_X T(\mu_X) \quad \text{and} \quad \mu_X \eta_{T(X)} = \mathrm{id}_{T(X)} = \mu_X T(\eta_X).$$

Monads are a very general notion: every adjunction defines a monad (and every monad comes from an adjunction). Let T be a monad on $\mathcal{C}$. An action of T on an object M of $\mathcal{C}$ is a morphism $r : T(M) \to M$ in $\mathcal{C}$ satisfying:

$$rT(r) = r\mu_M \quad \text{and} \quad r\eta_M = \mathrm{id}_M.$$

We call such a pair (M, r) a *T-module in $\mathcal{C}$* or, by abuse, a *T-module* (since the traditional term 'T-algebra' would be awkward in our context). Denote $T\text{-}\mathcal{C}$ the category of T-modules in $\mathcal{C}$, and $U_T : T\text{-}\mathcal{C} \to \mathcal{C}$ the forgetful functor defined by $U_T(M, r) = M$.

Now suppose that $\mathcal{C}$ is monoidal, and denote $\otimes$ its monoidal product and $\mathbb{1}$ its unit object. When does the monoidal structure of $\mathcal{C}$ lift to $T\text{-}\mathcal{C}$? The answer lies in the notion of bimonad introduced by Moerdijk [11]. A *bimonad* is a monad T which is comonoidal, that is, endowed with a natural transformation

$$T_2(X, Y) : T(X \otimes Y) \to T(X) \otimes T(Y)$$

(which plays the role of a coproduct) and a morphism $T_0 : T(\mathbb{1}) \to \mathbb{1}$ (which plays the role of a counit) satisfying certain compatibility axioms which generalize those of a bialgebra. For example, the axiom which corresponds with the compatibility of the product and the coproduct is:

$$T_2(X, Y)\, \mu_{X \otimes Y} = (\mu_X \otimes \mu_Y) T_2\big(T(X), T(Y)\big) T\big(T_2(X, Y)\big).$$

Note that no braiding is needed to write this down.

The next step is to define the notion of antipode for a bimonad. It comes out that the usual axioms for an antipode cannot be generalized to bimonads in a straightforward way. In order to bypass this difficulty, we use the categorical interpretation of an antipode in terms of duality. Let $\mathcal{C}$ be an autonomous category, that is, a monoidal category where each object X has a left dual ${}^\vee X$ and a right dual $X^\vee$. A *Hopf monad* on $\mathcal{C}$ is a bimonad T on $\mathcal{C}$ such that the category $T\text{-}\mathcal{C}$ is autonomous. This condition turns out to be equivalent to the existence of certain natural transformations

$$s^l_X : T\big({}^\vee T(X)\big) \to {}^\vee X \quad \text{and} \quad s^r_X : T\big(T(X)^\vee\big) \to X^\vee.$$

These are called the *left antipode* and the *right antipode* as they encode respectively the left and right autonomy of $T\text{-}\mathcal{C}$. The left and right duals of a T-module (M, r) are then given by:

$${}^\vee(M, r) = \big({}^\vee M, s^l_M T\big({}^\vee r\big)\big) \quad \text{and} \quad (M, r)^\vee = \big(M^\vee, s^r_M T\big(r^\vee\big)\big).$$

The notion of Hopf monad is very general. Firstly Hopf monads generalize Hopf algebras in braided autonomous categories. Indeed, every Hopf algebra A in a braided autonomous category

$\mathcal{C}$ yields a Hopf monad T on $\mathcal{C}$ defined by $T(X) = A \otimes X$. In particular a finite-dimensional Hopf algebra H over a field $\Bbbk$ yields a Hopf monad $T(V) = H \otimes V$ on $\mathrm{vect}(\Bbbk)$, and $T\text{-}\mathrm{vect}(\Bbbk) = {}_H\mathrm{mod}$. Secondly, Hopf monads do exist in a non-braided setting. In fact, any monoidal adjunction between autonomous categories gives rise to a Hopf monad. This general property allows us, for example, to give a Tannaka reconstruction theorem in terms of Hopf monads for fiber functors with values in a category of bimodules.

Quite surprisingly, many fundamental results of the theory of Hopf algebras (such as the decomposition of Hopf modules, the existence of integrals, Maschke's criterium of semisimplicity, etc.) extend to Hopf monads, sometimes in a straightforward way, sometimes at the price of some technical trick. Also, it turns out that many effective tools for the study of Hopf algebras have a monadic counterpart. In particular, we introduce the notions of sovereign grouplike element, R-matrix (and Drinfeld element), and twist for a Hopf monad T. These express the fact that the category $T\text{-}\mathcal{C}$ of T-modules is sovereign, or braided, or ribbon (recall that $\mathcal{C}$ itself need not be braided).

In a subsequent paper [3], we construct the *double* of a Hopf monad T on an autonomous category $\mathcal{C}$. This is a quasitriangular Hopf monad on $\mathcal{C}$ whose category of modules is canonically isomorphic (as a braided category) to the center of $T\text{-}\mathcal{C}$.

0.1. Organization of the paper

In Section 1, we review a few general facts about monads, which we use intensively throughout the text. In Section 2, we recall the definition of bimonads. In Section 3, we define antipodes and Hopf monads, and we establish their first properties. In Section 4, we introduce Hopf modules and prove the fundamental theorem for Hopf modules over a Hopf monad. We apply this in Section 5 to prove a theorem on the existence of universal integrals for a Hopf monad. In Section 6, we define semisimple and separable monads, and give a characterization of semisimple Hopf monads (which generalizes Maschke's theorem). In Section 7, we define sovereign grouplike elements and involutory Hopf monads. In Section 8, we define R-matrices and twists for a Hopf monad. Finally, in Section 9, we give other examples which illustrate the generality of the notion of Hopf monad, including a Tannaka reconstruction theorem.

0.2. Conventions and notations

Unless otherwise specified, categories are assumed to be small, and monoidal categories are assumed to be strict.

If $\mathcal{C}$ is a category, we denote $\mathrm{Ob}(\mathcal{C})$ the set of objects of $\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ the set of morphisms in $\mathcal{C}$ from an object X to an object Y. The identity functor of $\mathcal{C}$ will be denoted by $1_{\mathcal{C}}$. We denote $\mathcal{C}^{\mathrm{op}}$ the *opposite category* (where arrows are reversed).

Let $\mathcal{C}$, $\mathcal{D}$ be two categories. Functors from $\mathcal{C}$ to $\mathcal{D}$ are the objects of a category $\mathrm{FUN}(\mathcal{C}, \mathcal{D})$. Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, a morphism $\alpha : F \to G$ is a family $\{\alpha_X : F(X) \to G(X)\}_{X \in \mathrm{Ob}(\mathcal{C})}$ of morphisms in $\mathcal{D}$ satisfying the following naturality condition: $\alpha_Y F(f) = G(f)\alpha_X$ for every morphism $f : X \to Y$ in $\mathcal{C}$. Such a morphism is called a *natural transformation* (the term *functorial morphism* is also used). We denote $\mathrm{HOM}(F, G)$ the set $\mathrm{Hom}_{\mathrm{FUN}(\mathcal{C},\mathcal{D})}(F, G)$ of natural transformations from F to G.

If $\mathcal{C}$, $\mathcal{C}'$ are two categories, we denote $\sigma_{\mathcal{C},\mathcal{C}'}$ the flip functor $\mathcal{C} \times \mathcal{C}' \to \mathcal{C}' \times \mathcal{C}$ defined by $(X, Y) \mapsto (Y, X)$ and $(f, g) \mapsto (g, f)$.

1. Monads

In this section, we review a few general facts about monads, which we use intensively throughout the text.

1.1. Monads

Let $\mathcal{C}$ be a category. Recall that the category $\mathrm{End}(\mathcal{C})$ of endofunctors of $\mathcal{C}$ is strict monoidal with composition for monoidal product and identity functor $1_{\mathcal{C}}$ for unit object. A *monad* on $\mathcal{C}$ (also called a *triple*) is an algebra in $\mathrm{End}(\mathcal{C})$, that is, a triple (T, μ, η), where $T : \mathcal{C} \to \mathcal{C}$ is a functor, $\mu : T^2 \to T$ and $\eta : 1_{\mathcal{C}} \to T$ are natural transformations, such that:

$$\mu_X T(\mu_X) = \mu_X \mu_{T(X)}; \tag{1}$$

$$\mu_X \eta_{T(X)} = \mathrm{id}_{T(X)} = \mu_X T(\eta_X); \tag{2}$$

for any object X of $\mathcal{C}$.

Let (T, μ, η) be a monad on $\mathcal{C}$. An *action* of T on an object M of $\mathcal{C}$ is a morphism $r : T(M) \to M$ in $\mathcal{C}$ such that:

$$rT(r) = r\mu_M \quad \text{and} \quad r\eta_M = \mathrm{id}_M. \tag{3}$$

The pair (M, r) is then called a *T-module in $\mathcal{C}$*, or just a *T-module*.[3]

Given two T-modules (M, r) and (N, s) in $\mathcal{C}$, a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(M, N)$ is said to be *T-linear* if $fr = sT(f)$. Such an f is called a *morphism of T-modules* from (M, r) to (N, s). This gives rise to the category $T\text{-}\mathcal{C}$ of T-modules (with composition inherited from $\mathcal{C}$).

We will denote $U_T : T\text{-}\mathcal{C} \to \mathcal{C}$ the forgetful functor of T defined by $U_T(M, r) = M$ for any T-module (M, r) and $U_T(f) = f$ for any T-linear morphism f. Then U_T admits a left adjoint $F_T : \mathcal{C} \to T\text{-}\mathcal{C}$, which is given by $F_T(X) = (T(X), \mu_X)$ for any object X of $\mathcal{C}$ and $F_T(f) = T(f)$ for any morphism f in $\mathcal{C}$. Note that $T = U_T F_T$ is the monad of this adjunction (see example below).

Example 1.1 *(Monad of an adjunction).* Let $\mathcal{C}$ and $\mathcal{D}$ be categories. It is a standard fact (see [9]) that if $(F : \mathcal{C} \to \mathcal{D}, U : \mathcal{D} \to \mathcal{C})$ is a pair of adjoint functors, with adjunction morphisms $\eta : 1_{\mathcal{C}} \to UF$ and $\varepsilon : FU \to 1_{\mathcal{D}}$, then $T = UF$ is a monad on $\mathcal{C}$, with product $\mu = U(\varepsilon_F) : T^2 \to T$ and unit η. The monad (T, μ, η) is called the *monad of the adjunction* (F, U). Also there exists a unique functor $K : \mathcal{D} \to T\text{-}\mathcal{C}$ such that $U_T K = U$ and $KF = F_T$. The functor K is given by $A \mapsto (U(A), U(\varepsilon_A))$.

Example 1.2. Let $\mathcal{C}$ be a monoidal category and A be an object of $\mathcal{C}$. Let $A \otimes ?$ be the endofunctor of $\mathcal{C}$ defined by $(A \otimes ?)(X) = A \otimes X$ and $(A \otimes ?)(f) = \mathrm{id}_A \otimes f$. Let $m : A \otimes A \to A$ and $u : \mathbb{1} \to A$ be morphisms in $\mathcal{C}$. Define $\mu_X = m \otimes \mathrm{id}_X$ and $\eta_X = u \otimes \mathrm{id}_X$. Then $(A \otimes ?, \mu, \eta)$ is a monad on $\mathcal{C}$ if and only if (A, m, u) is an algebra in $\mathcal{C}$. If such is the case, then $(A \otimes ?)\text{-}\mathcal{C}$ is nothing but

[3] This is not standard terminology: pairs (M, r) are usually called T-algebras in the literature (see [9]). However, throughout this paper, pairs (M, r) are considered as the analogues of modules over an algebra, and so the term 'algebra' would be awkward in this context (in particular we must distinguish 'Hopf modules' from 'Hopf algebras').

the category of left A-modules in $\mathcal{C}$. Similarly, the endofunctor $? \otimes A$ is a monad on $\mathcal{C}$ if and only if A is an algebra in $\mathcal{C}$ and, if such is the case, $(? \otimes A)$-$\mathcal{C}$ is the category of right A-modules in $\mathcal{C}$.

The following classical lemma will be useful later on.

Lemma 1.3. *Let T be a monad on a category $\mathcal{C}$ and $U_T : T\text{-}\mathcal{C} \to \mathcal{C}$ be the forgetful functor. Let $\mathcal{D}$ be a second category and $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. Then we have a canonical bijection*

$$?^\sharp : \mathrm{Hom}(F, GT) \to \mathrm{Hom}(FU_T, GU_T), \quad f \mapsto f^\sharp$$

defined by $f^\sharp_{(M,r)} = G(r) f_M$ for any T-module (M,r). Its inverse

$$?^\flat : \mathrm{Hom}(FU_T, GU_T) \to \mathrm{Hom}(F, GT), \quad g \mapsto g^\flat$$

is given by $g^\flat_X = g_{(T(X),\mu_X)} F(\eta_X)$ for any object X of $\mathcal{C}$. If F, G are contravariant functors (that is, functors from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$), then the bijection becomes:

$$?^\sharp : \mathrm{Hom}\big(GT^{\mathrm{op}}, F\big) \to \mathrm{Hom}\big(GU_T^{\mathrm{op}}, FU_T^{\mathrm{op}}\big)$$

with $f^\sharp_{(M,r)} = f_M G(r)$ and $g^\flat_X = F(\eta_X) g_{(T(X),\mu_X)}$.

Proof. Let us verify the covariant case. We first remark that $?^\sharp$ and $?^\flat$ are well-defined. Indeed $g^\flat$ is clearly natural, and $f^\sharp$ is natural since, for any T-linear morphism $h : (M,r) \to (N,s)$, we have $f^\sharp_{(N,s)} F(h) = G(s) f_N F(h) = G(sT(h)) f_M = G(hr) f_M = G(h) f^\sharp_{(M,r)}$. Now $f^{\sharp\flat}_X = G(\mu_X) f_{T(X)} F(\eta_X) = G(\mu_X) GT(\eta_X) f_X = f_X$ for any object X of $\mathcal{C}$, and $g^{\flat\sharp}_{(M,r)} = G(r) g_{(T(M),\mu_M)} F(\eta_M) = g_{(M,r)} F(r) F(\eta_M) = g_{(M,r)}$ for any T-module (M,r). Hence $?^\sharp$ and $?^\flat$ are inverse to each other. The contravariant case is a mere reformulation. □

Given a functor $F : \mathcal{C} \to \mathcal{D}$ and a positive integer n, we denote $\mathcal{C}^n = \mathcal{C} \times \cdots \times \mathcal{C}$ and $F^{\times n} : \mathcal{C}^n \to \mathcal{D}^n$ the n-uple cartesian product of $\mathcal{C}$ and F. Note that if T is a monad on a category $\mathcal{C}$, then $T^{\times n}$ is a monad on $\mathcal{C}^n$, and we have $T^{\times n}\text{-}\mathcal{C}^n = (T\text{-}\mathcal{C})^n$ and $U_{T^{\times n}} = (U_T)^{\times n}$. Re-writing Lemma 1.3 for this monad, we get:

Lemma 1.4. *Let T be a monad on a category $\mathcal{C}$ and $U_T : T\text{-}\mathcal{C} \to \mathcal{C}$ be the forgetful functor. Fix an positive integer n. Let $\mathcal{D}$ be a second category and $F, G : \mathcal{C}^n \to \mathcal{D}$ be two functors. Then we have a canonical bijection*

$$?^\sharp : \mathrm{Hom}\big(F, GT^{\times n}\big) \to \mathrm{Hom}\big(FU_T^{\times n}, GU_T^{\times n}\big), \quad f \mapsto f^\sharp$$

defined by $f^\sharp_{(M_1,r_1),\dots,(M_n,r_n)} = G(r_1, \dots, r_n) f_{M_1,\dots,M_n}$. Its inverse

$$?^\flat : \mathrm{Hom}\big(FU_T^{\times n}, GU_T^{\times n}\big) \to \mathrm{Hom}\big(F, GT^{\times n}\big), \quad g \mapsto g^\flat$$

is given by $g^\flat_{X_1,\dots,X_n} = g_{(T(X_1),\mu_{X_1}),\dots,(T(X_n),\mu_{X_n})} F(\eta_{X_1}, \dots, \eta_{X_n})$. The contravariant case can be stated similarly (see Lemma 1.3).

1.2. Convolution product

Let $\mathcal{C}$, $\mathcal{D}$ be two categories and (T, μ, η) be a monad on $\mathcal{C}$. Let F, G, H be three functors $\mathcal{C}^n \to \mathcal{D}$. Given $f \in \mathrm{Hom}(F, GT^{\times n})$ and $g \in \mathrm{Hom}(G, HT^{\times n})$, define their *convolution product* $g * f \in \mathrm{Hom}(F, HT^{\times n})$ by setting, for any objects $X_1, \dots, X_n$ of $\mathcal{C}$,

$$(g * f)_{X_1,\dots,X_n} = H(\mu_{X_1}, \dots, \mu_{X_n})\, g_{T(X_1),\dots,T(X_n)}\, f_{X_1,\dots,X_n}. \tag{4}$$

This convolution product reflects the composition of natural transformations in the category $\mathrm{Fun}(\mathcal{C}^n, \mathcal{D})$ via the canonical bijection $\mathrm{Hom}(F, GT^{\times n}) \simeq \mathrm{Hom}(FU_T^{\times n}, GU_T^{\times n})$ given by Lemma 1.4.

We say that $f \in \mathrm{Hom}(F, GT^{\times n})$ is $*$-*invertible* if there exists $g \in \mathrm{Hom}(G, FT^{\times n})$ such that $g * f = F(\eta^{\times n})$ and $f * g = G(\eta^{\times n})$. This means that $f^\sharp$ is a natural isomorphism, with inverse $g^\sharp$. If such a g exists, then it is unique and we denote it f^{*-1}.

1.3. Central elements

Let T be a monad on a category $\mathcal{C}$. By Section 1.2, the set $\mathrm{Hom}(1_\mathcal{C}, T)$ is a monoid, with unit η, for the convolution product $*$ defined, for any $\phi, \psi \in \mathrm{Hom}(1_\mathcal{C}, T)$, by

$$(\phi * \psi)_X = \mu_X \phi_{T(X)} \psi_X = \mu_X T(\psi_X) \phi_X : X \to T(X). \tag{5}$$

Recall that, via the canonical bijection $?^\sharp : \mathrm{Hom}(1_\mathcal{C}, T) \to \mathrm{Hom}(U_T, U_T)$ of Lemma 1.3, this convolution product corresponds with composition of natural endomorphisms of the forgetful functor $U_T : T\text{-}\mathcal{C} \to T$.

Define maps $L, R : \mathrm{Hom}(1_\mathcal{C}, T) \to \mathrm{Hom}(T, T)$ as follows: given $a \in \mathrm{Hom}(1_\mathcal{C}, T)$, let $L_a, R_a \in \mathrm{Hom}(T, T)$ be the natural transformations defined, for any object X of $\mathcal{C}$, by:

$$(L_a)_X = \mu_X a_{T(X)} \quad \text{and} \quad (R_a)_X = \mu_X T(a_X). \tag{6}$$

Remark that $L_a b = a * b$ and $R_a b = b * a$ for all $a, b \in \mathrm{Hom}(1_\mathcal{C}, T)$.

A *central element* of T is a natural transformation $a \in \mathrm{Hom}(1_\mathcal{C}, T)$ such that $L_a = R_a$. For example, by (2), the unit η of T is a central element. Notice that any central element of T is in particular central in the monoid $(\mathrm{Hom}(1_\mathcal{C}, T), *, \eta)$.

Lemma 1.5. *Let T be a monad on a category $\mathcal{C}$ and $a \in \mathrm{Hom}(1_\mathcal{C}, T)$. The following conditions are equivalent:*

(i) *The morphism a is a central element of T;*
(ii) *For any T-module (M, r), the morphism $a^\sharp_{(M,r)} : M \to M$ is T-linear;*
(iii) *There exists a (necessarily unique) natural transformation $\tilde{a} : 1_{T\text{-}\mathcal{C}} \to 1_{T\text{-}\mathcal{C}}$ such that $U_T(\tilde{a}) = a^\sharp$.*

Proof. Clearly, (ii) is equivalent to (iii). Let (M, r) be a T-module. Then, using (3),

$$a^\sharp_{(M,r)} r = r a_M r = r T(r) a_{T(M)} = r \mu_M a_{T(M)} = r (L_a)_M$$

and

$$rT\left(a^{\sharp}_{(M,r)}\right) = rT(r)T(a_M) = r\mu_M T(a_M) = r(R_a)_M.$$

Therefore (i) is equivalent to (ii) by Lemma 1.3. □

1.4. The adjoint action

Let T be a monad on a category $\mathcal{C}$ and consider the maps $L, R: \mathrm{H{\scriptstyle OM}}(1_{\mathcal{C}}, T) \to \mathrm{H{\scriptstyle OM}}(T, T)$ defined as in (6).

Lemma 1.6. *The maps L and R are respectively a homomorphism and an anti-homomorphism of monoids from* $(\mathrm{H{\scriptstyle OM}}(1_{\mathcal{C}}, T), *, \eta)$ *to* $(\mathrm{H{\scriptstyle OM}}(T, T), \circ, \mathrm{id}_T)$. *Moreover $L_a R_b = R_b L_a$ for all $a, b \in \mathrm{H{\scriptstyle OM}}(1_{\mathcal{C}}, T)$.*

Proof. For any object X of $\mathcal{C}$, we have $(L_\eta)_X = \mu_X \eta_{T(X)} = \mathrm{id}_{T(X)}$ by (2) and, given $a, b \in \mathrm{H{\scriptstyle OM}}(1_{\mathcal{C}}, T)$,

$$\begin{aligned}
(L_{a*b})_X &= \mu_X \mu_{T(X)} a_{T^2(X)} b_{T(X)} \\
&= \mu_X T(\mu_X) a_{T^2(X)} b_{T(X)} \quad \text{by (1)} \\
&= \mu_X a_{T(X)} \mu_X b_{T(X)} = (L_a)_X (L_b)_X.
\end{aligned}$$

Therefore L is a homomorphism of monoids. Likewise one shows that R anti-homomorphism of monoids. Finally, given $a, b \in \mathrm{H{\scriptstyle OM}}(1_{\mathcal{C}}, T)$ and an object X of $\mathcal{C}$, we have:

$$\begin{aligned}
(L_a R_b)_X &= \mu_X a_{T(X)} \mu_X T(b_X) \\
&= \mu_X T(\mu_X) a_{T^2(X)} T(b_X) \\
&= \mu_X \mu_{T(X)} T^2(b_X) a_{T(X)} \quad \text{by (1)} \\
&= \mu_X T(b_X) \mu_X a_{T(X)} = (R_b)_X (L_a)_X,
\end{aligned}$$

and so $L_a R_b = R_b L_a$. □

Given $a \in \mathrm{H{\scriptstyle OM}}(1_{\mathcal{C}}, T)$, we get from Lemma 1.6 that L_a (respectively R_a) is invertible if and only if a is $*$-invertible and, if such the case, $L_a^{-1} = L_{a^{*-1}}$ (respectively $R_a^{-1} = R_{a^{*-1}}$). Denote $\mathrm{A{\scriptstyle UT}}(T)$ the group of natural automorphisms of T and $\mathrm{H{\scriptstyle OM}}(1_{\mathcal{C}}, T)^\times$ the group of $*$-invertible elements of the monoid $(\mathrm{H{\scriptstyle OM}}(1_{\mathcal{C}}, T), *, \eta)$. Define:

$$\mathrm{ad}: \quad \begin{cases} \mathrm{H{\scriptstyle OM}}(1_{\mathcal{C}}, T)^\times \to \mathrm{A{\scriptstyle UT}}(T), \\ \qquad a \mapsto \mathrm{ad}_a = L_a R_{a^{*-1}} = R_{a^{*-1}} L_a. \end{cases} \tag{7}$$

The map ad is a group morphism (by Lemma 1.6) and is called the *adjoint action* of T. Its kernel is made of the $*$-invertible central elements of T.

Notice that $\mathrm{ad}_a b = a * b * a^{*-1}$ for any $b \in \mathrm{H{\scriptstyle OM}}(1_{\mathcal{C}}, T)$.

1.5. Morphisms of monads

A *morphism of monads* between two monads (T, μ, η) and (T', μ', η') on a category $\mathcal{C}$ is a natural transformation $f : T \to T'$ such that:

$$f_X \mu_X = \mu'_X f_{T'(X)} T(f_X) \quad \text{and} \quad f_X \eta_X = \eta'_X \tag{8}$$

for any object X of $\mathcal{C}$.

Lemma 1.7. *Let T and T' be two monads on a category $\mathcal{C}$. Let $f : T \to T'$ be a natural transformation. The following conditions are equivalent*:

(i) *$f : T \to T'$ is a morphism of monads*;
(ii) *for any T'-module (M, r), the pair (M, rf_M) is a T-module.*

Moreover, if f is a morphism of monads, then the assignment $(M, r) \mapsto (M, rf_M)$ defines a functor $f^* : T'\text{-}\mathcal{C} \to T\text{-}\mathcal{C}$ which satisfies $U_T f^* = U_{T'}$. Lastly, for any functor $F : T'\text{-}\mathcal{C} \to T\text{-}\mathcal{C}$ such that $U_T F = U_{T'}$, there exists a unique morphism of monads $f : T \to T'$ such that $F = f^*$.

Proof. Let (M, r) be a T'-module. The pair (M, rf_M) is a T-module if and only if $rf_M \mu_M = rf_M T(rf_M) = rT'(r) f_{T'(M)} T(f_M)$ and $rf_M \eta_M = \mathrm{id}_M$. Using (3), this is equivalent to $rf_M \mu_M = r\mu'_M f_{T'(M)} T(f_M)$ and $rf_M \eta_M = r\eta_M$. By Lemma 1.3, this holds for each T'-module (M, r) if and only if f is a morphism of monads. Clearly, if f is a morphism of monads, then f^* is a well-defined functor. Let $F : T'\text{-}\mathcal{C} \to T\text{-}\mathcal{C}$ be a functor such that $U_T F = U_{T'}$. Given a T'-module (M, r), denote $\rho_{(M,r)}$ the T-action on M such that $F(M, r) = (M, \rho_{(M,r)})$. This defines a natural transformation $\rho \in \mathrm{HOM}(TU_{T'}, U_{T'})$. Let $f = \rho^\flat \in \mathrm{HOM}(T, T')$ (see Lemma 1.3). We have $\rho = f^\sharp$, that is, $\rho_{(M,r)} = rf_M$ for each T'-module (M, r). This shows simultaneously that f is a morphism of monads and $F = f^*$. $\square$

2. Bimonads

In this section, we review the definition and properties of a bimonad. This notion was introduced by Moerdijk in [11].

2.1. (Co-)monoidal functors

Let $(\mathcal{C}, \otimes, \mathbb{1})$ and $(\mathcal{D}, \otimes, \mathbb{1})$ be two monoidal categories. A *monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a triple (F, F_2, F_0), where $F : \mathcal{C} \to \mathcal{D}$ is a functor, $F_2 : F \otimes F \to F\otimes$ is a morphism of functors, and $F_0 : \mathbb{1} \to F(\mathbb{1})$ is a morphism in $\mathcal{D}$, such that:

$$F_2(X, Y \otimes Z)\big(\mathrm{id}_{F(X)} \otimes F_2(Y, Z)\big) = F_2(X \otimes Y, Z)\big(F_2(X, Y) \otimes \mathrm{id}_{F(Z)}\big); \tag{9}$$

$$F_2(X, \mathbb{1})(\mathrm{id}_{F(X)} \otimes F_0) = \mathrm{id}_{F(X)} = F_2(\mathbb{1}, X)(F_0 \otimes \mathrm{id}_{F(X)}); \tag{10}$$

for all objects X, Y, Z of $\mathcal{C}$.

A *comonoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a triple (F, F_2, F_0), where $F : \mathcal{C} \to \mathcal{D}$ is a functor, $F_2 : F\otimes \to F \otimes F$ is a natural transformation, and $F_0 : F(\mathbb{1}) \to \mathbb{1}$ is a morphism in $\mathcal{D}$, such that:

$$\big(\mathrm{id}_{F(X)} \otimes F_2(Y,Z)\big)F_2(X, Y\otimes Z) = \big(F_2(X,Y)\otimes \mathrm{id}_{F(Z)}\big)F_2(X\otimes Y, Z); \tag{11}$$

$$(\mathrm{id}_{F(X)} \otimes F_0)F_2(X,\mathbb{1}) = \mathrm{id}_{F(X)} = (F_0 \otimes \mathrm{id}_{F(X)})F_2(\mathbb{1}, X); \tag{12}$$

for all objects X, Y, Z of $\mathcal{C}$. Comonoidal functors are sometimes called *opmonoidal* in the literature.

A (co-)monoidal functor (F, F_2, F_0) is said to be *strong* (respectively *strict*) if F_2 and F_0 are isomorphisms (respectively identities). For example, the identity functor $1_{\mathcal{C}}$ of a monoidal category $\mathcal{C}$ is a strict (co-)monoidal functor.

Given a functor $F:\mathcal{C}\to\mathcal{D}$, a natural isomorphism $F_2: F\otimes F \to F\otimes$, and an isomorphism $F_0:\mathbb{1}\to F(\mathbb{1})$, the triple (F, F_2, F_0) is a monoidal functor if and only if (F, F_2^{-1}, F_0^{-1}) is a comonoidal functor.

Lemma 2.1. *Let F and G be two composable functors between monoidal categories.*

(a) *If F and G are monoidal functors, then GF is a monoidal functor with $(GF)_2 = G(F_2)G_2$ and $(GF)_0 = G(F_0)G_0$.*
(b) *If F and G are comonoidal functors, then GF is a comonoidal functor with $(GF)_2 = G_2G(F_2)$ and $(GF)_0 = G_0G(F_0)$.*

2.2. *(Co-)monoidal natural transformations*

A natural transformation $\varphi: F\to G$ between two monoidal functors $F:\mathcal{C}\to\mathcal{D}$ and $G:\mathcal{C}\to\mathcal{D}$ is *monoidal* if it satisfies:

$$\varphi_{X\otimes Y}F_2(X,Y) = G_2(X,Y)(\varphi_X\otimes\varphi_Y) \quad \text{and} \quad G_0 = \varphi_{\mathbb{1}}F_0 \tag{13}$$

for all objects X, Y of $\mathcal{C}$.

Likewise, a natural transformation $\varphi: F\to G$ between two comonoidal functors $F:\mathcal{C}\to\mathcal{D}$ and $G:\mathcal{C}\to\mathcal{D}$ is *comonoidal* if it satisfies:

$$G_2(X,Y)\varphi_{X\otimes Y} = (\varphi_X\otimes\varphi_Y)F_2(X,Y) \quad \text{and} \quad G_0\varphi_{\mathbb{1}} = F_0 \tag{14}$$

for all objects X, Y of $\mathcal{C}$.

2.3. *Bimonads*

A *bimonad* on a monoidal category $\mathcal{C}$ is a monad (T,μ,η) on $\mathcal{C}$ such that the functor $T:\mathcal{C}\to\mathcal{C}$ is comonoidal and the natural transformations $\mu: T^2\to T$ and $\eta: 1_{\mathcal{C}}\to T$ are comonoidal. Here $1_{\mathcal{C}}$ is the (strict) comonoidal identity functor of $\mathcal{C}$ and T^2 is the comonoidal functor obtained by composition of the comonoidal functor T with itself as in Lemma 2.1(b). Explicitly, μ and η are comonoidal if they satisfy, for all objects X, Y of $\mathcal{C}$,

$$T_2(X,Y)\mu_{X\otimes Y} = (\mu_X\otimes\mu_Y)T_2\big(T(X),T(Y)\big)T\big(T_2(X,Y)\big); \tag{15}$$

$$T_0\mu_{\mathbb{1}} = T_0T(T_0); \tag{16}$$

$$T_2(X,Y)\eta_{X\otimes Y} = (\eta_X\otimes\eta_Y); \tag{17}$$

$$T_0\eta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}. \tag{18}$$

Our notion of bimonad coincides exactly with the notion of 'Hopf monad' introduced in [11]. However, by analogy with the notions of bialgebra and Hopf algebra, we prefer to reserve the term 'Hopf monad' for bimonads with antipodes (see Section 3.3). This choice may be justified by the following example:

Example 2.2. Let $\mathcal{C}$ be a braided category, with braiding $\tau_{X,Y}: X \otimes Y \to Y \otimes X$, and A be an algebra in $\mathcal{C}$. Let $T = A \otimes ?$ be its associated monad on $\mathcal{C}$, see Example 1.2. Let $\Delta: A \to A \otimes A$ and $\varepsilon: A \to \mathbb{1}$ be morphisms in $\mathcal{C}$. Set

$$T_2(X, Y) = (\mathrm{id}_A \otimes \tau_{A,X} \otimes \mathrm{id}_Y)(\Delta \otimes \mathrm{id}_{X \otimes Y}): T(X \otimes Y) \to T(X) \otimes T(Y);$$

$$T_0 = \varepsilon: T(\mathbb{1}) \to \mathbb{1}.$$

Then (T, T_2, T_0) is a bimonad on $\mathcal{C}$ if and only if (A, Δ, ε) is a bialgebra in $\mathcal{C}$. Similarly, given a bialgebra A in $\mathcal{C}$, the endofunctor $? \otimes A$ is a bimonad on $\mathcal{C}$ with

$$(? \otimes A)_2(X, Y) = (\mathrm{id}_X \otimes \tau_{Y,A} \otimes \mathrm{id}_A)(\mathrm{id}_{X \otimes Y} \otimes \Delta) \quad \text{and} \quad (? \otimes A)_0 = \varepsilon.$$

In particular, any bialgebra H over a field $\Bbbk$ defines bimonads $H \otimes_{\Bbbk} ?$ and $? \otimes_{\Bbbk} H$ on the category $\mathrm{Vect}(\Bbbk)$ of $\Bbbk$-vector spaces.

We can reformulate the main result of [11] as follows (see also [10]):

Theorem 2.3 *(Moerdijk, 2002). Let T be a monad on a monoidal category $\mathcal{C}$. If T is a bimonad, then the category $T\text{-}\mathcal{C}$ of T-modules in $\mathcal{C}$ is monoidal by setting*:

$$(M, r) \otimes_{T\text{-}\mathcal{C}} (N, s) = \big(M \otimes N, (r \otimes s)T_2(M, N)\big) \quad \textit{and} \quad \mathbb{1}_{T\text{-}\mathcal{C}} = (\mathbb{1}, T_0).$$

Moreover this gives a bijective correspondence between:

- *bimonad structures for the monad T*;
- *monoidal structures of $T\text{-}\mathcal{C}$ such that the forgetful functor $U_T: T\text{-}\mathcal{C} \to \mathcal{C}$ is strict monoidal.*

Remark 2.4. Let T be a bimonad on a monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1})$. Then T can be viewed as a bimonad T^{op} on the monoidal category $\mathcal{C}^{\otimes \mathrm{op}} = (\mathcal{C}, \otimes^{\mathrm{op}}, \mathbb{1})$, with comonoidal structure $T_2^{\mathrm{op}} = T_2\, \sigma_{\mathcal{C},\mathcal{C}}$ and $T_0^{\mathrm{op}} = T_0$ (where $\sigma_{\mathcal{C},\mathcal{C}}$ is the flip functor). The bimonad T^{op} is called the *opposite* of the bimonad T. We have $T^{\mathrm{op}}\text{-}\mathcal{C}^{\otimes \mathrm{op}} = (T\text{-}\mathcal{C})^{\otimes \mathrm{op}}$.

Remark 2.5. Notice that the notion of bimonad is not 'self-dual': one may define a *bi-comonad* on a monoidal category $\mathcal{C}$ to be a bimonad of the opposite category $\mathcal{C}^{\mathrm{op}}$.

2.4. Monoidal adjunctions and bimonads

In this section, we extend to the monoidal setting the link between adjunctions and monads detailed in Example 1.1.

Theorem 2.6. *Let $\mathcal{C}$, $\mathcal{D}$ be two monoidal categories and $U:\mathcal{D}\to\mathcal{C}$ be a strong monoidal functor. Assume that the functor U has a left adjoint $F:\mathcal{C}\to\mathcal{D}$, and denote $T=UF$ the monad (on $\mathcal{C}$) of this adjunction. Then the functor F is a comonoidal functor, and so is T. Endowed with this comonoidal structure, the monad T is a bimonad. Moreover the canonical functor $K:\mathcal{D}\to T\text{-}\mathcal{C}$ is strong monoidal and satisfies $U_T K=U$ as monoidal functors and $KF=F_T$ as comonoidal functors.*

Remark 2.7. Any bimonad T is of the form of Theorem 2.6, since the forgetful functor U_T is strong monoidal, F_T is left adjoint to U_T, and $T=U_T F_T$.

Proof of Theorem 2.6. Denote $\eta:1_{\mathcal{C}}\to UF$ and $\varepsilon:FU\to 1_{\mathcal{D}}$ the adjunction morphisms. Define $F_2:F\otimes\to F\otimes F$ by setting, for any objects X,Y of $\mathcal{C}$,

$$F_2(X,Y)=\varepsilon_{F(X)\otimes F(Y)}F\big(U_2\big(F(X),F(Y)\big)\big)F(\eta_X\otimes\eta_Y)$$

and set $F_0=\varepsilon_{\mathbb{1}_{\mathcal{D}}}F(U_0):F(\mathbb{1}_{\mathcal{C}})\to\mathbb{1}_{\mathcal{D}}$. One verifies that (F,F_2,F_0) is a comonoidal functor. Since U is strong monoidal, we may also view it as a strong comonoidal functor (with comonoidal structure defined by U_2^{-1} and U_0^{-1}). Therefore both $T=UF$ and FU are comonoidal functors by Lemma 2.1(b). One checks that $\eta:1_{\mathcal{C}}\to T$, $\varepsilon:FU\to 1_{\mathcal{D}}$, and $\mu=U(\varepsilon_F):T^2\to T$ are comonoidal natural transformations. As a result, the monad $T=UF$ is a bimonad.

For any object A of $\mathcal{D}$, we have $K(A)=\big(U(A),U(\varepsilon_A)\big)$. For all objects A,B of $\mathcal{D}$, the morphism $U_2(A,B):U(A)\otimes U(B)\to U(A\otimes B)$ lifts to a (T-linear) morphism $K_2(A,B):K(A)\otimes K(B)\to K(A\otimes B)$. Likewise, $U_0:\mathbb{1}_{\mathcal{C}}\to U(\mathbb{1}_{\mathcal{D}})$ lifts to a (T-linear) morphism $K_0:\mathbb{1}_{T\text{-}\mathcal{C}}=(\mathbb{1}_{\mathcal{C}},T_0)\to K(\mathbb{1}_{\mathcal{D}})$. Moreover, (K,K_2,K_0) is a strong monoidal functor such that $U_TK=U$ as monoidal functors, because U_T is strict monoidal, faithful, and conservative. We also have $KF=F_T$ as comonoidal functors, since $KF=(UF,U(\varepsilon_F))=(T,\mu)=F_T$, $(KF)_2=U_2^{-1}(F,F)\,U(F_2)=T_2=(F_T)_2$, and $(KF)_0=U_0^{-1}U(F_0)=T_0=(F_T)_0$. □

Example 2.8. Szlachányi has shown that (left) bialgebroid, as defined in [13], may be interpreted in terms of bimonads. More precisely, let $\Bbbk$ is a commutative ring and B a $\Bbbk$-algebra. Denote ${}_B\mathrm{Mod}_B$ the category of B-bimodules, which is monoidal with tensor product $\otimes_B$ and unit object ${}_BB_B$. Then following data are equivalent:

- bimonads on ${}_B\mathrm{Mod}_B$ which commute with inductive limits;
- (left) bialgebroids with base B.

If A is a (left) bialgebroid, then the corresponding bimonad is $T=A\otimes_B$? and the monoidal categories $T\text{-}{}_B\mathrm{Mod}_B$ and ${}_A\mathrm{Mod}$ are equivalent. Note that in general the monoidal category ${}_B\mathrm{Mod}_B$ is not braided.

2.5. Morphisms of bimonads

A *morphism of bimonads* between two bimonads T and T' on a monoidal category $\mathcal{C}$ is a morphism of monads $f:T\to T'$ (see Section 1.5) which is comonoidal.

Lemma 2.9. *Let T and T' be two bimonads on a monoidal category $\mathcal{C}$. Let $f:T\to T'$ be a morphism of monads. Then f is a morphism of bimonads if and only if the functor $f^*:T'\text{-}\mathcal{C}\to$*

T-$\mathcal{C}$ induced by f (see Lemma 1.7) is monoidal strict. Moreover, for any strict monoidal functor $F : T'\text{-}\mathcal{C} \to T\text{-}\mathcal{C}$ such that $U_T F = U_{T'}$, there exists a unique morphism of bimonads $f : T \to T'$ such that $F = f^*$.

Proof. In view of Lemma 1.7, we have to show that the functor $F = f^*$ is monoidal strict if and only if f is comonoidal. We have $F(\mathbb{1}, T'_0) = (\mathbb{1}, T'_0 f_{\mathbb{1}})$ and

$$F\big((M,r)\otimes(N,s)\big) = \big(M\otimes N, (r\otimes s)T'_2(M,N) f_{M\otimes N}\big),$$
$$F(M,r)\otimes F(N,s) = \big(M\otimes N, (r\otimes s)(f_M\otimes f_N)T_2(M,N)\big),$$

for any T'-bimodules (M,r) and (N,s). We conclude by Lemma 1.4. □

3. Hopf monads

Let T be a bimonad on a monoidal category $\mathcal{C}$. By Theorem 2.3, the category T-$\mathcal{C}$ of T-modules is monoidal and the forgetful functor $U_T : T\text{-}\mathcal{C} \to \mathcal{C}$ is strict monoidal. Assuming $\mathcal{C}$ is autonomous (i.e., has duals), when is T-$\mathcal{C}$ autonomous as well? The answer lies in the notions of antipode and Hopf monad, which we introduce in this section. We first recall some properties of autonomous categories.

3.1. Autonomous categories

Recall that a *duality* in a monoidal category $\mathcal{C}$ is a quadruple (X, Y, e, d), where X, Y are objects of $\mathcal{C}$, $e : X\otimes Y \to \mathbb{1}$ (the *evaluation*) and $d : \mathbb{1} \to Y\otimes X$ (the *coevaluation*) are morphisms in $\mathcal{C}$, such that:

$$(e\otimes \mathrm{id}_X)(\mathrm{id}_X\otimes d) = \mathrm{id}_X \quad \text{and} \quad (\mathrm{id}_Y\otimes e)(d\otimes \mathrm{id}_Y) = \mathrm{id}_Y. \tag{19}$$

Then (X, e, d) is a *left dual* of Y, and (Y, e, d) is a *right dual* of X.

If $D = (X, Y, e, d)$ and $D' = (X', Y', e', d')$ are two dualities, two morphisms $f : X \to X'$ and $g : Y' \to Y$ are *in duality with respect to D and D'* if

$$e'(f\otimes \mathrm{id}_{Y'}) = e(\mathrm{id}_X\otimes g) \quad \big(\text{or, equivalently, } (\mathrm{id}_{Y'}\otimes f)d = (g\otimes \mathrm{id}_X)d'\big).$$

In that case we write $f = {}^\vee g_{D,D'}$ and $g = f^\vee_{D,D'}$, or simply $f = {}^\vee g$ and $g = f^\vee$ if the context justifies a more relaxed notation. Note that this defines a bijection between $\mathrm{Hom}_{\mathcal{C}}(X, X')$ and $\mathrm{Hom}_{\mathcal{C}}(Y', Y)$.

Left and right duals, if they exist, are essentially unique: if (Y, e, d) and (Y', e', d') are right duals of some object X, then there exists a unique isomorphism $u : Y \to Y'$ such that $e' = e(\mathrm{id}_X\otimes u^{-1})$ and $d' = (u\otimes \mathrm{id}_X)d$.

A *left autonomous* (respectively *right autonomous*, respectively *autonomous*) category is a monoidal category for which every object admits a left dual (respectively a right dual, respectively both a left and a right dual). Note that autonomous categories are also called rigid categories in the literature.

Assume $\mathcal{C}$ is a left autonomous category and, for each object X, pick a left dual $({}^\vee X, \mathrm{ev}_X, \mathrm{coev}_X)$. This data defines a strong monoidal functor ${}^\vee ? : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$, where $\mathcal{C}^{\mathrm{op}}$ is the opposite category to $\mathcal{C}$ with opposite monoidal structure. This monoidal functor is called the *left dual* *functor*. Notice that the actual choice of left duals is innocuous in the sense that different choices of left duals define canonically isomorphic left dual functors.

Likewise, if $\mathcal{C}$ is a right autonomous category, picking a right dual $(X^\vee, \widetilde{\mathrm{ev}}_X, \widetilde{\mathrm{coev}}_X)$ for each object X defines a strong monoidal functor $?^\vee : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$, called the *right dual functor*.

Remark 3.1. Subsequently, when dealing with left or right autonomous categories, we shall always assume tacitly that left duals or right duals have been chosen. Moreover, in formulae, we will often abstain (by abuse) from writing down the following canonical isomorphisms:

$$
\begin{aligned}
&{}^\vee ?_2(X, Y) : {}^\vee Y \otimes {}^\vee X \to {}^\vee (X \otimes Y), \quad {}^\vee ?_0 : \mathbb{1} \to {}^\vee \mathbb{1},\\
&?^\vee_2(X, Y) : Y^\vee \otimes X^\vee \to (X \otimes Y)^\vee, \quad ?^\vee_0 : \mathbb{1} \to \mathbb{1}^\vee.
\end{aligned}
$$

Remark 3.2. If $\mathcal{C}$ is autonomous, then the functors $?^{\vee \mathrm{op}}$ and ${}^\vee ?$ are canonically quasi-inverse. More precisely, for any object X of $\mathcal{C}$, we have the following canonical natural isomorphisms:

$$
\begin{aligned}
&(\widetilde{\mathrm{ev}}_X \otimes \mathrm{id}_{{}^\vee (X^\vee)})(\mathrm{id}_X \otimes \mathrm{coev}_{X^\vee}) : X \to {}^\vee\big(X^\vee\big),\\
&(\mathrm{id}_{({}^\vee X)^\vee} \otimes \mathrm{ev}_X)(\widetilde{\mathrm{coev}}_{{}^\vee X} \otimes \mathrm{id}_X) : X \to \big({}^\vee X\big)^\vee.
\end{aligned}
$$

Again, we will often abstain from writing down these isomorphisms.

Let $\mathcal{C}$, $\mathcal{D}$ be autonomous categories (with chosen left and right duals). For any functor $F : \mathcal{C} \to \mathcal{D}$, we define two functors ${}^! F : \mathcal{C} \to \mathcal{D}$ and $F^! : \mathcal{C} \to \mathcal{D}$ by setting:

$$
{}^! F(X) = {}^\vee F\big(X^\vee\big), \qquad {}^! F(f) = {}^\vee F\big(f^\vee\big), \qquad F^!(X) = F\big({}^\vee X\big)^\vee, \qquad F^!(f) = F\big({}^\vee f\big)^\vee
$$

for all object X and morphism f in $\mathcal{C}$. For any natural transformation $\alpha : F \to G$ between functors from $\mathcal{C}$ to $\mathcal{D}$, we define two natural transformations ${}^!\alpha : {}^! G \to {}^! F$ and $\alpha^! : G^! \to F^!$ by setting:

$$
{}^!\alpha_X = {}^\vee(\alpha_{X^\vee}) : {}^! G(X) \to {}^! F(X) \quad \text{and} \quad \alpha^!{}_X = (\alpha_{{}^\vee X})^\vee : G^!(X) \to F^!(X)
$$

for any object X of $\mathcal{C}$. These assignments lead to functors

$$
{}^! ? : \mathrm{FUN}(\mathcal{C}, \mathcal{D})^{\mathrm{op}} \to \mathrm{FUN}(\mathcal{C}, \mathcal{D}) \quad \text{and} \quad ?^! : \mathrm{FUN}(\mathcal{C}, \mathcal{D})^{\mathrm{op}} \to \mathrm{FUN}(\mathcal{C}, \mathcal{D}).
$$

Remark 3.3. The functors ${}^! ?$ and $?^!$ enjoy the following properties: given two composable functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ between autonomous categories, we have ${}^! 1_{\mathcal{C}} = 1_{\mathcal{C}} = 1_{\mathcal{C}}{}^!$, ${}^!(FG) = {}^! F {}^! G$ and $(FG)^! = F^! G^!$ (up to the canonical isomorphisms of Remark 3.2). Moreover, the functors ${}^! ?$ and $(?^!)^{\mathrm{op}}$ are quasi-inverse to each other.

3.2. Strong monoidal functors and duality

Let $F : \mathcal{C} \to \mathcal{D}$ be a strong monoidal functor between monoidal categories. If $D = (X, Y, e, d)$ is a duality in $\mathcal{C}$, then

$$
F(D) = \big(F(X), F(Y), F_0^{-1} F(e) F_2(X, Y), F_2(Y, X)^{-1} F(h) F_0\big)
$$

is a duality in $\mathcal{D}$. In particular, if $\mathcal{C}$ and $\mathcal{D}$ are left (respectively right) autonomous, we have a canonical isomorphism:

$$F_1^l(X): F\left({}^\vee X\right) \xrightarrow{\sim} {}^\vee F(X) \quad \left(\text{respectively } F_1^r(X): F\left(X^\vee\right) \xrightarrow{\sim} F(X)^\vee\right).$$

Lemma 3.4. *Let $F, G: \mathcal{C} \to \mathcal{D}$ be strong monoidal functors and $\alpha: F \to G$ be a monoidal natural transformation. If $\mathcal{C}$ is left or right autonomous, then α is an isomorphism. More precisely, if $\mathcal{C}$ is left* (*respectively right*) *autonomous then, for any object X of $\mathcal{C}$,*

$$\alpha_X^{-1} = (\alpha_{{}^\vee X})^\vee_{G(D),F(D)} \quad \left(\textit{respectively } \alpha_X^{-1} = {}^\vee(\alpha_{X^\vee})_{F(D),G(D)}\right)$$

where D is the duality $({}^\vee X, X, \mathrm{ev}_X, \mathrm{coev}_X)$ (respectively $(X, X^\vee, \widetilde{\mathrm{ev}}_X, \widetilde{\mathrm{coev}}_X)$). In particular, if $\mathcal{C}$ is autonomous, then $\alpha^{-1} = \alpha^! = {}^!\alpha$ (up to the canonical isomorphisms of Remark 3.2), *where ${}^!?$ and $?^!$ are the functors of Remark* 3.3.

Proof. Let $D = (X, Y, e, d)$ be a duality in $\mathcal{C}$. Denote ${}^\vee\alpha_Y: G(X) \to F(X)$ the left dual of $\alpha_Y: F(Y) \to G(Y)$ with respect to the dualities $F(D) = (F(X), F(Y), e_F, d_F)$ and $G(D) = (G(X), G(Y), e_G, d_G)$. We have

$$\begin{aligned} e_G(\alpha_X \otimes \alpha_Y) &= G_0^{-1} G(e) G_2(X, Y)(\alpha_X \otimes \alpha_Y) = G_0^{-1} G(e) \alpha_{X \otimes Y} F_2(X, Y) \\ &= G_0^{-1} \alpha_{\mathbb{1}} F(e) F_2(X, Y) = F_0^{-1} F(e) F_2(X, Y) = e_F \end{aligned}$$

and similarly $(\alpha_Y \otimes \alpha_X) d_F = d_G$. Now

$$\begin{aligned} {}^\vee\alpha_Y \alpha_X &= \left(e_G(\alpha_X \otimes \alpha_Y) \otimes \mathrm{id}_{F(X)}\right)(\mathrm{id}_{F(X)} \otimes d_F) \\ &= (e_F \otimes \mathrm{id}_{F(X)})(\mathrm{id}_{F(X)} \otimes d_F) = \mathrm{id}_{F(X)} \end{aligned}$$

and

$$\begin{aligned} \alpha_X {}^\vee\alpha_Y &= (e_G \otimes \mathrm{id}_{G(X)})\left(\mathrm{id}_{G(X)} \otimes (\alpha_Y \otimes \alpha_X) d_F\right) \\ &= (e_G \otimes \mathrm{id}_{G(X)})(\mathrm{id}_{G(X)} \otimes d_G) = \mathrm{id}_{G(X)}. \end{aligned}$$

Hence ${}^\vee\alpha_Y$ is inverse to α_X. $\square$

3.3. Antipodes

Let (T, μ, η) be a bimonad on a monoidal category $\mathcal{C}$.

If $\mathcal{C}$ is left autonomous, then a *left antipode for T* is a natural transformation $s^l = \{s^l_X: T({}^\vee T(X)) \to {}^\vee X\}_{X \in \mathrm{Ob}(\mathcal{C})}$ satisfying:

$$T_0 T(\mathrm{ev}_X) T\left({}^\vee\eta_X \otimes \mathrm{id}_X\right) = \mathrm{ev}_{T(X)}\left(s^l_{T(X)} T\left({}^\vee\mu_X\right) \otimes \mathrm{id}_{T(X)}\right) T_2\left({}^\vee T(X), X\right); \tag{20}$$

$$(\eta_X \otimes \mathrm{id}_{{}^\vee X}) \mathrm{coev}_X T_0 = \left(\mu_X \otimes s^l_X\right) T_2\left(T(X), {}^\vee T(X)\right) T(\mathrm{coev}_{T(X)}); \tag{21}$$

for every object X of $\mathcal{C}$.

If $\mathcal{C}$ is right autonomous, then a *right antipode for* T is a natural transformation $s^r = \{s^r_X : T(T(X)^\vee) \to X^\vee\}_{X \in \mathrm{Ob}(\mathcal{C})}$ satisfying:

$$T_0 T(\widetilde{\mathrm{ev}}_X) T\big(\mathrm{id}_X \otimes \eta^\vee_X\big) = \widetilde{\mathrm{ev}}_{T(X)}\big(\mathrm{id}_{T(X)} \otimes s^r_{T(X)} T\big(\mu^\vee_X\big)\big) T_2\big(X, T(X)^\vee\big); \tag{22}$$

$$(\mathrm{id}_{X^\vee} \otimes \eta_X)\widetilde{\mathrm{coev}}_X T_0 = \big(s^r_X \otimes \mu_X\big) T_2\big(T(X)^\vee, T(X)\big) T(\widetilde{\mathrm{coev}}_{T(X)}); \tag{23}$$

for every object X of $\mathcal{C}$.

Remark 3.5. This apparently complicated definition is justified by Theorem 3.8. The notion of left and right antipodes generalize the classical notion of an antipode and its inverse for a bialgebra. For details, see Example 3.10 below.

Remark 3.6. Let T be a bimonad on a left autonomous category $\mathcal{C}$, endowed with a left antipode s^l. Then s^l is a right antipode for the bimonad T^{op} on the right autonomous category $\mathcal{C}^{\otimes\,\mathrm{op}}$ (as defined in Remark 2.4). Likewise, if T is a bimonad on a right autonomous category $\mathcal{C}$ endowed with a right antipode s^r, then s^r is a left antipode for T^{op}.

The next theorem translates the fact that the antipode of a (classical) Hopf algebra is an anti-homomorphism of bialgebras.

Theorem 3.7. *Let T be a bimonad on a monoidal category $\mathcal{C}$. If s^l is a left antipode of T (assuming $\mathcal{C}$ is left autonomous), then we have*:

$$s^l_X \mu_{{}^\vee T(X)} = s^l_X T\big(s^l_{T(X)}\big) T^2\big({}^\vee\mu_X\big); \tag{24}$$

$$s^l_X \eta_{{}^\vee T(X)} = {}^\vee\eta_X; \tag{25}$$

$$s^l_{X\otimes Y} T\big({}^\vee T_2(X,Y)\big) = \big(s^l_Y \otimes s^l_X\big) T_2\big({}^\vee T(Y), {}^\vee T(X)\big); \tag{26}$$

$$s^l_{\mathbb{1}} T\big({}^\vee T_0\big) = T_0. \tag{27}$$

Likewise, if s^r is a right antipode of T (assuming $\mathcal{C}$ is right autonomous), then we have:

$$s^r_X \mu_{T(X)^\vee} = s^r_X T\big(s^r_{T(X)}\big) T^2\big(\mu^\vee_X\big); \tag{28}$$

$$s^r_X \eta_{T(X)^\vee} = \eta^\vee_X; \tag{29}$$

$$s^r_{X\otimes Y} T\big(T_2(X,Y)^\vee\big) = \big(s^r_Y \otimes s^r_X\big) T_2\big(T(Y)^\vee, T(X)^\vee\big); \tag{30}$$

$$s^r_{\mathbb{1}} T\big(T_0{}^\vee\big) = T_0. \tag{31}$$

Proof. The 'right part' can be deduced from the 'left part' by Remark 3.6. We prove here the 'multiplicative' assertions (24) and (25). The 'comultiplicative' assertions (26), (27) (and (30), (31)), which are stated here for convenience, will be proved in Section 3.5. Note that we will not use these assertions until then!

Assume $\mathcal{C}$ is left autonomous and T has a left antipode s^l. Let us show (24). Fix an object X of $\mathcal{C}$. Setting $\mathcal{L}_X = s^l_X \mu_{{}^\vee T(X)}$ and $\mathcal{R}_X = s^l_X T(s^l_{T(X)}) T^2({}^\vee\mu_X)$, we must prove $\mathcal{L}_X = \mathcal{R}_X$.

Recall (see Lemma 2.1(b)) that T^2 is a comonoidal functor. Define $\mu_X^{(2)} : T^3(X) \to T(X)$ and $D_X : T^2(\mathbb{1}) \to T^3(X) \otimes T^2({}^\vee T(X))$ by

$$\mu_X^{(2)} = \mu_X T(\mu_X) \quad \text{and} \quad D_X = T_2^2\big(T(X), {}^\vee T(X)\big) T^2(\mathrm{coev}_{T(X)}).$$

Firstly, we have:

$$\big(\mu_X^{(2)} \otimes \mathcal{L}_X\big) D_X = \big(\mu_X^{(2)} \otimes \mathcal{R}_X\big) D_X. \tag{32}$$

Indeed

$$\begin{aligned}
&\big(\mu_X^{(2)} \otimes \mathcal{L}_X\big) D_X \\
&\quad = \big(\mu_X \mu_{T(X)} \otimes s_X^l \mu_{{}^\vee T(X)}\big) T_2^2\big(T(X), {}^\vee T(X)\big) T^2(\mathrm{coev}_{T(X)}) \quad \text{by (1)} \\
&\quad = \big(\mu_X \otimes s_X^l\big) T_2\big(T(X), {}^\vee T(X)\big) \mu_{T(X) \otimes {}^\vee T(X)} T^2(\mathrm{coev}_{T(X)}) \quad \text{by (15)} \\
&\quad = \big(\mu_X \otimes s_X^l\big) T_2\big(T(X), {}^\vee T(X)\big) T(\mathrm{coev}_{T(X)}) \mu_{\mathbb{1}} \\
&\quad = (\eta_X \otimes \mathrm{id}_{{}^\vee X}) \,\mathrm{coev}_X\, T_0 \mu_{\mathbb{1}} \quad \text{by (21)} \\
&\quad = (\eta_X \otimes \mathrm{id}_{{}^\vee X}) \,\mathrm{coev}_X\, T_0 T(T_0) \quad \text{by (16)} \\
&\quad = \big(\mu_X \otimes s_X^l\big) T_2\big(T(X), {}^\vee T(X)\big) T(\mathrm{coev}_{T(X)}\, T_0) \quad \text{by (21)} \\
&\quad = \big(\mu_X \otimes s_X^l\big) T_2\big(T(X), {}^\vee T(X)\big) T\big((\mu_X \eta_{T(X)} \otimes \mathrm{id}_{{}^\vee T(X)}) \,\mathrm{coev}_{T(X)}\, T_0\big) \quad \text{by (2)} \\
&\quad = \big(\mu_X \otimes s_X^l\big) T_2\big(T(X), {}^\vee T(X)\big) \\
&\qquad \circ T\big(\big(\mu_X \mu_{T(X)} \otimes s_{T(X)}^l\big) T_2\big(T^2(X), {}^\vee T^2(X)\big) T(\mathrm{coev}_{T^2(X)})\big) \quad \text{by (21)} \\
&\quad = \big(\mu_X \otimes s_X^l\big) T_2\big(T(X), {}^\vee T(X)\big) \\
&\qquad \circ T\big(\big(\mu_X T(\mu_X) \otimes s_{T(X)}^l\big) T_2\big(T^2(X), {}^\vee T^2(X)\big) T(\mathrm{coev}_{T^2(X)})\big) \quad \text{by (1)} \\
&\quad = \big(\mu_X T(\mu_X) \otimes s_X^l T\big(s_{T(X)}^l\big) T^2\big({}^\vee \mu_X\big)\big) T_2\big(T^2(X), T\big({}^\vee T(X)\big)\big) \\
&\qquad \circ T\big(T_2\big(T(X), {}^\vee T(X)\big)\big) T^2(\mathrm{coev}_{T(X)}) \\
&\quad = \big(\mu_X^{(2)} \otimes \mathcal{R}_X\big) D_X.
\end{aligned}$$

Secondly, setting:

$$\nu_X = s_{T(X)}^l T\big({}^\vee \mu_X\big), \qquad \nu_X^{(2)} = \nu_X T(\nu_X), \quad \text{and} \quad E_X = \mathrm{ev}_{T(X)}\big(\nu_X^{(2)} \otimes \mu_X^{(2)}\big),$$

we have

$$(E_X \otimes \mathrm{id}_{T^2({}^\vee T(X))})(\mathrm{id}_{T^2({}^\vee T(X))} \otimes D_X) T_2^2\big({}^\vee T(X), \mathbb{1}\big) = \mathrm{id}_{T^2({}^\vee T(X))}. \tag{33}$$

Indeed, on the one hand, we have:

$$\begin{aligned}
&(\mathrm{id}_{T^2({}^\vee T(X))} \otimes D_X) T_2^2\big({}^\vee T(X), \mathbb{1}\big)\\
&\quad = \big(\mathrm{id}_{T^2({}^\vee T(X))} \otimes T_2^2\big(T(X), {}^\vee T(X)\big)\big)\\
&\qquad \circ T_2^2\big({}^\vee T(X), T(X) \otimes {}^\vee T(X)\big) T^2(\mathrm{id}_{{}^\vee T(X)} \otimes \mathrm{coev}_{T(X)})\\
&\quad = \big(T_2^2\big({}^\vee T(X), T(X)\big) \otimes \mathrm{id}_{T^3(X)}\big)\\
&\qquad \circ T_2^2\big({}^\vee T(X) \otimes T(X), {}^\vee T(X)\big) T^2(\mathrm{id}_{{}^\vee T(X)} \otimes \mathrm{coev}_{T(X)}) \quad \text{by (11).}
\end{aligned}$$

On the other hand, from (20) and (2), we obtain:

$$\mathrm{ev}_{T(X)}(\nu_X \otimes \mu_X) T_2\big({}^\vee T(X) \otimes T(X)\big) = T_0 T(\mathrm{ev}_{T(X)})$$

and so, using this twice,

$$\begin{aligned}
&E_X T_2^2\big({}^\vee T(X), T(X)\big)\\
&\quad = \mathrm{ev}_{T(X)}(\nu_X \otimes \mu_X) T_2\big({}^\vee T(X), T(X)\big) T(\nu_X \otimes \mu_X) T\big(T_2\big({}^\vee T(X), T(X)\big)\big)\\
&\quad = T_0 T(\mathrm{ev}_{T_X}) T(\nu_X \otimes \mu_X) T\big(T_2\big({}^\vee T(X), T(X)\big)\big) = T_0^2 T^2(\mathrm{ev}_{T(X)}).
\end{aligned}$$

Hence

$$\begin{aligned}
&(E_X \otimes \mathrm{id}_{T^2({}^\vee T(X))})(\mathrm{id}_{T^2({}^\vee T(X))} \otimes D_X) T_2^2\big({}^\vee T(X), \mathbb{1}\big)\\
&\quad = \big(T_0^2 T^2(\mathrm{ev}_{T(X)}) \otimes \mathrm{id}_{T^2({}^\vee T(X))}\big)\\
&\qquad \circ T_2^2\big({}^\vee T(X) \otimes T(X), {}^\vee T(X)\big) T^2(\mathrm{id}_{{}^\vee T(X)} \otimes \mathrm{coev}_{T(X)})\\
&\quad = \big(T_0^2 \otimes \mathrm{id}_{T^2({}^\vee T(X))}\big) T_2^2\big(\mathbb{1}, {}^\vee T(X)\big)\\
&\qquad \circ T^2\big((\mathrm{ev}_{T(X)} \otimes \mathrm{id}_{{}^\vee T(X)})(\mathrm{id}_{{}^\vee T(X)} \otimes \mathrm{coev}_{T(X)})\big)\\
&\quad = \mathrm{id}_{T^2({}^\vee T(X))} \quad \text{by (12) and (19),}
\end{aligned}$$

that is (33). Finally, we conclude:

$$\begin{aligned}
\mathcal{L}_X &= (E_X \otimes \mathcal{L}_X)(\mathrm{id}_{T^2({}^\vee T(X))} \otimes D_X) T_2^2\big({}^\vee T(X), \mathbb{1}\big) \quad \text{by (33)}\\
&= (E_X \otimes \mathcal{R}_X)(\mathrm{id}_{T^2({}^\vee T(X))} \otimes D_X) T_2^2\big({}^\vee T(X), \mathbb{1}\big) \quad \text{by (32)}\\
&= \mathcal{R}_X \quad \text{by (33).}
\end{aligned}$$

Let us prove (25). For any object X of $\mathcal{C}$, we have:

$$\begin{aligned}
\big(\mathrm{id}_{T(X)} \otimes {}^\vee\eta_X\big) \mathrm{coev}_{T(X)} &= (\eta_X \otimes \mathrm{id}_{{}^\vee X}) \mathrm{coev}_X\, T_0 \eta_{\mathbb{1}} \quad \text{by (18)}\\
&= \big(\mu_X \otimes s_X^l\big) T_2\big(T(X), {}^\vee T(X)\big) T(\mathrm{coev}_{T(X)}) \eta_{\mathbb{1}} \quad \text{by (21)}\\
&= \big(\mu_X \otimes s_X^l\big) T_2\big(T(X), {}^\vee T(X)\big) \eta_{T(X) \otimes {}^\vee T(X)} \mathrm{coev}_{T(X)}\\
&= \big(\mu_X \eta_{T(X)} \otimes s_X^l \eta_{{}^\vee T(X)}\big) \mathrm{coev}_{T(X)} \quad \text{by (17)}\\
&= \big(\mathrm{id}_{T(X)} \otimes s_X^l \eta_{{}^\vee T(X)}\big) \mathrm{coev}_{T(X)} \quad \text{by (2).}
\end{aligned}$$

Hence $s_X^l \eta_{{}^\vee T(X)} = {}^\vee\eta_X$ by (19). $\square$

The following theorem relates the existence of a left (respectively right) antipode for a bimonad to the existence of left (respectively right) duals for the category of modules over the bimonad.

Theorem 3.8. *Let T be a bimonad on a monoidal category $\mathcal{C}$.*

(a) *Assume $\mathcal{C}$ is left autonomous. Then T has a left antipode s^l if and only if the category T-$\mathcal{C}$ of T-modules is left autonomous. Moreover, if a left antipode exists, then it is unique. In terms of a left antipode s^l, left duals in T-$\mathcal{C}$ are given by*:

$$^\vee(M,r) = \left(^\vee M, s^l_M T\left(^\vee r\right)\right), \quad \mathrm{ev}_{(M,r)} = \mathrm{ev}_M, \quad \mathrm{coev}_{(M,r)} = \mathrm{coev}_M.$$

(b) *Assume $\mathcal{C}$ is right autonomous. Then T has a right antipode s^l if and only if the category T-$\mathcal{C}$ of T-modules is right autonomous. Moreover, if a right antipode exists, then it is unique. In terms of a right antipode s^r, right duals in T-$\mathcal{C}$ are given by*:

$$(M,r)^\vee = \left(M^\vee, s^r_M T\left(r^\vee\right)\right), \quad \widetilde{\mathrm{ev}}_{(M,r)} = \widetilde{\mathrm{ev}}_M, \quad \widetilde{\mathrm{coev}}_{(M,r)} = \widetilde{\mathrm{coev}}_M.$$

We prove Theorem 3.8 in the next section.

3.4. Proof of Theorem 3.8

Part (b) is just a re-writing of Part (a) applied to the bimonad T^{op} (see Remark 3.6). Let us show Part (a). Fix a left autonomous category $\mathcal{C}$ and a bimonad T on $\mathcal{C}$. Each object X of $\mathcal{C}$ has a left dual $(^\vee X, \mathrm{ev}_X, \mathrm{coev}_X)$. We first establish the following lemma:

Lemma 3.9. *If a T-module (M,r) has a left dual, then there exists a unique action $\delta : T(^\vee M) \to {}^\vee M$ such that $((^\vee M, \delta), \mathrm{ev}_M, \mathrm{coev}_M)$ is a left dual of (M,r).*

Proof. Assume (M,r) has a left dual $((N,\rho), e, d)$. The forgetful functor $U_T : T\text{-}\mathcal{C} \to \mathcal{C}$ being strict monoidal, (N,e,d) is a left dual of M in $\mathcal{C}$. Hence there is a unique isomorphism $u : N \to {}^\vee M$ such that $e = \mathrm{ev}_M(u \otimes \mathrm{id}_M)$ and $d = (\mathrm{id}_M \otimes u^{-1})\,\mathrm{coev}_M$. Define $\delta : T(^\vee M) \to {}^\vee M$ by $\delta = u\rho T(u^{-1})$. Then clearly $((^\vee M, \delta), \mathrm{ev}_M, \mathrm{coev}_M)$ is a left dual of (M,r). Now if we have another left dual of this form, say $((^\vee M, \delta'), \mathrm{ev}_M, \mathrm{coev}_M)$, then we have an isomorphism $v : (^\vee M, \delta) \to (^\vee M, \delta')$ such that $\mathrm{ev}_M = \mathrm{ev}_M(v \otimes \mathrm{id}_M)$, and so $v = \mathrm{id}_{^\vee M}$ and $\delta' = \delta$. □

Assume T-$\mathcal{C}$ is left autonomous. For any T-module (M,r), there exists a unique action $\delta_{(M,r)} : T(^\vee M) \to {}^\vee M$ such that $((^\vee M, \delta_{(M,r)}), \mathrm{ev}_M, \mathrm{coev}_M)$ is a left dual of (M,r) (by Lemma 3.9). If $f : M \to N$ is a T-linear morphism between two T-modules (M,r) and (N,s), then $^\vee f : {}^\vee N \to {}^\vee M$ is T-linear too. Hence δ satisfies the following naturality property: $^\vee f \delta_{(N,s)} = \delta_{(M,r)} T(^\vee f)$. In other words, we have: $\delta \in \mathrm{H{\scriptsize OM}}(T^\vee ?U_T^{\mathrm{op}}, {}^\vee ?U_T^{\mathrm{op}})$.

The conclusion of this discussion is that T-$\mathcal{C}$ is left autonomous if and only if there exits $\delta \in \mathrm{H{\scriptsize OM}}(T^\vee ?U_T^{\mathrm{op}}, {}^\vee ?U_T^{\mathrm{op}})$ such that, for any T-module (M,r), the triple $((^\vee M, \delta_{(M,r)}), \mathrm{ev}_M, \mathrm{coev}_M)$ is a left dual of (M,r).

Now consider an arbitrary element $\delta \in \mathrm{H{\scriptsize OM}}(T^\vee ?U_T^{\mathrm{op}}, {}^\vee ?U_T^{\mathrm{op}})$. By Lemma 1.3 (contravariant case) we have a canonical bijection:

$$?^\sharp : \mathrm{H{\scriptsize OM}}\left(T^\vee ?T^{\mathrm{op}}, {}^\vee ?\right) \to \mathrm{H{\scriptsize OM}}\left(T^\vee ?U_T^{\mathrm{op}}, {}^\vee ?U_T^{\mathrm{op}}\right), \quad f \mapsto f^\sharp.$$

Set $s^l = \delta^\flat \in \mathrm{H{\scriptsize OM}}(T^\vee ?T^{\mathrm{op}}, {}^\vee ?)$, where $?^\flat$ is the inverse of $?^\sharp$. Recall that $s^l_X = {}^\vee\eta_X \delta_{(T(X),\mu_X)}$ for any object X in $\mathcal{C}$. Moreover $\delta = (s^l)^\sharp$ and so $\delta_{(M,r)} = s^l_M T({}^\vee r)$ for any T-module (M, r). We have the following equivalences:

(A) the pair $({}^\vee M, \delta_{(M,r)})$ is a T-module for any T-module (M, r) if and only if s^l satisfies (24) and (25);

and, assuming the equivalent assertions of (A),

(B) the evaluation ev_M is T-linear for any T-module (M, r) if and only if s^l satisfies (20);
(C) the coevaluation coev_M is T-linear for any T-module (M, r) if and only if s^l satisfies (21).

Let us show (A). Recall that $({}^\vee M, \delta_{(M,r)})$ is a T-module if and only if both identities $\delta_{(M,r)}\mu_{{}^\vee M} = \delta_{(M,r)} T(\delta_{(M,r)})$ and $\delta_{(M,r)}\eta_{{}^\vee M} = \mathrm{id}_{{}^\vee M}$ hold. Replacing $\delta_{(M,r)}$ by $s^l_M T({}^\vee r)$ in the first identity, we get:

$$s^l_M T\big({}^\vee r\big)\mu_{{}^\vee M} = s^l_M T\big({}^\vee r\big) T\big(s^l_M\big) T^2\big({}^\vee r\big).$$

The left-hand side may be rewritten as $s^l_M \mu_{{}^\vee T(M)} T^2({}^\vee r)$. The right-hand side may be rewritten as $s^l_M T(s^l_{T(M)}) T^2({}^\vee T(r) {}^\vee r) = s^l_M T(s^l_{T(M)}) T^2({}^\vee \mu_M {}^\vee r)$. Therefore we finally get:

$$s^l_M \mu_{{}^\vee T(M)} T^2\big({}^\vee r\big) = s^l_M T\big(s^l_{T(M)}\big) T^2\big({}^\vee \mu_M\big) T^2\big({}^\vee r\big),$$

which is equivalent to (24) by Lemma 1.3. Likewise, the second identity is equivalent to (25) by a straightforward application of Lemma 1.3.

Let us show (B). Recall that ev_M is T-linear if an only if we have $T_0 T(\mathrm{ev}_M) = \mathrm{ev}_M(\delta_{(M,r)} \otimes r) T_2({}^\vee M, M)$. Replacing $\delta_{(M,r)}$ by $s^l_M T({}^\vee r)$, we get:

$$T_0 T(\mathrm{ev}_M) = \mathrm{ev}_M\big(s^l_M T\big({}^\vee r\big) \otimes r\big) T_2\big({}^\vee M, M\big). \tag{34}$$

Now, we have:

$$\begin{aligned}
&\mathrm{ev}_M\big(s^l_M T\big({}^\vee r\big) \otimes r\big) T_2\big({}^\vee M, M\big)\\
&\quad = \mathrm{ev}_{T(M)}\big({}^\vee r s^l_M T\big({}^\vee r\big) \otimes \mathrm{id}_{T(M)}\big) T_2\big({}^\vee M, M\big)\\
&\quad = \mathrm{ev}_{T(M)}\big(s^l_{T(M)} T\big({}^\vee T(r)\big) T\big({}^\vee r\big) \otimes \mathrm{id}_{T(M)}\big) T_2\big({}^\vee M, M\big)\\
&\quad = \mathrm{ev}_{T(M)}\big(s^l_{T(M)} T\big({}^\vee \mu_M\big) T\big({}^\vee r\big) \otimes \mathrm{id}_{T(M)}\big) T_2\big({}^\vee M, M\big) \quad \text{by (3)}\\
&\quad = \mathrm{ev}_{T(M)}\big(s^l_{T(M)} T\big({}^\vee \mu_M\big) \otimes \mathrm{id}_{T(M)}\big) T_2\big({}^\vee T(M), M\big) T\big({}^\vee r \otimes \mathrm{id}_M\big).
\end{aligned}$$

On the other hand, $T_0 T(\mathrm{ev}_M) = T_0 T(\mathrm{ev}_M) T({}^\vee \eta_M \otimes \mathrm{id}_M) T({}^\vee r \otimes \mathrm{id}_M)$ by (3). Therefore, by Lemma 1.3 and duality, (34) is equivalent to Axiom (20).

Let us show (C). Recall that coev_M is T-linear if and only if we have $\mathrm{coev}_M T_0 = (r \otimes \delta_{(M,r)}) T_2(M, {}^\vee M) T(\mathrm{coev}_M)$. By a computation similar to that of the proof of (B), this is equivalent to:

$$(r \otimes \mathrm{id}_{^\vee M})(\eta_M \otimes \mathrm{id}_{^\vee M})\,\mathrm{coev}_M\, T_0$$
$$= (r \otimes \mathrm{id}_{^\vee M})(\mu_M \otimes s_M) T_2\big(T(M), {}^\vee T(M)\big) T(\mathrm{coev}_{T(M)}),$$

and so, by Lemma 1.3 and duality, to Axiom (21).

Let us conclude. If s^l is a left antipode then, using only the 'multiplicative' part of Theorem 3.7 (which we have already proved) and setting $\delta = s^{l\sharp}$, we see by (A), (B), (C) that $(({}^\vee M, \delta_{(M,r)}), \mathrm{ev}_M, \mathrm{coev}_M)$ is a left dual of (M,r) in T-$\mathcal{C}$, and so T-$\mathcal{C}$ is left autonomous. Conversely, if T-$\mathcal{C}$ is left autonomous, then there exists δ such that $(({}^\vee M, \delta_{(M,r)}), \mathrm{ev}_M, \mathrm{coev}_M)$ is a duality in T-$\mathcal{C}$ and so, by (B) and (C), $s^l = \delta^\flat$ satisfies the axioms of a left antipode. By Lemma 3.9, such a δ is unique. Hence the uniqueness of a left antipode, since the correspondence $\delta \leftrightarrow s^l$ is bijective. This completes the proof of Theorem 3.8.

3.5. *End of the proof of Theorem 3.7*

We still have to prove assertions (26) and (27) of Theorem 3.7 (from which (30) and (31) can be deduced via Remark 3.6).

To show (26), let (M,r) and (N,s) be two T-modules. Recall that

$${}^\vee(N,s) \otimes {}^\vee(M,r) = \big({}^\vee N \otimes {}^\vee M, \big(s^l_N T\big({}^\vee s\big) \otimes s^l_M T\big({}^\vee r\big)\big) T_2\big({}^\vee N, {}^\vee M\big)\big)$$

and

$${}^\vee\big((M,r) \otimes (N,s)\big) = \big({}^\vee(M \otimes N), s^l_{M\otimes N} T\big({}^\vee T_2(M,N)^\vee(r \otimes s)\big)\big)$$

are canonically isomorphic via the isomorphism ${}^\vee ?_2(M,N) : {}^\vee N \otimes {}^\vee M \to {}^\vee(M \otimes N)$. By Lemma 3.9, we get (up to suitable identifications):

$$\big(s^l_N \otimes s^l_M\big) T_2\big({}^\vee T(N), {}^\vee T(M)\big) T\big({}^\vee(r \otimes s)\big) = s^l_{M\otimes N} T\big({}^\vee T_2(M,N)\big) T\big({}^\vee(r \otimes s)\big).$$

Hence (26) by applying Lemma 1.3.

Finally, via the isomorphism ${}^\vee ?_0 : \mathbb{1} \to {}^\vee\mathbb{1}$, the T-modules $(\mathbb{1}, T_0)$ and ${}^\vee(\mathbb{1}, T_0) = ({}^\vee\mathbb{1}, s^l_{\mathbb{1}} T({}^\vee T_0))$ are isomorphic. Hence $s^l_{\mathbb{1}} T({}^\vee T_0) = T_0$, that is, (27).

3.6. *Hopf monads*

A *left* (*respectively right*) *Hopf monad* is a bimonad on a left (respectively right) autonomous category which has a left (respectively right) antipode.

A *Hopf monad* is a bimonad on an autonomous category which has a left antipode and a right antipode. In particular, by Theorem 3.8, the category of modules over a Hopf monad is autonomous.

Example 3.10. Let $\mathcal{C}$ be a braided autonomous category with braiding by τ. Let A be a bialgebra in $\mathcal{C}$, with product m, unit u, coproduct Δ, and counit ε. Consider the bimonad $A \otimes ?$ (see Example 2.2). Firstly, let $S : A \to A$ be a morphism in $\mathcal{C}$ and define:

$$s^l_X = (\mathrm{ev}_A\, \tau_{^\vee A, A} \otimes \mathrm{id}_{^\vee X})\big(S \otimes \tau^{-1}_{^\vee A, {}^\vee X}\big) : A \otimes {}^\vee X \otimes {}^\vee A \to {}^\vee X.$$

Then s^l is a left antipode for the bimonad $A \otimes ?$ if and only if S is an antipode of the bialgebra A, that is, if and only if S satisfies:

$$m(S \otimes \mathrm{id}_A)\Delta = u\varepsilon = m(\mathrm{id}_A \otimes S)\Delta.$$

Secondly, let $S' : A \to A$ be another morphism in $\mathcal{C}$ and define:

$$s^r_X = (\widetilde{\mathrm{ev}}_A \otimes \mathrm{id}_{X^\vee})\big(S' \otimes \tau^{-1}_{A^\vee, X^\vee}\big) : A \otimes X^\vee \otimes A^\vee \to X^\vee.$$

Then s^r is a right antipode for the bimonad $A \otimes ?$ if and only if S' is an 'inverse of the antipode,' that is, setting $m^{\mathrm{op}} = m\tau^{-1}_{A,A}$, if and only if S' satisfies:

$$m^{\mathrm{op}}(S' \otimes \mathrm{id}_A)\Delta = u\varepsilon = m^{\mathrm{op}}(\mathrm{id}_A \otimes S')\Delta.$$

Thus $A \otimes ?$ is a Hopf monad if and only if A is a Hopf algebra in $\mathcal{C}$ with invertible antipode. Similarly, a right antipode for the bimonad $? \otimes A$ corresponds with an antipode for the bialgebra A, and a left antipode for $? \otimes A$ corresponds with an 'inverse of the antipode' for A. In particular, any *finite-dimensional* Hopf algebra H over a field $\Bbbk$ yields Hopf monads $H \otimes_\Bbbk ?$ and $? \otimes_\Bbbk H$ on the category $\mathrm{vect}(\Bbbk)$ of finite-dimensional $\Bbbk$-vector spaces.

Proposition 3.11. *Let T be a Hopf monad on an autonomous category $\mathcal{C}$. Then its left antipode s^l and its right antipode s^r are 'inverse' to each other in the sense*:

$$\mathrm{id}_{T(X)} = s^r_{{}^\vee T(X)} T\big(\big(s^l_X\big)^\vee\big) = s^l_{T(X)^\vee} T\big({}^\vee\big(s^r_X\big)\big)$$

for any object X of $\mathcal{C}$ (*up to the canonical isomorphisms of Remark* 3.2).

Proof. Let (M, r) be a T-module. We have ${}^\vee(M, r) = ({}^\vee M, s^l_M T({}^\vee r))$ and so $({}^\vee(M, r))^\vee = (({}^\vee M)^\vee, s^r_{{}^\vee M} T(T({}^\vee r)^\vee) T((s^l_M)^\vee))$. Via the canonical isomorphism of Remark 3.2, we have $r = s^r_{{}^\vee M} T(T({}^\vee r)^\vee) T((s^l_M)^\vee) = r s^r_{{}^\vee T(M)} T((s^l_M)^\vee)$. So, by Lemma 1.3, we have $\mathrm{id}_{T(X)} = s^r_{{}^\vee T(X)} T((s^l_X)^\vee)$. The second identity is obtained by replacing T with T^{op}. □

Recall that any a functor $F : \mathcal{C} \to \mathcal{D}$ between autonomous categories gives rise to functors ${}^!F : \mathcal{C} \to \mathcal{D}$ and $F^! : \mathcal{C} \to \mathcal{D}$ defined by ${}^!F(X) = {}^\vee F(X^\vee)$ and $F^!(X) = F({}^\vee X)^\vee$ (see Section 3.1).

Corollary 3.12. *Let T be a Hopf monad on an autonomous category $\mathcal{C}$. Then $T^!$ is right-adjoint to T, with adjunction morphisms $e : TT^! \to 1_\mathcal{C}$ and $h : 1_\mathcal{C} \to T^!T$ given by $e_X = s^r_{{}^\vee X} : T(T^!(X)) \to ({}^\vee X)^\vee \simeq X$ and $h_X = (s^l_X)^\vee : X \simeq ({}^\vee X)^\vee \to T^!(T(X))$. Likewise the functor ${}^!T$ is a right-adjoint to T.*

Remark 3.13. An interesting consequence of Corollary 3.12 is that a Hopf monad always preserves direct limits.

3.7. Monoidal adjunctions and Hopf monads

In this section, we show that the bimonad of a monoidal adjunction between autonomous categories is a Hopf monad.

Theorem 3.14. *Let $\mathcal{C}, \mathcal{D}$ be two monoidal categories and $U : \mathcal{D} \to \mathcal{C}$ be a strong monoidal functor which admits a left adjoint $F : \mathcal{C} \to \mathcal{D}$. Denote $T = UF$ the bimonad of this adjunction* (*see* Theorem 2.6). *Assume that the category $\mathcal{C}$ is left* (*respectively right*) *autonomous and that, for any object X of $\mathcal{C}$, $F(X)$ has a left* (*respectively right*) *dual in $\mathcal{D}$. Then T is a left* (*respectively right*) *Hopf monad. In particular if $\mathcal{C}$ and $\mathcal{D}$ are both left* (*respectively right*) *autonomous, then T is a left* (*respectively right*) *Hopf monad.*

As an immediate consequence, we have:

Corollary 3.15. *Let $\mathcal{C}, \mathcal{D}$ be two autonomous categories and $U : \mathcal{D} \to \mathcal{C}$ be a strong monoidal functor which admits a left adjoint $F : \mathcal{C} \to \mathcal{D}$. Then $T = UF$ is a Hopf monad on $\mathcal{C}$.*

Remark 3.16. Any Hopf monad T is of the form of Corollary 3.15, since the forgetful functor U_T is strong monoidal, F_T is left adjoint to U_T, and $T = U_T F_T$.

Proof of Theorem 3.14. Let us prove the "left" version, as left and right are exchanged by taking the opposite monoidal products on $\mathcal{C}$ and $\mathcal{D}$. We first prove it in the special case where $\mathcal{D}$ is left autonomous by constructing a left antipode s^l for the bimonad T. Denote $\eta : 1_{\mathcal{C}} \to UF$ and $\varepsilon : FU \to 1_{\mathcal{D}}$ the adjunction morphisms, and set $K(A) = \big(U(A), U(\varepsilon_A)\big)$ for any object A of $\mathcal{D}$. Recall that K is strong monoidal. Let A be an object of $\mathcal{D}$. Since A has a left dual in $\mathcal{D}$ and K is strong monoidal, the T-module $K(A)$ has a left dual in T-$\mathcal{C}$. By Lemma 3.9, we may choose this left dual of the form $(({}^\vee U(A), \rho_A), \mathrm{ev}_{U(A)}, \mathrm{coev}_{U(A)})$, with $\rho_A : T({}^\vee U(A)) \to {}^\vee U(A)$ uniquely determined. One verifies that ρ_A is natural in A. Note that the T-linearity of $\mathrm{ev}_{U(A)}$ and $\mathrm{coev}_{U(A)}$ translate respectively as:

$$T_0 T(\mathrm{ev}_{U(A)}) = \mathrm{ev}_{U(A)}\big(\rho_A \otimes U(\varepsilon_A)\big) T_2\big({}^\vee U(A), U(A)\big), \tag{35}$$

$$\mathrm{coev}_{U(A)}\, T_0 = \big(U(\varepsilon_A) \otimes \rho_A\big) T_2\big(U(A), {}^\vee U(A)\big) T(\mathrm{coev}_{U(A)}). \tag{36}$$

Now, for any object X of $\mathcal{C}$, set $s^l_X = {}^\vee\eta_X \rho_{F(X)} : T({}^\vee T(X)) \to {}^\vee X$. Clearly s^l is natural. Fix an object X of $\mathcal{C}$. By (2) and the naturality of ρ, we have:

$$\begin{aligned}\rho_{F(X)} &= {}^\vee\eta_{T(X)}{}^\vee\mu_X \rho_{F(X)} = {}^\vee\eta_{T(X)}{}^\vee U(\varepsilon_{F(X)})\rho_{F(X)} \\ &= {}^\vee\eta_{T(X)}\rho_{FT(X)} T\big({}^\vee U(\varepsilon_{F(X)})\big) = s^l_{T(X)} T\big({}^\vee\mu_X\big).\end{aligned}$$

Therefore we get:

$$\begin{aligned}&T_0 T(\mathrm{ev}_X) T\big({}^\vee\eta_X \otimes \mathrm{id}_X\big) \\ &\quad= T_0 T(\mathrm{ev}_{T(X)}) T(\mathrm{id}_{{}^\vee T(X)} \otimes \eta_X) \\ &\quad= \mathrm{ev}_{T(X)}\big(s^l_{T(X)} T\big({}^\vee\mu_X\big) \otimes \mu_X\big) T_2\big({}^\vee T(X), T(X)\big) T(\mathrm{id}_{{}^\vee T(X)} \otimes \eta_X) \quad \text{by (35)} \\ &\quad= \mathrm{ev}_{T(X)}\big(s^l_{T(X)} T\big({}^\vee\mu_X\big) \otimes \mathrm{id}_{T(X)}\big) T_2\big({}^\vee T(X), X\big) \quad \text{by (2).}\end{aligned}$$

Likewise we have:

$$
\begin{aligned}
&(\eta_X \otimes \mathrm{id}_{^\vee X})\,\mathrm{coev}_X\, T_0\\
&\quad = \big(\mathrm{id}_{T(X)} \otimes {}^\vee\eta_X\big)\,\mathrm{coev}_{T(X)}\, T_0\\
&\quad = \big(\mu_X \otimes {}^\vee\eta_X s^l_{T(X)} T\big({}^\vee\mu_X\big)\big) T_2\big(T(X), {}^\vee T(X)\big) T(\mathrm{coev}_{T(X)}) \quad \text{by (36)}\\
&\quad = \big(\mu_X \otimes s^l_X\big) T_2\big(T(X), {}^\vee T(X)\big) T(\mathrm{coev}_{T(X)}) \quad \text{by (2).}
\end{aligned}
$$

Hence s^l satisfies (20) and (21), that is, s^l is a left antipode for T.

Finally, let us prove the general case. Let $\mathcal{D}_0$ be the full subcategory of $\mathcal{D}$ formed by the objects of $\mathcal{D}$ which admit a left dual. The category $\mathcal{D}_0$ is left autonomous (see Section 3.1). By assumption, the functor F factors through $\mathcal{D}_0$. Denote $F_0 : \mathcal{C} \to \mathcal{D}_0$ the resulting functor and $U_0 : \mathcal{D}_0 \to \mathcal{C}$ the restriction of U to $\mathcal{D}_0$. Then U_0 is strong monoidal, F_0 is left adjoint to U_0, and $T = U_0 F_0$. Hence T is a left Hopf monad by the previous case. □

3.8. Morphisms of Hopf monads

A *morphism of Hopf monads* on an autonomous category is a morphism of their underlying bimonads (see Section 2.5).

Lemma 3.17. *A morphism $f : T \to T'$ of Hopf monads preserves the antipodes. More precisely, if T has a left antipode s^l and T' has a left antipode s'^l, then $s^l_X T({}^\vee f_X) = s'^l_X f_{^\vee T'(X)}$ for any object X of $\mathcal{C}$, and similarly for right antipodes.*

Proof. Let $f^* : T'\text{-}\mathcal{C} \to T\text{-}\mathcal{C}$ be the strict monoidal functor induced by f. Recall it is given by $f^*(M, r) = (M, r f_M)$. Let (M, r) be a T'-module. Since f^* is monoidal strict, $f^*({}^\vee(M, r)) = ({}^\vee M, s'^l_M T'({}^\vee r) f_{^\vee M})$ is a left dual of (M, r) and so canonically isomorphic to ${}^\vee f^*(M, r) = ({}^\vee M, s^l_M T({}^\vee f_M {}^\vee r))$. Therefore $s^l_M T({}^\vee f_M) T({}^\vee r) = s'^l_M f_{^\vee T'(M)} T({}^\vee r)$ by Lemma 3.9. Hence, by Lemma 1.3, we get $s^l_X T({}^\vee f_X) = s'^l_X f_{^\vee T'(X)}$ for any object X of $\mathcal{C}$. □

3.9. Convolution product and antipodes

Let T be a Hopf monad on an autonomous category $\mathcal{C}$. Let $?^\sharp : \mathrm{HOM}(1_\mathcal{C}, T) \to \mathrm{HOM}(U_T, U_T)$ be the isomorphism of Lemma 1.3 (with inverse $?^\flat$) and let ${}^!?, ?^! : \mathrm{End}(\mathcal{C})^{\mathrm{op}} \to \mathrm{End}(\mathcal{C})$ be the functors of Remark 3.3. Define two maps:

$$
S\colon \quad \begin{cases} \mathrm{HOM}(1_\mathcal{C}, T) \to \mathrm{HOM}(1_\mathcal{C}, T), \\ \qquad f \mapsto S(f) = \big({}^!\big(f^\sharp\big)\big)^\flat \end{cases} \tag{37}
$$

and

$$
S^{-1}\colon \quad \begin{cases} \mathrm{HOM}(1_\mathcal{C}, T) \to \mathrm{HOM}(1_\mathcal{C}, T), \\ \qquad f \mapsto S(f) = \big(\big(f^\sharp\big)^!\big)^\flat. \end{cases} \tag{38}
$$

Explicitly, using Theorem 3.8 and Lemma 1.3, we have:

$$S(f)_X = \left(s^l_X f_{^\vee T(X)}\right)^\vee \quad \text{and} \quad S^{-1}(f)_X = {}^\vee\left(s^r_X f_{T(X)^\vee}\right)$$

for any object X of $\mathcal{C}$ (up to the canonical isomorphisms of Remark 3.2), where s^l and s^r are the left and right antipodes of T respectively.

Lemma 3.18. *Let T be a Hopf monad on an autonomous category $\mathcal{C}$. Then the map S is an anti-automorphism of the monoid* $(\mathrm{HOM}(1_\mathcal{C}, T), *, \eta)$, *and S^{-1} is its inverse.*

Proof. Since the convolution product $*$ corresponds to composition of endomorphisms of U_T, and since the functors ${}^!?, ?^! : \mathrm{End}(\mathcal{C}) \to \mathrm{End}(\mathcal{C})^{\mathrm{op}}$ are strong monoidal, the maps S and S^{-1} are anti-endomorphisms of $\mathrm{HOM}(1_\mathcal{C}, T)$. Since the functors ${}^!?$ and $(?^!)^{\mathrm{op}}$ are inverse to each other (up to the canonical isomorphisms of Remark 3.2), the maps S and S^{-1} are inverse to each other. □

Example 3.19. For the Hopf monad $A \otimes ?$ (see Example 3.10), where A is a Hopf algebra in an autonomous braided category, the maps S and S^{-1} are given by $S(f) = (S_A \otimes 1_\mathcal{C}) f$ and $S^{-1}(f) = (S_A^{-1} \otimes 1_\mathcal{C}) f$, where S_A is the antipode of A.

3.10. Grouplike elements

A *grouplike element* of a bimonad T on a monoidal category $\mathcal{C}$ is a natural transformation $g : 1_\mathcal{C} \to T$ satisfying:

$$T_2(X, Y) g_{X \otimes Y} = g_X \otimes g_Y; \tag{39}$$

$$T_0 g_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}. \tag{40}$$

We will denote by $G(T)$ the set of grouplike elements of T. Using (15)–(18), we see that $(G(T), *, \eta)$ is a monoid, where $*$ is the convolution product (5).

Lemma 3.20. *Let T be a bimonad on a monoidal category $\mathcal{C}$. Via the canonical bijection* $\mathrm{HOM}(1_\mathcal{C}, T) \simeq \mathrm{HOM}(U_T, U_T)$ *of Lemma* 1.3, *grouplike elements of T correspond exactly with monoidal endomorphisms of the strict monoidal functor U_T.*

Proof. Let $g \in \mathrm{HOM}(1_\mathcal{C}, T)$. Then $g^\sharp \in \mathrm{HOM}(U_T, U_T)$ is monoidal if and only if, for all (M, r) and (N, s) in T-$\mathcal{C}$, we have $(r \otimes s) T_2(X, Y) g_{X \otimes Y} = (r \otimes s)(g_X \otimes g_Y)$ and $T_0 g_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$, which is equivalent to $g \in G(T)$ by Lemma 1.3. □

Lemma 3.21. *Let T be a Hopf monad on an autonomous category $\mathcal{C}$. Then $(G(T), *, \eta)$ is a group. Moreover the inverse of $g \in G(T)$ is $g^{*-1} = S(g) = S^{-1}(g)$, with S and S^{-1} as in* (37) *and* (38).

Proof. Let $g \in G(T)$. By Lemma 3.4, $g^\sharp \in \mathrm{HOM}(U_T, U_T)$ is a monoidal natural isomorphism with inverse ${}^!(g^\sharp) = (g^\sharp)^!$. Hence, by Lemma 3.20, g is invertible with inverse $S(g) = S^{-1}(g)$. □

4. Hopf modules

In this section, we introduce Hopf modules and prove the fundamental theorem for Hopf modules over a Hopf monad.

4.1. Comodules

Let C be a coalgebra in a monoidal category $\mathcal{C}$, with coproduct $\Delta : C \to C \otimes C$ and counit $\varepsilon : C \to \mathbb{1}$. Recall that a right C-comodule is a pair (M, ρ), where M is an object of $\mathcal{C}$ and $\rho : M \to M \otimes C$ is a morphism in $\mathcal{C}$, satisfying:

$$(\rho \otimes \mathrm{id}_C)\rho = (\mathrm{id}_M \otimes \Delta)\rho \quad \text{and} \quad (\mathrm{id}_M \otimes \varepsilon)\rho = \mathrm{id}_M. \tag{41}$$

A morphism $f : (M, \rho) \to (N, \varrho)$ of right C-comodules is a morphism $f : M \to N$ in $\mathcal{C}$ such that $\varrho f = (f \otimes \mathrm{id}_C)\rho$. Thus the category of right C-comodules. Likewise one defines the category of left C-comodules.

Lemma 4.1. *Let C be a coalgebra in a monoidal category $\mathcal{C}$. If (M, ρ) is a left C-comodule and $\mathcal{C}$ is right autonomous, then $(M, \rho)^\vee = (M^\vee, \varrho^r)$ is a right C-comodule, where*

$$\varrho^r = (\mathrm{id}_{M^\vee \otimes C} \otimes \widetilde{\mathrm{ev}}_M)(\mathrm{id}_{M^\vee} \otimes \rho \otimes \mathrm{id}_{M^\vee})(\widetilde{\mathrm{coev}}_M \otimes \mathrm{id}_{M^\vee}).$$

Moreover, this construction defines a contravariant functor form the category of left C-comodules to the category of right C-comodules. Similarly, if (M, ρ) is a right C-comodule and $\mathcal{C}$ is left autonomous, then ${}^\vee(M, \rho) = ({}^\vee M, \varrho^l)$ is a left C-comodule, where

$$\varrho^l = (\mathrm{ev}_M \otimes \mathrm{id}_{C \otimes {}^\vee M})(\mathrm{id}_{{}^\vee M} \otimes \rho \otimes \mathrm{id}_{{}^\vee M})(\mathrm{id}_{{}^\vee M} \otimes \mathrm{coev}_M).$$

This construction is functorial too.

Proof. Left to the reader. $\square$

Let T be a comonoidal endofunctor of a monoidal category $\mathcal{C}$. By (11) and (12), the object $T(\mathbb{1})$ is a coalgebra in $\mathcal{C}$, with coproduct $T_2(\mathbb{1}, \mathbb{1})$ and counit T_0. By a *left* (respectively *right*) *T-comodule*, we mean a left (respectively right) $T(\mathbb{1})$-comodule.

Note that if T is a bimonad, then every object X becomes a left (respectively right) T-comodule with *trivial coaction* given by $\eta_{\mathbb{1}} \otimes \mathrm{id}_X$ (respectively $\mathrm{id}_X \otimes \eta_{\mathbb{1}}$).

4.2. Hopf modules

Let T be a bimonad on a monoidal category $\mathcal{C}$. The axioms of a bimonad ensure that $(T(\mathbb{1}), \mu_{\mathbb{1}})$ is a coalgebra in the category T-$\mathcal{C}$ of T-modules, with coproduct $T_2(\mathbb{1}, \mathbb{1})$ and counit T_0. A *right Hopf T-module* is a right $(T(\mathbb{1}), \mu_{\mathbb{1}})$-comodule in T-$\mathcal{C}$, that is, a triple (M, r, ρ) such that (M, r) is a T-module, (M, ρ) is a right T-comodule, and:

$$\rho r = (r \otimes \mu_{\mathbb{1}}) T_2\big(M, T(\mathbb{1})\big) T(\rho). \tag{42}$$

A *morphism of Hopf T-modules* between two right Hopf T-modules (M, r, ρ) and (N, s, ϱ) is a morphism of $(T(\mathbb{1}), \mu_{\mathbb{1}})$-comodules in T-$\mathcal{C}$, that is, a morphism $f : M \to N$ in $\mathcal{C}$ such that

$$fr = sT(f) \quad \text{and} \quad (f \otimes \mathrm{id}_{T(\mathbb{1})})\rho = \varrho f. \tag{43}$$

Remark 4.2. As is the classical case, any morphism of Hopf T-modules which is an isomorphism in $\mathcal{C}$ is an isomorphism of Hopf T-modules.

Similarly, one can define the notion of *left Hopf T-module*, which is a right Hopf T-module for the bimonad T^{op} (see Remark 2.4).

Lemma 4.3. *Let T be a bimonad on a monoidal category $\mathcal{C}$. If (M, ρ) is a right T-comodule then, setting $\varrho = (\mathrm{id}_{T(M)} \otimes \mu_{\mathbb{1}})T_2(M, T(\mathbb{1}))T(\rho)$, the triple $(T(M), \mu_M, \varrho)$ is a right Hopf T-module. In particular $(T(X), \mu_X, T_2(X, \mathbb{1}))$ is a right Hopf T-module for any object X of $\mathcal{C}$.*

Proof. Let (M, ρ) be a right T-comodule. Firstly, we have:

$$\begin{aligned}
(\varrho \otimes \mathrm{id}_{T(\mathbb{1})})\varrho &= \big((\mathrm{id}_{T(M)} \otimes \mu_{\mathbb{1}})T_2\big(M, T(\mathbb{1})\big)T(\rho) \otimes \mu_{\mathbb{1}}\big)T_2\big(M, T(\mathbb{1})\big)T(\rho) \\
&= \big((\mathrm{id}_{T(M)} \otimes \mu_{\mathbb{1}})T_2\big(M, T(\mathbb{1})\big) \otimes \mu_{\mathbb{1}}\big)T_2\big(M \otimes T(\mathbb{1}), T(\mathbb{1})\big)T\big((\rho \otimes \mathrm{id}_{T(\mathbb{1})})\rho\big) \\
&= \big(\mathrm{id}_{T(M)} \otimes (\mu_{\mathbb{1}} \otimes \mu_{\mathbb{1}})T_2\big(T(\mathbb{1}), T(\mathbb{1})\big)\big)T_2\big(M, T(\mathbb{1}) \otimes T(\mathbb{1})\big) \\
&\quad \times T\big((\mathrm{id}_M \otimes T_2(\mathbb{1}, \mathbb{1}))\rho\big) \quad \text{by (11) and (41)} \\
&= \big(\mathrm{id}_{T(M)} \otimes (\mu_{\mathbb{1}} \otimes \mu_{\mathbb{1}})T_2^2(\mathbb{1}, \mathbb{1})\big)T_2\big(M, T(\mathbb{1})\big)T(\rho) \\
&= \big(\mathrm{id}_{T(M)} \otimes T_2(\mathbb{1}, \mathbb{1})\mu_{\mathbb{1}}\big)T_2\big(M, T(\mathbb{1})\big)T(\rho) \quad \text{by (15)} \\
&= \big(\mathrm{id}_{T(M)} \otimes T_2(\mathbb{1}, \mathbb{1})\big)\varrho
\end{aligned}$$

and

$$\begin{aligned}
(\mathrm{id}_{T(M)} \otimes T_0)\varrho &= (\mathrm{id}_{T(M)} \otimes T_0\mu_{\mathbb{1}})T_2\big(M, T(\mathbb{1})\big)T(\rho) \\
&= \big(\mathrm{id}_{T(M)} \otimes T_0 T(T_0)\big)T_2\big(M, T(\mathbb{1})\big)T(\rho) \quad \text{by (16)} \\
&= (\mathrm{id}_{T(M)} \otimes T_0)T_2(M, \mathbb{1})T\big((\mathrm{id}_M \otimes T_0)\rho\big) \\
&= \mathrm{id}_{T(M)} \quad \text{by (12) and (41),}
\end{aligned}$$

so that $(T(M), \varrho)$ is a right T-comodule. Secondly,

$$\begin{aligned}
\varrho\mu_M &= (\mathrm{id}_{T(M)} \otimes \mu_{\mathbb{1}})T_2\big(M, T(\mathbb{1})\big)T(\rho)\mu_M \\
&= (\mathrm{id}_{T(M)} \otimes \mu_{\mathbb{1}})T_2\big(M, T(\mathbb{1})\big)\mu_{M \otimes T(\mathbb{1})}T^2(\rho) \\
&= (\mu_M \otimes \mu_{\mathbb{1}}\mu_{T(\mathbb{1})})T_2\big(T(M), T^2(\mathbb{1})\big)T\big(T_2(M, \mathbb{1})\big)T^2(\rho) \quad \text{by (15)} \\
&= \big(\mu_M \otimes \mu_{\mathbb{1}}T(\mu_{\mathbb{1}})\big)T_2\big(T(M), T^2(\mathbb{1})\big)T\big(T_2(M, \mathbb{1})T(\rho)\big) \quad \text{by (1)} \\
&= (\mu_M \otimes \mu_{\mathbb{1}})T_2\big(T(M), T(\mathbb{1})\big)T(\varrho).
\end{aligned}$$

Hence $(T(M), \mu_M, \varrho)$ is a right Hopf T-module. Now, for any object X of $\mathcal{C}$, the pair $(X, \mathrm{id}_X \otimes \eta_{\mathbb{1}})$ is a right T-comodule, so that $(T(X), \mu_X, \varrho)$ is a right Hopf T-module, with

$$\begin{aligned}\varrho &= (\mathrm{id}_{T(X)} \otimes \mu_{\mathbb{1}}) T_2\big(X, T(\mathbb{1})\big) T(\mathrm{id}_X \otimes \eta_{\mathbb{1}}) \\ &= \big(\mathrm{id}_{T(X)} \otimes \mu_{\mathbb{1}} T(\eta_{\mathbb{1}})\big) T_2(X, \mathbb{1}) = T_2(X, \mathbb{1})\end{aligned}$$

by (2), which completes the proof of Lemma 4.3. □

Lemma 4.4. *Let T be a right Hopf monad on a right autonomous category $\mathcal{C}$. If M is a left Hopf module, then $M^\vee$ is a right Hopf T-module, with the structure of T-module defined in Theorem* 3.8(b) *and the structure of right T-comodule defined in Lemma* 4.1. *This defines a contravariant functor from the category of left Hopf T-modules to the category of right Hopf T-modules.*

Proof. This results from Lemma 4.1. Indeed, recall that $(T(\mathbb{1}), \mu_{\mathbb{1}})$ is a coalgebra in T-$\mathcal{C}$, with coproduct $T_2(\mathbb{1}, \mathbb{1})$ and counit T_0. Let (M, r, ρ) be a left Hopf T-module. This means that $((M, r), \rho)$ is a left $(T(\mathbb{1}), \mu_{\mathbb{1}})$-comodule, so $((M, r)^\vee, \varrho^l)$ is a right $(T(\mathbb{1}), \mu_{\mathbb{1}})$-comodule, in the notations of Lemma 4.1. In other words, $(M, s^l_M T({}^\vee r), \varrho^l)$ is a right Hopf T-module. This construction is functorial since morphisms of Hopf T-modules are nothing but morphisms of $(T(\mathbb{1}), \mu_{\mathbb{1}})$-comodules. □

4.3. Coinvariants

Let $\mathcal{D}$ be a category and $f, g : X \to Y$ be parallel morphisms in $\mathcal{D}$. A morphism $i : E \to X$ in $\mathcal{D}$ *equalizes* the pair (f, g) if $fi = gi$. An *equalizer* (also called *difference kernel*) of the pair (f, g) is a morphism $i : E \to X$ which equalizes the pair (f, g) and which is universal for this property in the following sense: for any morphism $j : F \to X$ in $\mathcal{C}$ equalizing the pair (f, g), there exists a unique morphism $p : F \to E$ in $\mathcal{C}$ such that $j = pi$. We say that *equalizers exist in $\mathcal{D}$* if each pair of parallel morphisms in $\mathcal{D}$ admits an equalizer.

We say that a functor $F : \mathcal{D} \to \mathcal{D}'$ *preserves equalizers* if, whenever i is an equalizer of a pair (f, g) of parallel morphisms in $\mathcal{D}$, then $F(i)$ is an equalizer of the pair $(F(f), F(g))$. Notice that a left exact functor preserves equalizers.

Let T be a bimonad on a monoidal category $\mathcal{C}$. We say that a right T-comodule (M, ρ) *admits coinvariants* if the pair of parallel morphisms $(\rho, \mathrm{id}_M \otimes \eta_{\mathbb{1}})$ admits an equalizer:

$$\begin{tikzcd} N \arrow[r, "i"] & M \arrow[r, shift left, "\rho"] \arrow[r, shift right, "\mathrm{id}_M \otimes \eta_{\mathbb{1}}"'] & M \otimes T(\mathbb{1}). \end{tikzcd}$$

If such is the case, N is called the *coinvariant part of M,* and is denoted $M^{\mathrm{co}\,T}$. In fact $M^{\mathrm{co}\,T}$ is a right T-comodule (with trivial coaction) and $i : (N, \mathrm{id}_N \otimes \eta_{\mathbb{1}}) \to (M, \rho)$ is a morphism of T-comodules.

Similarly, one defines the *coinvariant part* of a left T-comodule (M, ρ) which is, when it exists, an equalizer of the pair $(\rho, \eta_{\mathbb{1}} \otimes \mathrm{id}_M)$.

If a right or left T-comodule (M, ρ) admits a coinvariant part $i : M^{\mathrm{co}\,T} \to M$, we say that T *preserves the coinvariant part of* (M, ρ) if $T(i)$ is an equalizer of the pair $(T(\rho), T(\mathrm{id}_M \otimes \eta_{\mathbb{1}}))$ or $(T(\rho), T(\eta_{\mathbb{1}} \otimes \mathrm{id}_M))$ respectively. Note this is the case when T preserves equalizers.

We say that a right (respectively left) Hopf T-module (M, r, ρ) *admits coinvariants* if the underlying right (respectively left) T-comodule (M, ρ) admits coinvariants. If such the case, the *coinvariant part of* (M, r, ρ) is the coinvariant part of (M, ρ).

4.4. Decomposition of Hopf modules

In this section we show that, under certain assumptions on equalizers, Hopf modules can be decomposed as in the classical case.

Theorem 4.5. *Let T be a right Hopf monad on a right autonomous category. Let (M, r, ρ) be a right Hopf T-module admitting a coinvariant part $i : M^{\mathrm{co}\,T} \to M$ which is preserved by T. Then*

$$rT(i) : (M, r, \rho) \to \big(T\big(M^{\mathrm{co}\,T}\big), \mu_{M^{\mathrm{co}\,T}}, T_2\big(M^{\mathrm{co}\,T}, \mathbb{1}\big)\big)$$

is an isomorphism of right Hopf T-modules.

Proof. See Section 4.5. □

Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *conservative* if any morphism f in $\mathcal{C}$ such that $F(f)$ is an isomorphism in $\mathcal{D}$, is an isomorphism in $\mathcal{C}$.

Theorem 4.6. *Let T be a right Hopf monad on a right autonomous category $\mathcal{C}$. Suppose that right Hopf T-modules admit coinvariants which are preserved by T. Then the assignments*

$$X \mapsto \big(T(X), \mu_X, T_2(X, \mathbb{1})\big), \quad f \mapsto T(f)$$

define a functor from $\mathcal{C}$ to the category of right Hopf T-modules, which is an equivalence of categories if and only if T is conservative.

Proof. See Section 4.6. □

Remark 4.7. For a left Hopf monad T over a left autonomous category, one may formulate a similar decomposition theorem for left Hopf T-modules, which may be deduced from Theorem 4.6 applied to the Hopf monad T^{op}, in virtue of Remark 3.6.

Example 4.8. Let A be a Hopf algebra in a braided right autonomous category $\mathcal{C}$. Consider the right Hopf monad $T = ? \otimes A$ on $\mathcal{C}$, see Example 2.2. A right Hopf T-module is nothing but a right Hopf module over A in the usual sense, that is, a triple $(M, r : M \otimes A \to M, \rho : M \to M \otimes A)$ such that (M, r) is a right A-module, (M, ρ) is a right A-comodule, and $\rho r = (m \otimes r)(\mathrm{id}_A \otimes \tau_{A,A} \otimes \mathrm{id}_M)(\rho \otimes \Delta)$, where τ is the braiding of $\mathcal{C}$, m is the product of A, and Δ is coproduct of A. Assume now that $\mathcal{C}$ splits idempotents (see [1]). Then M admits a coinvariant part, which is the object splitting the idempotent $r(S_A \otimes \mathrm{id}_M)\rho$, where S_A denotes the antipode of A. Moreover, T preserves coinvariants (because $\otimes$ is exact) and T is conservative (because A is a bialgebra). Therefore Theorems 4.5 and 4.6 apply in this setting: we obtain the fundamental theorem of Hopf modules for categorical Hopf algebras. In the case where S_A is invertible, it was first stated in [1].

4.5. *Proof of Theorem 4.5*

Let T be a right Hopf monad on a right autonomous category $\mathcal{C}$, with right antipode s^r. Our proof will rely very strongly on the properties of the natural transformation $\Gamma_X : X \otimes T(\mathbb{1}) \to T^2(X)$ defined by:

$$\Gamma_X = \big(\widetilde{\mathrm{ev}}_X\big(\mathrm{id}_X \otimes s^r_X\big) \otimes \mathrm{id}_{T^2(X)}\big)\big(\mathrm{id}_X \otimes T_2\big(T(X)^\vee, T(X)\big)T(\widetilde{\mathrm{coev}}_{T(X)})\big) \tag{44}$$

for any object X of $\mathcal{C}$.

Notice that, if T is of the form $? \otimes A$, where A is a Hopf algebra in a braided autonomous category (see Example 3.10), then $\Gamma_X = \mathrm{id}_X \otimes (S \otimes \mathrm{id}_A)\Delta$.

Lemma 4.9. *For any object X of $\mathcal{C}$, we have*

(a) $\mu_X \Gamma_X = \eta_X \otimes T_0$;
(b) $T(\mu_X)\Gamma_{T(X)}T_2(X, \mathbb{1}) = T(\eta_X)$;
(c) $T\big((\mathrm{id}_{T(X)} \otimes \mu_{\mathbb{1}})T_2(X, T(\mathbb{1}))\big)\Gamma_{X \otimes T(\mathbb{1})}(\mathrm{id}_X \otimes T_2(\mathbb{1}, \mathbb{1})) = T(\mathrm{id}_{T(X)} \otimes \eta_{\mathbb{1}})\Gamma_X$;
(d) $\Gamma_X(\mathrm{id}_X \otimes \eta_{\mathbb{1}}) = \eta_{T(X)}\eta_X$.

Proof. Let us prove Part (a). We have:

$$\begin{aligned}
\mu_X \Gamma_X &= (\widetilde{\mathrm{ev}}_X \otimes \mathrm{id}_{T(X)})\big(\mathrm{id}_X \otimes \big(s^r_X \otimes \mu_X\big)T_2\big(T(X)^\vee, T(X)\big)T(\widetilde{\mathrm{coev}}_{T(X)})\big)\\
&= (\widetilde{\mathrm{ev}}_X \otimes \eta_X)(\mathrm{id}_X \otimes \widetilde{\mathrm{coev}}_X T_0) \quad \text{by (23)}\\
&= \eta_X \otimes T_0.
\end{aligned}$$

Let us prove Part (b). We have:

$$\begin{aligned}
&T(\mu_X)\Gamma_{T(X)}T_2(X, \mathbb{1})\\
&\quad = \big(\widetilde{\mathrm{ev}}_{T(X)}\big(\mathrm{id}_{T(X)} \otimes s^r_{T(X)}\big) \otimes T(\mu_X)\big)\\
&\qquad \circ \big(\mathrm{id}_{T(X)} \otimes T_2\big(T^2(X)^\vee, T^2(X)\big)T(\widetilde{\mathrm{coev}}_{T^2(X)})\big)T_2(X, \mathbb{1})\\
&\quad = \big(\widetilde{\mathrm{ev}}_{T(X)}\big(\mathrm{id}_{T(X)} \otimes s^r_{T(X)}T\big(\mu_X^\vee\big)\big) \otimes \mathrm{id}_{T^2(X)}\big)\big(\mathrm{id}_{T(X)} \otimes T_2\big(T(X)^\vee, T(X)\big)\big)\\
&\qquad \circ T_2\big(X, T(X)^\vee \otimes T(X)\big)T(\mathrm{id}_X \otimes \widetilde{\mathrm{coev}}_{T(X)})\\
&\quad = \big(\widetilde{\mathrm{ev}}_{T(X)}\big(\mathrm{id}_{T(X)} \otimes s^r_{T(X)}T\big(\mu_X^\vee\big)\big)T_2\big(X, T(X)^\vee\big) \otimes \mathrm{id}_{T^2(X)}\big)\\
&\qquad \circ T_2\big(X \otimes T(X)^\vee, T(X)\big)T(\mathrm{id}_X \otimes \widetilde{\mathrm{coev}}_{T(X)}) \quad \text{by (11)}\\
&\quad = \big(T_0 T(\widetilde{\mathrm{ev}}_X)T\big(\mathrm{id}_X \otimes \eta_X^\vee\big) \otimes \mathrm{id}_{T^2(X)}\big)\\
&\qquad \circ T_2\big(X \otimes T(X)^\vee, T(X)\big)T(\mathrm{id}_X \otimes \widetilde{\mathrm{coev}}_{T(X)}) \quad \text{by (22)}\\
&\quad = \big(T_0 \otimes T(\eta_X)\big)T_2(\mathbb{1}, X)T\big((\widetilde{\mathrm{ev}}_X \otimes \mathrm{id}_X)(\mathrm{id}_X \otimes \widetilde{\mathrm{coev}}_{T(X)})\big) = T(\eta_X).
\end{aligned}$$

Let us prove Part (c). Denote by $\mathcal{L}_X$ the left-hand side of Part (c). Firstly, using the naturality of T_2, we have:

$$\begin{aligned}\mathcal{L}_X = \big(&\widetilde{\mathrm{ev}}_{X\otimes T(\mathbb{1})}(\mathrm{id}_{X\otimes T(\mathbb{1})}\otimes\alpha_X)\otimes\mathrm{id}_{T(T(X)\otimes T(\mathbb{1}))}\big)\\ &\circ\big(\mathrm{id}_{X\otimes T(\mathbb{1})}\otimes T_2\big(T(\mathbb{1})^\vee\otimes T(X)^\vee, T(X)\otimes T(\mathbb{1})\big)\big)\\ &\circ\big(\mathrm{id}_X\otimes T_2\big(\mathbb{1}, T(\mathbb{1})^\vee\otimes T(X)^\vee\otimes T(X)\otimes T(\mathbb{1})\big)T\big(\widetilde{\mathrm{coev}}_{T(X)\otimes T(\mathbb{1})}\big)\big)\end{aligned}$$

where $\alpha_X = s^r_{X\otimes T(\mathbb{1})}T(T_2(X,T(\mathbb{1}))^\vee)T(\mu_{\mathbb{1}}^\vee\otimes\mathrm{id}_{T(X)^\vee})$. Now

$$\begin{aligned}\alpha_X &= \big(s^r_{T(\mathbb{1})}\otimes s^r_X\big)T_2\big(T^2(\mathbb{1})^\vee, T(X)^\vee\big)T\big(\mu_{\mathbb{1}}^\vee\otimes\mathrm{id}_{T(X)^\vee}\big) \quad \text{by (30)}\\ &= \big(s^r_{T(\mathbb{1})}T\big(\mu_{\mathbb{1}}^\vee\big)\otimes s^r_X\big)T_2\big(T(\mathbb{1})^\vee, T(X)^\vee\big)\end{aligned}$$

and, using (11),

$$\begin{aligned}&\big(\mathrm{id}_{X\otimes T(\mathbb{1})}\otimes T_2\big(T(\mathbb{1})^\vee, T(X)^\vee\big)\otimes\mathrm{id}_{T(X)\otimes T(\mathbb{1})}\big)\\ &\qquad\circ\big(\mathrm{id}_{X\otimes T(\mathbb{1})}\otimes T_2\big(T(\mathbb{1})^\vee\otimes T(X)^\vee, T(X)\otimes T(\mathbb{1})\big)\big)\\ &\qquad\circ\big(\mathrm{id}_X\otimes T_2\big(\mathbb{1}, T(\mathbb{1})^\vee\otimes T(X)^\vee\otimes T(X)\otimes T(\mathbb{1})\big)\big)\\ &\quad=\big(\mathrm{id}_X\otimes T_2\big(\mathbb{1}, T(\mathbb{1})^\vee\big)\otimes T_2\big(T(X)^\vee, T(X)\otimes T(\mathbb{1})\big)\big)\\ &\qquad\circ\big(\mathrm{id}_X\otimes T_2\big(T(\mathbb{1})^\vee, T(X)^\vee\otimes T(X)\otimes T(\mathbb{1})\big)\big).\end{aligned}$$

Therefore, since $\widetilde{\mathrm{ev}}_{X\otimes T(\mathbb{1})} = \widetilde{\mathrm{ev}}_X(\mathrm{id}_X\otimes\widetilde{\mathrm{ev}}_{T(\mathbb{1})}\otimes\mathrm{id}_{X^\vee})$, we have:

$$\begin{aligned}\mathcal{L}_X = \big(&\widetilde{\mathrm{ev}}_X\big(\mathrm{id}_X\otimes s^r_X\big)\otimes\mathrm{id}_{T(T(X)\otimes T(\mathbb{1}))}\big)\\ &\circ\big(\mathrm{id}_X\otimes\widetilde{\mathrm{ev}}_{T(\mathbb{1})}\big(\mathrm{id}_{T(\mathbb{1})}\otimes s^r_{T(\mathbb{1})}T\big(\mu_{\mathbb{1}}^\vee\big)\big)T_2\big(\mathbb{1}, T(\mathbb{1})^\vee\big)\otimes T_2\big(T(X)^\vee, T(X)\otimes T(\mathbb{1})\big)\big)\\ &\circ\big(\mathrm{id}_X\otimes T_2\big(T(\mathbb{1})^\vee, T(X)^\vee\otimes T(X)\otimes T(\mathbb{1})\big)T(\widetilde{\mathrm{coev}}_{T(X)\otimes T(\mathbb{1})})\big).\end{aligned}$$

Now $\widetilde{\mathrm{ev}}_{T(\mathbb{1})}(\mathrm{id}_{T(\mathbb{1})}\otimes s^r_{T(\mathbb{1})}T(\mu_{\mathbb{1}}^\vee))T_2(\mathbb{1}, T(\mathbb{1})^\vee) = T_0T(\eta_{\mathbb{1}}^\vee)$ by (22). Hence, using (12),

$$\begin{aligned}\mathcal{L}_X &= \big(\widetilde{\mathrm{ev}}_X\big(\mathrm{id}_X\otimes s^r_X\big)\otimes T(\mathrm{id}_{T(X)}\otimes\eta_{\mathbb{1}})\big)\\ &\qquad\circ\big(\mathrm{id}_X\otimes T_2\big(T(X)^\vee, T(X)\big)T(\widetilde{\mathrm{coev}}_{T(X)})\big)\\ &= T(\mathrm{id}_{T(X)}\otimes\eta_{\mathbb{1}})\Gamma_X.\end{aligned}$$

Let us prove Part (d). We have:

$$\begin{aligned}&\Gamma_X(\mathrm{id}_X\otimes\eta_{\mathbb{1}})\\ &\quad=\big(\widetilde{\mathrm{ev}}_X\big(\mathrm{id}_X\otimes s^r_X\big)\otimes\mathrm{id}_{T^2(X)}\big)\big(\mathrm{id}_X\otimes T_2\big(T(X)^\vee, T(X)\big)T(\widetilde{\mathrm{coev}}_{T(X)})\eta_{\mathbb{1}}\big)\\ &\quad=\big(\widetilde{\mathrm{ev}}_X\big(\mathrm{id}_X\otimes s^r_X\big)\otimes\mathrm{id}_{T^2(X)}\big)\big(\mathrm{id}_X\otimes T_2\big(T(X)^\vee, T(X)\big)\eta_{T(X)^\vee\otimes T(X)}\widetilde{\mathrm{coev}}_{T(X)}\big)\\ &\quad=\big(\widetilde{\mathrm{ev}}_X\big(\mathrm{id}_X\otimes s^r_X\eta_{T(X)^\vee}\big)\otimes\eta_{T(X)}\big)(\mathrm{id}_X\otimes\widetilde{\mathrm{coev}}_{T(X)}) \quad \text{by (17)}\\ &\quad=\big(\widetilde{\mathrm{ev}}_X\big(\mathrm{id}_X\otimes\eta_X^\vee\big)\otimes\eta_{T(X)}\big)(\mathrm{id}_X\otimes\widetilde{\mathrm{coev}}_{T(X)}) \quad \text{by (29)}\\ &\quad=\eta_{T(X)}\eta_X. \qquad \square\end{aligned}$$

Lemma 4.10. *For any T-module (M,r), $T(r)\Gamma_M:(M,r)\otimes(T(\mathbb{1}),\mu_{\mathbb{1}})\to(T(M),\mu_M)$ is a morphism of T-modules.*

Proof. Since

$$\begin{aligned}
&T_2\big(T(M)^\vee,T(M)\big)T(\widetilde{\mathrm{coev}}_{T(M)})\mu_{\mathbb{1}}\\
&\quad=T_2\big(T(M)^\vee,T(M)\big)\mu_{T(M)^\vee\otimes T(M)}T^2(\widetilde{\mathrm{coev}}_{T(M)})\\
&\quad=(\mu_{T(M)^\vee}\otimes\mu_{T(M)})T_2^2\big(T(M)^\vee,T(M)\big)T^2(\widetilde{\mathrm{coev}}_{T(M)})\quad\text{by (15)},
\end{aligned}$$

we have

$$\begin{aligned}
&T(r)\Gamma_M(r\otimes\mu_{\mathbb{1}})\\
&\quad=\big(\widetilde{\mathrm{ev}}_M\big(\mathrm{id}_M\otimes s^r_M\big)\otimes T(r)\big)\big(r\otimes T_2\big(T(M)^\vee,T(M)\big)T(\widetilde{\mathrm{coev}}_{T(M)})\mu_{\mathbb{1}}\big)\\
&\quad=\big(\widetilde{\mathrm{ev}}_M\big(r\otimes s^r_M\mu_{T(M)^\vee}\big)\otimes T(r)\mu_{T(M)}\big)\\
&\qquad\circ\big(\mathrm{id}_{T(M)}\otimes T_2^2\big(T(M)^\vee,T(M)\big)T^2(\widetilde{\mathrm{coev}}_{T(M)})\big)\\
&\quad=\big(\widetilde{\mathrm{ev}}_M\big(r\otimes s^r_M T\big(s^r_{T(M)}\big)T^2\big(\mu_M^\vee\big)\big)\otimes T(r)\mu_{T(M)}\big)\\
&\qquad\circ\big(\mathrm{id}_{T(M)}\otimes T_2^2\big(T(M)^\vee,T(M)\big)T^2(\widetilde{\mathrm{coev}}_{T(M)})\big)\quad\text{by (28)}\\
&\quad=\big(\widetilde{\mathrm{ev}}_M\big(r\otimes s^r_M T\big(s^r_{T(M)}\big)\big)\otimes T(r)\mu_{T(M)}T^2(\mu_M)\big)\\
&\qquad\circ\big(\mathrm{id}_{T(M)}\otimes T_2^2\big(T^2(M)^\vee,T^2(M)\big)T^2(\widetilde{\mathrm{coev}}_{T^2(M)})\big).
\end{aligned}$$

Now

$$\begin{aligned}
&\big(T\big(s^r_{T(M)}\big)\otimes\mathrm{id}_{T^3(M)}\big)T_2^2\big(T^2(M)^\vee,T^2(M)\big)T^2(\widetilde{\mathrm{coev}}_{T^2(M)})\\
&\quad=T_2\big(T(M)^\vee,T^3(M)\big)T\big(\big(s^r_{T(M)}\otimes\mathrm{id}_{T^3(M)}\big)T_2\big(T^2(M)^\vee,T^2(M)\big)T(\widetilde{\mathrm{coev}}_{T^2(M)})\big)\\
&\quad=T_2\big(T(M)^\vee,T^3(M)\big)T\big((\mathrm{id}_{T(M)}\otimes\Gamma_{T(M)})(\widetilde{\mathrm{coev}}_{T(M)}\otimes\mathrm{id}_{T(\mathbb{1})})\big)\\
&\quad=\big(\mathrm{id}_{T(T(M)^\vee)}\otimes T(\Gamma_{T(M)})\big)T_2\big(T(M)^\vee,T(M)\otimes T(\mathbb{1})\big)T(\widetilde{\mathrm{coev}}_{T(M)}\otimes\mathrm{id}_{T(\mathbb{1})}),
\end{aligned}$$

and, using (3),

$$T(r)\mu_{T(M)}T^2(\mu_M)=\mu_M T^2(r\mu_M)=\mu_M T^2\big(rT(r)\big)=T(r)\mu_{T(M)}T^3(r).$$

Therefore, we get

$$\begin{aligned}
&T(r)\Gamma_M(r\otimes\mu_{\mathbb{1}})T_2\big(M,T(\mathbb{1})\big)\\
&\quad=\big(\widetilde{\mathrm{ev}}_M\big(r\otimes s^r_M\big)\otimes T(r)\mu_{T(M)}T^3(r)T(\Gamma_{T(M)})\big)\\
&\qquad\circ\big(\mathrm{id}_{T(M)}\otimes T_2\big(T(M)^\vee,T(M)\otimes T(\mathbb{1})\big)T(\widetilde{\mathrm{coev}}_{T(M)}\otimes\mathrm{id}_{T(\mathbb{1})})\big)T_2\big(M,T(\mathbb{1})\big)\\
&\quad=\big(\widetilde{\mathrm{ev}}_M\big(r\otimes s^r_M\big)T_2\big(M,T(M)^\vee\big)\otimes T(r)\mu_{T(M)}T\big(T^2(r)\Gamma_{T(M)}\big)\big)\\
&\qquad\circ T_2\big(M\otimes T(M)^\vee,T(M)\otimes T(\mathbb{1})\big)T(\mathrm{id}_M\otimes\widetilde{\mathrm{coev}}_{T(M)}\otimes\mathrm{id}_{T(\mathbb{1})})\quad\text{by (11)}
\end{aligned}$$

$$
\begin{aligned}
&= \big(\widetilde{\mathrm{ev}}_M\big(r \otimes s^r_M T\big(r^\vee\big)\big)T_2\big(M, M^\vee\big) \otimes T(r)\mu_{T(M)}T(\Gamma_M)\big)\\
&\quad \circ T_2\big(M \otimes M^\vee, M \otimes T(\mathbb{1})\big)T(\mathrm{id}_M \otimes \widetilde{\mathrm{coev}}_M \otimes \mathrm{id}_{T(\mathbb{1})}) \quad \text{by naturality of } \Gamma\\
&= \big(T_0 T(\widetilde{\mathrm{ev}}_M) \otimes T(r)\mu_{T(M)}T(\Gamma_M)\big)\\
&\quad \circ T_2\big(M \otimes M^\vee, M \otimes T(\mathbb{1})\big)T(\mathrm{id}_M \otimes \widetilde{\mathrm{coev}}_M \otimes \mathrm{id}_{T(\mathbb{1})}) \quad \text{by Theorem 3.8(b)}\\
&= \big(T_0 \otimes \mu_M T^2(r)T(\Gamma_M)\big)T_2\big(\mathbb{1}, M \otimes T(\mathbb{1})\big)\\
&\quad \circ T\big(\big(\widetilde{\mathrm{ev}}_M \otimes \mathrm{id}_{M \otimes T(\mathbb{1})}\big)(\mathrm{id}_M \otimes \widetilde{\mathrm{coev}}_M \otimes \mathrm{id}_{T(\mathbb{1})})\big)\\
&= \mu_M T\big(T(r)\Gamma_M\big) \quad \text{by (12).}
\end{aligned}
$$

Hence $T(r)\Gamma_M$ is a morphism of T-modules. □

Now we are ready to prove Theorem 4.5. Let (M, r, ρ) be a right Hopf T-module and $i : M^{\mathrm{co}T} \to M$ be an equalizer of the pair $(\rho, \mathrm{id}_M \otimes \eta_{\mathbb{1}})$. We will show that $rT(i)$ is an isomorphism in $\mathcal{C}$ (by constructing an inverse) and we will check that $rT(i)$ is a morphism of right Hopf T-modules. By Remark 4.2, this will prove the theorem.

Set $\psi_M = T(r)\Gamma_M\rho : M \to T(M)$.

Lemma 4.11. *The morphism ψ_M enjoys the following properties*:

(a) $r\psi_M = \mathrm{id}_M$;
(b) $\psi_M r = \mu_M T(\psi_M)$;
(c) $T(\rho)\psi_M = T(\mathrm{id}_M \otimes \eta_{\mathbb{1}})\psi_M$;
(d) $\psi_M i = \eta_M i$.

Proof. Since M is a T-module and a right T-comodule, we have, by Lemma 4.9(a), $r\psi_M = rT(r)\Gamma_M\rho = r\mu_M\Gamma_M\rho = r(\eta_M \otimes T_0)\rho = r\eta_M = \mathrm{id}_M$. Hence Part (a). Moreover, we have:

$$
\begin{aligned}
\psi_M r &= T(r)\Gamma_M\rho r\\
&= T(r)\Gamma_M(r \otimes \mu_{\mathbb{1}})T_2\big(M, T(\mathbb{1})\big)T(\rho) \quad \text{by (42)}\\
&= \mu_M T\big(T(r)\Gamma_M\big)T(\rho) \quad \text{by Lemma 4.10}\\
&= \mu_M T(\psi_M).
\end{aligned}
$$

Hence Part (b). Now, since M is a right Hopf T-module, we have, by Lemma 4.9(c),

$$
\begin{aligned}
T(\rho)\psi_M &= T(\rho r)\Gamma_M\rho\\
&= T(r \otimes \mu_{\mathbb{1}})T\big(T_2\big(M, T(\mathbb{1})\big)\big)T^2(\rho)\Gamma_M\rho\\
&= T(r \otimes \mu_{\mathbb{1}})T\big(T_2\big(M, T(\mathbb{1})\big)\big)\Gamma_{M \otimes T(\mathbb{1})}(\rho \otimes \mathrm{id}_{T(\mathbb{1})})\rho\\
&= T(r \otimes \mathrm{id}_{T(\mathbb{1})})T\big((\mathrm{id}_{T(M)} \otimes \mu_{\mathbb{1}})T_2\big(M, T(\mathbb{1})\big)\big)\Gamma_{M \otimes T(\mathbb{1})}\big(\mathrm{id}_M \otimes T_2(\mathbb{1}, \mathbb{1})\big)\rho\\
&= T(r \otimes \mathrm{id}_{T(\mathbb{1})})T(\mathrm{id}_{T(M)} \otimes \eta_{\mathbb{1}})\Gamma_M\rho\\
&= T(\mathrm{id}_M \otimes \eta_{\mathbb{1}})\psi_M.
\end{aligned}
$$

Hence Part (c). Lastly, we have

$$\begin{aligned}\psi_M i &= T(r)\Gamma_M \rho i = T(r)\Gamma_M(\mathrm{id}_M \otimes \eta_{\mathbb{1}})i\\ &= T(r)\eta_{T(M)}\eta_M i \quad \text{by Lemma 4.9(d)}\\ &= \eta_M r \eta_M i = \eta_M i.\end{aligned}$$

Hence Part (d). □

By Lemma 4.11(c), ψ_M equalizes the pair $(T(\rho), T(\mathrm{id}_M \otimes \eta_{\mathbb{1}}))$. Since, by assumption, $T(i)$ is an equalizer of the pair $(T(\rho), T(\mathrm{id}_M \otimes \eta_{\mathbb{1}}))$, there exists a (unique) map $\phi_M : M \to T(M^{\mathrm{co}\,T})$ such that $\psi_M = T(i)\phi_M$.

Let us check that ϕ_M is inverse to $rT(i)$. We have $rT(i)\phi_M = r\psi_M = \mathrm{id}_{T(M)}$ by Lemma 4.11(a). In order to show that $\phi_M rT(i) = \mathrm{id}_{M^{\mathrm{co}\,T}}$, it is enough to check that $T(i)\phi_M rT(i) = T(i)$ because $T(i)$, being an equalizer, is a monomorphism. Now

$$\begin{aligned}T(i)\phi_M rT(i) &= \psi_M rT(i) = \mu_M T(\psi_M)T(i) \quad \text{by Lemma 4.11(b)}\\ &= \mu_M T(\eta_M)T(i) \quad \text{by Lemma 4.11(d)}\\ &= T(i).\end{aligned}$$

Hence $rT(i)$ is an isomorphism in $\mathcal{C}$.

Finally, let us check that $rT(i)$ is a morphism of right Hopf modules. Firstly, we have $rT(rT(i)) = rT(r)T^2(i) = r\mu_M T^2(i) = rT(i)\mu_{M^{\mathrm{co}\,T}}$. Therefore $rT(i)$ is a morphism of T-modules. Secondly, we have:

$$\begin{aligned}\rho rT(i) &= (r \otimes \mu_{\mathbb{1}})T_2\big(M, T(\mathbb{1})\big)T(\rho)T(i)\\ &= (r \otimes \mu_{\mathbb{1}})T_2\big(M, T(\mathbb{1})\big)T(\mathrm{id}_M \otimes \eta_{\mathbb{1}})T(i)\\ &= \big(r \otimes \mu_{\mathbb{1}}T(\eta_{\mathbb{1}})\big)T_2(M, \mathbb{1})T(i)\\ &= \big(rT(i) \otimes \mathrm{id}_{T(\mathbb{1})}\big)T_2\big(M^{\mathrm{co}\,T}, \mathbb{1}\big).\end{aligned}$$

So $rT(i)$ is also a morphism of right T-comodules. This completes the proof of Theorem 4.5.

4.6. Proof of Theorem 4.6

Firstly, by Lemma 4.3 and naturality of μ and T_2, the assignments $X \mapsto (T(X), \mu_X, T_2(X, \mathbb{1}))$ and $f \mapsto T(f)$ define a functor, denoted $\tilde{T}$, from $\mathcal{C}$ to the category of right Hopf T-modules.

Assume $\tilde{T}$ is an equivalence. In particular $\tilde{T}$ is conservative. If f is a morphism in $\mathcal{C}$ such that $T(f)$ is an isomorphism in $\mathcal{C}$, then $\tilde{T}(f)$ is an isomorphism (by Remark 4.2) and so is f (since $\tilde{T}$ is conservative). Hence T is conservative.

Let us prove the converse. Let (M, r, ρ) be a right Hopf T-module and $M^{\mathrm{co}\,T}$ be its coinvariant part, which exists by assumption. By the universal property of equalizers, any morphism of right Hopf modules $f : (M, r, \rho) \to (M', r', \rho')$ induces a morphism $M^{\mathrm{co}\,T} \to M'^{\mathrm{co}\,T}$. This defines a functor $?^{\mathrm{co}\,T}$ from the category of right Hopf T-modules to $\mathcal{C}$. By Theorem 4.5, the functor $?^{\mathrm{co}\,T}$ is a right quasi-inverse of $\tilde{T}$. Assume now that T is conservative. It is enough to prove that, for any object X of $\mathcal{C}$, $\eta_X : X \to T(X)$ is the coinvariant part of the right T-comodule $(T(X), T_2(X, \mathbb{1}))$. Indeed, if this is true, then η_X induces a natural isomorphism $X \xrightarrow{\sim} \tilde{T}(X)^{\mathrm{co}\,T}$, so that $?^{\mathrm{co}\,T}$ is also a left quasi-inverse of $\tilde{T}$. We have the following lemma:

Lemma 4.12. *Let T be a right Hopf monad on a right autonomous category $\mathcal{C}$. Then $T(\eta_X)$ is an equalizer of the pair $(T(T_2(X, \mathbb{1})), T(\mathrm{id}_{T(X)} \otimes \eta_{\mathbb{1}}))$.*

Proof. Let $f : Y \to T^2(X)$ be a morphism in $\mathcal{C}$ equalizing the morphisms $T(T_2(X, \mathbb{1}))$ and $T(\mathrm{id}_{T(X)} \otimes \eta_{\mathbb{1}})$. If there exists $g : Y \to T^2(X)$ such that $f = T(\eta_X)g$, then $g = \mu_X T(\eta_X)g = \mu_X f$, and so g is unique. All we have to check is that $f = T(\eta_X)\mu_X f$. We have:

$$\begin{aligned}
T^2(\eta_X)f &= T\big(T(\mu_X)\Gamma_{T(X)}\big)T\big(T_2(X, \mathbb{1})\big)f \quad \text{by Lemma 4.9(b)}\\
&= T\big(T(\mu_X)\Gamma_{T(X)}\big)T(\mathrm{id}_{T(X)} \otimes \eta_{\mathbb{1}})f \quad \text{by assumption}\\
&= T\big(T(\mu_X)\eta_{T^2(X)}\eta_{T(X)}\big)f \quad \text{by Lemma 4.9(d)}\\
&= T(\eta_{T(X)}\mu_X\eta_{T(X)})f\\
&= T(\eta_{T(X)})f \quad \text{by (2).}
\end{aligned}$$

Hence $f = \mu_{T(X)}T(\eta_{T(X)})f = \mu_{T(X)}T^2(\eta_X)f = T(\eta_X)\mu_X f$. $\square$

Now let X be an object of $\mathcal{C}$. The right Hopf T-module $(T(X), \mu_X, T_2(X, \mathbb{1}))$ admits a coinvariant part $i : T(X)^{\mathrm{co}\,T} \to T(X)$ (by assumption) which is an equalizer of the pair $(T_2(X, \mathbb{1}), \mathrm{id}_X \otimes \eta_{\mathbb{1}})$. Since η_X equalizes this pair by (17), there exists a unique morphism $j : X \to T(X)^{\mathrm{co}\,T}$ such that $\eta_X = ij$. To prove that η_X is an equalizer, we need to show that j is an isomorphism (since i is an equalizer). Now, applying T to this situation, we have two equalizers for the pair $(T(T_2(X, \mathbb{1})), T(\mathrm{id}_X \otimes \eta_{\mathbb{1}}))$, namely $T(\eta_X)$ (by Lemma 4.12) and $T(i)$ (because T preserves coinvariants of right Hopf T-modules). Therefore, since $T(\eta_X) = T(i)T(j)$, the morphism $T(j)$ is an isomorphism, and so is j because T is conservative. This completes the proof of Theorem 4.6.

5. Integrals

In this section, we introduce integrals for bimonads and, using the decomposition theorem for Hopf modules, we prove the existence of universal integrals of a Hopf monad.

5.1. Integrals

Let T be a bimonad on a monoidal category $\mathcal{C}$ and K be an endofunctor of $\mathcal{C}$. A (K*-valued*) *left integral of* T is a natural transformation $c : T \to K$ such that:

$$(\mathrm{id}_{T(\mathbb{1})} \otimes c_X)T_2(\mathbb{1}, X) = \eta_{\mathbb{1}} \otimes c_X. \tag{45}$$

A (K*-valued*) *right integral of* T is a natural transformation $c : T \to K$ such that:

$$(c_X \otimes \mathrm{id}_{T(\mathbb{1})})T_2(X, \mathbb{1}) = c_X \otimes \eta_{\mathbb{1}}. \tag{46}$$

Example 5.1. Let A be a bialgebra in a braided category $\mathcal{C}$. Consider the bimonad $T = A \otimes ?$ on $\mathcal{C}$, see Example 2.2. Let $\chi : A \to k$ be a morphism in $\mathcal{C}$. Set $K = k \otimes ?$ and define $c : T \to K$ by $c_X = \chi \otimes \mathrm{id}_X$. Then c is a K-valued left (respectively right) integral of T if and only if χ is a k-valued left (respectively right) integral of A.

Let T be a bimonad on a monoidal category $\mathcal{C}$. A left (respectively right) integral $\lambda : T \to I$ of T is *universal* if, for any left (respectively right) integral $c : T \to K$ of T, there exists a unique natural transformation $f : I \to K$ such that $c = f\lambda$.

Note that a universal left (respectively right) integral of T is unique up to unique natural isomorphism.

5.2. Existence of universal integrals

Recall that, according to Lemma 4.1, if T is a comonoidal endofunctor of an autonomous category $\mathcal{C}$ and X an object of $\mathcal{C}$, then we have a right T-comodule $(T(X), T_2(\mathbb{1}, X))^\vee$ and a left T-comodule ${}^\vee(T(X), T_2(X, \mathbb{1}))$.

Proposition 5.2. *Let T be a bimonad on an autonomous category $\mathcal{C}$.*

(a) *Assume that, for any object X of $\mathcal{C}$, the right T-module $(T(X), T_2(\mathbb{1}, X))^\vee$ admits coinvariants. Then T admits a universal left integral $\lambda^l : T \to I_l$, which is characterized by the fact that $(\lambda^l_X)^\vee : I_l(X)^\vee \to T(X)^\vee$ is the coinvariant part of $(T(X), T_2(\mathbb{1}, X))^\vee$ for each object X of $\mathcal{C}$.*
(b) *Assume that, for any object X of $\mathcal{C}$, the left T-module ${}^\vee(T(X), T_2(X, \mathbb{1}))$ admits coinvariants. Then T admits a universal right integral $\lambda^r : T \to I_r$, which is characterized by the fact that ${}^\vee(\lambda^r_X) : {}^\vee I_r(X) \to {}^\vee T(X)$ is the coinvariant part of ${}^\vee(T(X), T_2(X, \mathbb{1}))$ for each object X of $\mathcal{C}$.*

Proof. We prove Part (a), from which Part (b) can be deduced using the opposite bimonad. For an object X of $\mathcal{C}$, we have $(T(X), T_2(\mathbb{1}, X))^\vee = (T(X)^\vee, \rho^r_X)$, with $\rho^r_X = T_2(\mathbb{1}, X)^\vee(\mathrm{id}_{T(X)^\vee} \otimes \widetilde{\mathrm{coev}}_{T(\mathbb{1})})$.

Firstly, observe that a natural transformation $c : T \to K$ is a left K-valued integral of T if and only if, for any object X of $\mathcal{C}$, the morphism $c^\vee_X : K(X)^\vee \to T(X)^\vee$ equalizes the pair $(\rho^r_X, \mathrm{id}_{T(X)^\vee} \otimes \eta_{\mathbb{1}})$. Indeed, we have $(\mathrm{id}_{T(\mathbb{1})} \otimes c_X)T_2(\mathbb{1}, X) = \eta_{\mathbb{1}} \otimes c_X$ if and only if

$$\begin{aligned}
&(\mathrm{id}_{T(X)^\vee \otimes T(\mathbb{1})} \otimes \widetilde{\mathrm{ev}}_{K(X)})(\mathrm{id}_{T(\mathbb{1})} \otimes c_X)T_2(\mathbb{1}, X)(\widetilde{\mathrm{coev}}_{T(X)} \otimes \mathrm{id}_{K(X)^\vee}) \\
&\quad = (\mathrm{id}_{T(X)^\vee \otimes T(\mathbb{1})} \otimes \widetilde{\mathrm{ev}}_{K(X)})(\eta_{\mathbb{1}} \otimes c_X)(\widetilde{\mathrm{coev}}_{T(X)} \otimes \mathrm{id}_{K(X)^\vee}),
\end{aligned}$$

that is, if and only if $\rho^r_X c^\vee_X = c^\vee_X \otimes \eta_{\mathbb{1}}$.

Now assume that, for any object X of $\mathcal{C}$, $(T(X), T_2(\mathbb{1}, X))^\vee$ admits a coinvariant part $i_X : E(X) \to T(X)^\vee$. The morphism i_X is an equalizer of the pair $(\rho^r_X, \mathrm{id}_{T(X)^\vee} \otimes \eta_{\mathbb{1}})$. Define $\lambda^l_X = {}^\vee i_X : T(X) \cong {}^\vee(T(X)^\vee) \to {}^\vee E(X)$. Using the universal property of equalizers, one checks easily that the assignment $X \mapsto {}^\vee E(X)$ defines an endofunctor $I_l = {}^\vee E$ of $\mathcal{C}$ and that $\lambda^l : T \to I_l$ is a natural transformation. By the initial remark, λ^l is a left integral for T.

Let K be an endofunctor of $\mathcal{C}$ and c be a left K-valued integral. Again by the initial remark, there exists a unique morphism $a_X : K(X)^\vee \to E(X)$ such that $c_X{}^\vee = i_X a_X$. Using the universal property of equalizers, one checks that $a : K^\vee \to E$ is a natural transformation. Dualizing, we obtain that there exists a unique natural transformation $f = {}^\vee a : I_l = {}^\vee E \to K$ such that $c = f\lambda^l$. Hence λ^l is a universal left integral. $\square$

Recall that, for any endofunctor K of an autonomous category $\mathcal{C}$, we form two endofunctors $K^! = ?^\vee \circ K^{\mathrm{op}} \circ {}^\vee ?^{\mathrm{op}}$ and ${}^!K = {}^\vee ? \circ K^{\mathrm{op}} \circ ?^{\vee\mathrm{op}}$, see Section 3.1. This defines two functors ${}^!?, ?^! : \mathrm{End}(\mathcal{C})^{\mathrm{op}} \to \mathrm{End}(\mathcal{C})$ such that ${}^!?$ and $?^{!\mathrm{op}}$ are quasi-inverse.

Theorem 5.3. *Let T be a Hopf monad on an autonomous category $\mathcal{C}$. Assume that left Hopf T-modules and right Hopf T-modules admit coinvariants which are preserved by T (such is the case if equalizers exist in $\mathcal{C}$ and are preserved by T). Suppose moreover that T is conservative. Then there exist two auto-equivalences I_l and I_r of the category $\mathcal{C}$, a universal I_l-valued left integral of T, and a universal I_r-valued right integral of T. Moreover $I_l^!$ is quasi-inverse to I_r and ${}^!I_r$ is quasi-inverse to I_l.*

Example 5.4. Let A be a Hopf algebra, with invertible antipode S_A, in a braided autonomous category $\mathcal{C}$. Consider the Hopf monad $T = A \otimes ?$ on $\mathcal{C}$, and assume that $\mathcal{C}$ splits idempotents as in Example 4.8. Then Theorem 5.3 applies, and there exists a universal left integral $\lambda^l : T \to I_l$ and a universal right integral $\lambda^r : T \to I_r$ on T, where I_l and I_r are equivalences of $\mathcal{C}$ such that $I_l^!$ is quasi-inverse to I_r. Moreover, by Proposition 5.2 and since $\otimes$ commutes with equalizers, there exist objects k_l and k_r and morphisms $\int^l : A \to k_l$ and $\int^r : A \to k_r$ in $\mathcal{C}$ such that $I_l = k_l \otimes ?$, $\lambda^l_X = \int^l \otimes \mathrm{id}_X$, $I_r = k_r \otimes ?$ and $\lambda^r_X = \int^r \otimes \mathrm{id}_X$. The morphisms $\int^l$ and $\int^r$ are universal left and right integrals of the Hopf algebra A respectively. Since $I_l^! = ? \otimes k_l{}^\vee$ is quasi-inverse to $I_r = k_r \otimes ?$, we see that $k_r \otimes k_l{}^\vee \cong \mathbb{1}$. Hence we may assume $k_r = k_l$ and this object, denoted Int, is $\otimes$-invertible. Let us summarize this discussion: there exists a $\otimes$-invertible object Int of $\mathcal{C}$, a universal left integral $\int^l : A \to \mathrm{Int}$ and a universal right integral $\int^r : A \to \mathrm{Int}$ on A. This result was first proven in [1].

Remark 5.5. In Theorem 5.3, the auto-equivalences I_l and I_r are in general not isomorphic to $1_{\mathcal{C}}$. Such is already the case in the setting of Example 5.4 (since in Example 3.1 of [1] the object Int is not isomorphic to $\mathbb{1}$).

Proof of Theorem 5.3. By Proposition 5.2, T admits universal left and right integrals, which we denote $\lambda^l : T \to I_l$ and $\lambda^r : T \to I_r$ respectively.

Let X be an object of $\mathcal{C}$. By Lemma 4.3, $(T(X), \mu_X, T_2(X, \mathbb{1}))$ is a right Hopf T-module. So ${}^\vee(T(X), \mu_X, T_2(X, \mathbb{1}))$ is a left Hopf T-module (by Lemma 4.4) whose coinvariant part is ${}^\vee(\lambda^r_X) : {}^\vee I_r(X) \to {}^\vee T(X)$ (by Proposition 5.2(b)). Therefore, by Theorem 4.5, we have an isomorphism $T({}^\vee I_r(X)) \to {}^\vee T(X)$ of left Hopf T-modules. Applying the right dual functor of Lemma 4.4 to this isomorphism, we obtain an isomorphism of right Hopf T-modules $T(X) \xrightarrow{\sim} T({}^\vee I_r(X))^\vee$. Now, by Proposition 5.2(a), $I_l(Y)^\vee$ is the coinvariant part of the right Hopf T-module $(T(Y), \mu_Y, T_2(\mathbb{1}, Y))^\vee$ for any object Y of $\mathcal{C}$. Hence, using Theorem 4.6, we deduce a natural isomorphism

$$X \simeq T(X)^{\mathrm{co}\,T} \xrightarrow{\sim} \bigl(T\bigl({}^\vee I_r(X)\bigr)^\vee\bigr)^{\mathrm{co}\,T} \simeq I_l\bigl({}^\vee I_r(X)\bigr)^\vee = I_l^! I_r(X).$$

Similarly, applying the previous construction to T^{op}, we obtain a natural isomorphism $1_{\mathcal{C}} \xrightarrow{\sim} {}^!I_r I_l$. Hence, using Remark 3.3, we conclude that I_l and I_r are auto-equivalences of $\mathcal{C}$, $I_l^!$ is quasi-inverse to I_r and ${}^!I_r$ is quasi-inverse to I_l. □

5.3. Integrals and antipodes

In this section we show that, as in the classical case, left (respectively right) integrals are transported to right (respectively left) integrals via the antipode. It turns out that this works only for integrals with values in endofunctors admitting a right adjoint.

Proposition 5.6. *Let T be a bimonad on an autonomous category $\mathcal{C}$ and J, K be endofunctors of $\mathcal{C}$.*

(a) *Assume T is a right Hopf monad. Let $c : T \to J$ be a left integral of T and suppose we have a natural transformation $\varepsilon : J^! K \to 1_{\mathcal{C}}$. For any object X of $\mathcal{C}$, set:*

$$c_X^{(\varepsilon)} = s^r_{{}^\vee K(X)} T\big(c^!{}_{K(X)} \varepsilon^!{}_X\big) : T(X) \to K(X).$$

Then the natural transformation $c^{(\varepsilon)} : T \to K$ is right integral of T.

(b) *Assume T is a left Hopf monad. Let $d : T \to K$ be a right integral of T and suppose we have a natural transformation $\varepsilon' : K J^! \to 1_{\mathcal{C}}$. For any object X of $\mathcal{C}$, set:*

$${}^{(\varepsilon')}d_X = s^l_{J(X)^\vee} T\big({}^!d_{J(X)} {}^!\varepsilon'_X\big) : T(X) \to J(X).$$

Then the natural transformation ${}^{(\varepsilon')}d : T \to J$ is a left integral of T.

(c) *Assume that T is a Hopf monad. Suppose that ${}^!K$ is right adjoint to J, with adjunction morphisms $\alpha : J^! K \to 1_{\mathcal{C}}$ and $\beta : 1_{\mathcal{C}} \to {}^!K J$. Then the assignment $c \mapsto c^{(\alpha)}$ defines a bijection between J-valued left integrals of T and K-valued right integrals of T, whose inverse is given by $d \mapsto {}^{(\beta^!)}d$.*

Proof. Let us prove Part (a). Set $d = c^{(\varepsilon)}$. Let X be an object of $\mathcal{C}$ and set $Y = {}^\vee K(X)$ and $y = c^!{}_{K(X)} \varepsilon^!{}_X$. By (45), we have $T_2(\mathbb{1}, Y)^\vee (y \otimes \mathrm{id}_{T(\mathbb{1})^\vee}) = y \otimes \eta_{\mathbb{1}}{}^\vee$. Therefore:

$$\begin{aligned}
&d_X \otimes T(\widetilde{\mathrm{coev}}_{\mathbb{1}})\eta_{\mathbb{1}} \\
&\quad = (d_X \otimes \eta_{\mathbb{1}} \widetilde{\mathrm{coev}}_{\mathbb{1}} T_0) T_2(X, \mathbb{1}) \quad \text{by (12)} \\
&\quad = \big(s^r_Y T(y) \otimes \big(s^r_{\mathbb{1}} \otimes \mu_{\mathbb{1}}\big) T_2\big(T(\mathbb{1})^\vee, T(\mathbb{1})\big) T(\widetilde{\mathrm{coev}}_{T(\mathbb{1})})\big) T_2(X, \mathbb{1}) \quad \text{by (23)} \\
&\quad = \big(s^r_Y \otimes \big(s^r_{\mathbb{1}} \otimes \mu_{\mathbb{1}}\big) T_2\big(T(\mathbb{1})^\vee, T(\mathbb{1})\big)\big) T_2\big(T(Y)^\vee, T(\mathbb{1})^\vee \otimes T(\mathbb{1})\big) T(y \otimes \widetilde{\mathrm{coev}}_{T(\mathbb{1})}) \\
&\quad = \big(\big(s^r_Y \otimes s^r_{\mathbb{1}}\big) T_2\big(T(Y)^\vee, T(\mathbb{1})^\vee\big) \otimes \mu_{\mathbb{1}}\big) \\
&\qquad \circ T_2\big(T(Y)^\vee \otimes T(\mathbb{1})^\vee, T(\mathbb{1})\big) T(y \otimes \widetilde{\mathrm{coev}}_{T(\mathbb{1})}) \quad \text{by (11)} \\
&\quad = \big(s^r_Y T\big(T_2(\mathbb{1}, Y)^\vee\big) \otimes \mu_{\mathbb{1}}\big) T_2\big(T(Y)^\vee \otimes T(\mathbb{1})^\vee, T(\mathbb{1})\big) T(y \otimes \widetilde{\mathrm{coev}}_{T(\mathbb{1})}) \quad \text{by (30)}
\end{aligned}$$

and so

$$
\begin{aligned}
&d_X \otimes T(\widetilde{\mathrm{coev}}_{\mathbb{1}})\eta_{\mathbb{1}} \\
&\quad = \big(s^r_Y \otimes \mu_{\mathbb{1}}\big) T\big(T_2(\mathbb{1}, Y)^\vee\big) T_2\big(T(Y)^\vee, T(\mathbb{1})\big) \\
&\qquad \circ T\big(\big(T_2(\mathbb{1}, Y)^\vee \otimes \mathrm{id}_{T(\mathbb{1})}\big)(y \otimes \widetilde{\mathrm{coev}}_{T(\mathbb{1})})\big) \\
&\quad = \big(s^r_Y \otimes \mu_{\mathbb{1}}\big) T\big(T_2(\mathbb{1}, Y)^\vee\big) T_2\big(T(Y)^\vee, T(\mathbb{1})\big) \\
&\qquad \circ T\big(y \otimes \big(\eta_{\mathbb{1}}^\vee \otimes \mathrm{id}_{T(\mathbb{1})}\big)\widetilde{\mathrm{coev}}_{T(\mathbb{1})}\big) \quad \text{by (45)} \\
&\quad = \big(s^r_Y \otimes \mu_{\mathbb{1}}\big) T\big(T_2(\mathbb{1}, Y)^\vee\big) T_2\big(T(Y)^\vee, T(\mathbb{1})\big) T(y \otimes \eta_{\mathbb{1}}\widetilde{\mathrm{coev}}_{\mathbb{1}}) \\
&\quad = \big(s^r_Y T(y) \otimes \mu_{\mathbb{1}} T(\eta_{\mathbb{1}}) T(\widetilde{\mathrm{coev}}_{\mathbb{1}})\big) T(X, \mathbb{1}) \\
&\quad = \big(d_X \otimes T(\widetilde{\mathrm{coev}}_{\mathbb{1}})\big) T(X, \mathbb{1}) \quad \text{by (2).}
\end{aligned}
$$

Since $\widetilde{\mathrm{coev}}_{\mathbb{1}}$ is an isomorphism, we get $d_X \otimes \eta_{\mathbb{1}} = (d_X \otimes \mathrm{id}_{T(\mathbb{1})})T(X, \mathbb{1})$. Hence d is a K-valued right integral of T.

Part (b) is obtained by applying Part (a) to the opposite Hopf monad. Let us prove Part (c). Let $c : T \to J$ be a left integral of T. Let us check that ${}^{(\beta^!)}(c^{(\alpha)}) = c$. For any object X of $\mathcal{C}$, we have:

$$
\begin{aligned}
{}^{(\beta^!)}\big(c^{(\alpha)}\big)_X &= s^l_{J(X)^\vee} T\big({}^\vee\big(s^r_{{}^\vee K(J(X)^\vee)} T\big(c^!{}_{K(J(X)^\vee)} \alpha^\vee_{J(X)}\big)\big)\beta_X\big) \\
&= s^l_{J(X)^\vee} T\big({}^\vee T\big(c^!{}_{K(J(X)^\vee)} \alpha^\vee_{J(X)}\big)^\vee \big(s^r_{{}^\vee K(J(X)^\vee)}\big)\beta_X\big) \\
&= s^l_{J(X)^\vee} T\big({}^\vee T\big(T(\beta_X)^\vee c^!{}_{K(J(X)^\vee)} \alpha^\vee_{J(X)}\big)^\vee s^r_X\big) \\
&= \alpha_{J(X)} c_{{}^! K J(X)} T(\beta_X) s^l_{T(X)^\vee} T\big({}^\vee s^r_X\big) \\
&= \alpha_{J(X)} c_{{}^! K J(X)} T(\beta_X) \quad \text{by Proposition 3.11,} \\
&= \alpha_{J(X)} J(\beta_X) c_X = c_X \quad \text{by adjunction.}
\end{aligned}
$$

Finally, applying this to the opposite Hopf monad, we obtain that $({}^{(\beta^!)}d)^{(\alpha)} = d$ for any right integral $d : T \to K$. □

Proposition 5.7. *Under the hypotheses of Theorem* 5.3*, in the bijective correspondence of Proposition* 5.6(c)*, a universal left integral of T is transformed into a universal right integral of T, and conversely.*

Proof. By Theorem 5.3, there exist a universal left integral $\lambda^l : T \to J$ of T and a universal right integral $\lambda^r : T \to K$ of T such that ${}^!K$ is quasi-inverse (and, in particular, right adjoint) to J. Denote $\alpha : J^!K \to 1_{\mathcal{C}}$ and $\beta : 1_{\mathcal{C}} \to {}^!KJ$ the adjunction isomorphisms. Using Proposition 5.6, define a right integral $c : T \to K$ and a left integral $d : T \to J$ by $c = (\lambda^l)^{(\alpha)}$ and $d = {}^{(\beta^!)}(\lambda^r)$. We have to show that they are universal. Since λ^l and λ^r are universal, there exist unique natural transformations $f : K \to K$ and $g : J \to J$ such that $d = f\lambda^l$ and $c = g\lambda^r$. It is sufficient to prove that f and g are isomorphisms. Since $d^{(\alpha)} = \lambda^r$ by Proposition 5.6(c), we have, for any object X of $\mathcal{C}$,

$$
\lambda^r_X = d^{(\alpha)}_X = \big(\lambda^l\big)^{(\alpha)}_X T\big(f^!{}_{K(X)}\big) = c_X T\big(f^!{}_{K(X)}\big) = K\big(f^!{}_{K(X)}\big) c_X = K\big(f^!{}_{K(X)}\big) g_X \lambda^r_X.
$$

Thus $K(f^!{}_K)g = \mathrm{id}_K$ by the universal property of λ^r. By naturality of g, we also have $gK(f^!{}_K) = \mathrm{id}_K$. Hence g is an isomorphism. Similarly one shows that f is an isomorphism. □

6. Semisimplicity

In this section, we define semisimple and separable monads, and give a characterization of semisimple Hopf monads (which generalizes Maschke's theorem).

6.1. Semisimple monads

Let T be a monad on a category $\mathcal{C}$. Recall that for any object Y of $\mathcal{C}$, $(T(Y), \mu_Y)$ is a T-module. Such a T-module is said to be *free*. If (M, r) is a T-module, then r is a T-linear morphism from the free module $(T(M), \mu_M)$ to (M, r). Note that $\eta_M : M \to T(M)$ is a section of r in $\mathcal{C}$, but in general η_M is not T-linear. Recall that a *section* of a morphism $f : X \to Y$ is a morphism $g : Y \to X$ such that $fg = \mathrm{id}_Y$.

Proposition 6.1. *Let T be a monad on a category $\mathcal{C}$. The following conditions are equivalent*:

(i) *for any T-module (M, r), the T-linear morphism r has a T-linear section*;
(ii) *any T-linear morphism has a T-linear section if and only if the underlying morphism in $\mathcal{C}$ has a section*;
(iii) *any T-module is a T-linear retract of a free T-module.*

A *semisimple* monad is a monad satisfying the equivalent conditions of Proposition 6.1.

Remark 6.2. Assume that $\mathcal{C}$ is abelian semisimple, and T is additive. Then T is semisimple if and only if the category $T\text{-}\mathcal{C}$ of T-modules is abelian semisimple.

Proof of Proposition 6.1. We have (ii) implies (i) since η_M is a section of r in $\mathcal{C}$. Clearly (i) implies (iii). Let us show that (iii) implies (ii). Let $f : (M, r) \to (N, s)$ be a T-linear morphism between two T-modules and $g : N \to M$ be a section of f in $\mathcal{C}$. By assumption, (N, s) is a retract of $(T(X), \mu_X)$ for some object X of $\mathcal{C}$. Let $p : T(X) \to N$ and $i : N \to T(X)$ be T-linear morphisms such that $pi = \mathrm{id}_N$. Set $g' = rT(gp\eta_X)i : N \to M$. We have:

$$\begin{aligned} g's &= rT(gp\eta_X)is = rT(gp\eta_X)\mu_X T(i) = r\mu_M T^2(gp\eta_X)T(i) \\ &= rT(r)T\big(T(gp\eta_X)i\big) = rT(g') \end{aligned}$$

and $fg' = frT(gp\eta_X)i = sT(fgp\eta_X)i = sT(p)T(\eta_X)i = p\mu_X T(\eta_X)i = pi = \mathrm{id}_N$. Hence g' is a T-linear section of f. □

6.2. Separable monads

Let A be an algebra in a monoidal category. In particular $A \otimes A$ is a A-bimodule, and the multiplication $m : A \otimes A \to A$ of A is a morphism of A-bimodules. Recall that A is *separable* if m has a section $\varsigma : A \to A \otimes A$ as a morphism of A-bimodules, which means that:

$$(m \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \varsigma) = \varsigma m = (\mathrm{id}_A \otimes m)(\varsigma \otimes \mathrm{id}_A) \quad \text{and} \quad m\varsigma = \mathrm{id}_A.$$

In this case, set $\gamma = \varsigma u : \mathbb{1} \to A \otimes A$, where $u : \mathbb{1} \to A$ is the unit of A. Then the morphism γ satisfies:

$$(m \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \gamma) = (\mathrm{id}_A \otimes m)(\gamma \otimes \mathrm{id}_A) \quad \text{and} \quad m\gamma = u.$$

Conversely if $\gamma : \mathbb{1} \to A \otimes A$ satisfies the above equation, then A is separable and the section of m is $\varsigma = (m \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \gamma) : A \to A \otimes A$. We extend this notion to monads.

Proposition 6.3. *Let T be a monad on a category $\mathcal{C}$. The following conditions are equivalent*:

(i) *one may choose naturally for each T-module (M, r) a T-linear section $\sigma_{(M,r)}$ of the morphism $r : T(M) \to M$. Here 'naturally' means that, for any T-linear morphism $f : (M, r) \to (N, s)$, we have*:

$$\sigma_{(N,s)} f = T(f)\sigma_{(M,r)};$$

(ii) *there exists a natural transformation $\varsigma : T \to T^2$ such that*:

$$T(\mu_X)\varsigma_{T(X)} = \varsigma_X \mu_X = \mu_{T(X)} T(\varsigma_X) \quad \textit{and} \quad \mu_X \varsigma_X = \mathrm{id}_{T(X)};$$

(iii) *there exists a natural transformation $\gamma : 1_{\mathcal{C}} \to T^2$ such that*

$$T(\mu_X)\gamma_{T(X)} = \mu_{T(X)} T(\gamma_X) \quad \textit{and} \qquad \mu_X \gamma_X = \eta_X.$$

A *separable* monad is a monad satisfying the equivalent conditions of Proposition 6.3.

Proof. Let us show that (i) implies (ii). Define $\varsigma = \sigma_{(T,\mu)} : T \to T^2$, which is clearly a natural transformation such that $\varsigma_X \mu_X = \mathrm{id}_{T(X)}$. Since μ_X is T-linear, the naturality of σ gives $\varsigma_X \mu_X = T(\mu_X)\varsigma_{T(X)}$. Finally, using the T-linearity of σ, we have $\varsigma_X \mu_X = \mu_{T(X)} T(\varsigma_X)$.

Let us show that (ii) implies (iii). Set $\gamma = \varsigma\eta : 1_{\mathcal{C}} \to T^2$. Then

$$\begin{aligned} T(\mu_X)\gamma_{T(X)} &= T(\mu_X)\varsigma_{T(X)}\eta_{T(X)} = \varsigma_X \mu_X \eta_{T(X)} \\ &= \varsigma_X \mu_X T(\eta_X) = \mu_{T(X)} T(\varsigma_X) T(\eta_X) = \mu_{T(X)} T(\gamma_X), \end{aligned}$$

and $\mu_X \gamma_X = \mu_X \varsigma_X \eta_X = \eta_X$.

Let us show that (iii) implies (i). For any T-module (M, r), set $\sigma_{(M,r)} = T(r)\gamma_M$. We have:

$$\begin{aligned} \sigma_{(M,r)} r &= T(r)\gamma_M r = T(r)T^2(r)\gamma_{T(M)} = T(r)T(\mu_M)\gamma_{T(M)} \\ &= T(r)\mu_{T(M)} T(\gamma_M) = \mu_M T^2(r) T(\gamma_M) = \mu_M T(\sigma_{(M,r)}) \end{aligned}$$

and $r\sigma_{(M,r)} = rT(r)\gamma_M = r\mu_M \gamma_M = r\eta_M = \mathrm{id}_M$. Therefore $\sigma_{(M,r)}$ is a T-linear section of r. Finally, for any T-linear morphism $f : (M, r) \to (N, s)$, we have:

$$\sigma_{(N,s)} f = T(s)\gamma_N f = T(s)T^2(f)\gamma_M = T\big(sT(f)\big)\gamma_M = T(fr)\gamma_M = T(f)\sigma_{(M,r)}.$$

Hence σ is natural. $\square$

6.3. Cointegrals

Let (T, μ, η) be a bimonad on a monoidal category $\mathcal{C}$. A *cointegral* of T is a morphism $\Lambda : \mathbb{1} \to T(\mathbb{1})$ satisfying:

$$\mu_{\mathbb{1}} T(\Lambda) = \Lambda T_0. \tag{47}$$

This condition means that Λ is a morphism of T-modules from $(\mathbb{1}, T_0)$ to $(T(\mathbb{1}), \mu_{\mathbb{1}})$.

Example 6.4. Let A be a bialgebra in a braided category $\mathcal{C}$ and $\Lambda : \mathbb{1} \to A$ be a morphism in $\mathcal{C}$. Then Λ is an cointegral of the bimonad $A \otimes ?$ (respectively $? \otimes A$) of Example 2.2 if and only if Λ is a left (respectively right) integral *in* A.

6.4. Maschke theorem

In this section, we extend the theorem of Maschke, which characterizes semisimple Hopf algebras in terms of (co)integrals, to the (non-linear) setting of Hopf monads.

Theorem 6.5 *(Maschke theorem for Hopf monads). Let T be a right Hopf monad on a right autonomous category. The following assertions are equivalent*:

(i) *T is semisimple*;
(ii) *T is separable*;
(iii) *T admits a cointegral $\Lambda : \mathbb{1} \to T(\mathbb{1})$ such that $T_0 \Lambda = \mathrm{id}_{\mathbb{1}}$.*

Proof. We have (ii) implies (i) by Propositions 6.1 and 6.3.

Let us show that (i) implies (iii). Consider the T-module $(\mathbb{1}, T_0)$. Since T is semisimple, there exists a T-linear morphism $\Lambda : (\mathbb{1}, T_0) \to (T(\mathbb{1}), \mu_{\mathbb{1}})$ such that $T_0 \Lambda = \mathrm{id}_{\mathbb{1}}$. The T-linearity of Λ means $\mu_{\mathbb{1}} T(\Lambda) = \Lambda T_0$, that is, Λ is a cointegral.

Finally, let us show that (iii) implies (ii). Consider the morphisms $\Gamma_X : X \otimes T(\mathbb{1}) \to T^2(X)$ as defined in (44). Set $\gamma_X = \Gamma_X(\mathrm{id}_X \otimes \Lambda) : X \to T^2(X)$. By Lemma 4.10 applied to the T-module $(T(X), \mu_X)$, we have:

$$T(\mu_X)\Gamma_{T(X)}(\mu_X \otimes \mu_{\mathbb{1}})T_2\big(T(X), T(\mathbb{1})\big) = \mu_{T(X)} T\big(T(\mu_X)\Gamma_{T(X)}\big). \tag{48}$$

Composing the left-hand side of (48) with $T(\eta_X \otimes \Lambda)$ gives:

$$\begin{aligned}
&T(\mu_X)\Gamma_{T(X)}(\mu_X \otimes \mu_{\mathbb{1}})T_2\big(T(X), T(\mathbb{1})\big)T(\eta_X \otimes \Lambda)\\
&\quad = T(\mu_X)\Gamma_{T(X)}\big(\mu_X T(\eta_X) \otimes \mu_{\mathbb{1}} T(\Lambda)\big)T_2(X, \mathbb{1})\\
&\quad = T(\mu_X)\Gamma_{T(X)}(\mathrm{id}_{T(X)} \otimes \Lambda T_0)T_2(X, \mathbb{1})\\
&\quad = T(\mu_X)\gamma_{T(X)}.
\end{aligned}$$

Composing the right-hand side of (48) with $T(\eta_X \otimes \Lambda)$ gives:

$$\begin{aligned}
&\mu_{T(X)}T\big(T(\mu_X)\Gamma_{T(X)}\big)T(\eta_X\otimes\Lambda)\\
&\quad=\mu_{T(X)}T\big(T(\mu_X)T^2(\eta_X)\Gamma_X(\mathrm{id}_X\otimes\Lambda)\big)\\
&\quad=\mu_{T(X)}T(\gamma_X).
\end{aligned}$$

Hence $T(\mu_X)\gamma_{T(X)}=\mu_{T(X)}T(\gamma_X)$. Moreover, using Lemma 4.9(a) and since $T_0\Lambda=\mathrm{id}_{\mathbb{1}}$, we have $\mu_X\gamma_X=\mu_X\Gamma_X(\mathrm{id}_X\otimes\Lambda)=(\eta_X\otimes T_0\Lambda)=\eta_X$. We conclude that T is separable by Proposition 6.3. □

7. Sovereign and involutory Hopf monads

In this section, we introduce and study sovereign and involutory Hopf monads.

7.1. Sovereign categories

Let $\mathcal{C}$ be a left autonomous category. Recall that the choice of a left dual $({}^\vee X,\mathrm{ev}_X,\mathrm{coev}_X)$ for each object X of $\mathcal{C}$ defines a left dual functor ${}^\vee ?:\mathcal{C}^{\mathrm{op}}\to\mathcal{C}$ which is strong monoidal. Hence a *double left dual functor* ${}^{\vee\vee}?$, defined by $X\mapsto{}^\vee({}^\vee X)$ and $f\mapsto{}^\vee({}^\vee f)$, which is a strong monoidal endofunctor of $\mathcal{C}$.

The choice of left duals is innocuous in that different choices of left duals define canonically isomorphic double left dual functors (see Section 3.1). Subsequently we will refer to *the* double left dual functor ${}^{\vee\vee}?$.

A *sovereign structure* on a left autonomous category $\mathcal{C}$ is a monoidal natural transformation $\phi:1_{\mathcal{C}}\to{}^{\vee\vee}?$. By Lemma 3.4, such a ϕ is an isomorphism.

A *sovereign category* is a left autonomous category endowed with a sovereign structure. Note that a sovereign category is autonomous. Indeed, let $\mathcal{C}$ be a sovereign category, with chosen left duals $({}^\vee X,\mathrm{ev}_X,\mathrm{coev}_X)$ and sovereign structure $\phi_X:X\xrightarrow{\sim}{}^{\vee\vee}X$. For each object X of $\mathcal{C}$, set:

$$\begin{aligned}
&\widetilde{\mathrm{ev}}_X=\mathrm{ev}_{{}^\vee X}(\phi_X\otimes\mathrm{id}_{{}^\vee X}):X\otimes{}^\vee X\to\mathbb{1},\\
&\widetilde{\mathrm{coev}}_X=\big(\mathrm{id}_{{}^\vee X}\otimes\phi_X^{-1}\big)\,\mathrm{coev}_{{}^\vee X}:\mathbb{1}\to{}^\vee X\otimes X.
\end{aligned}$$

Then $({}^\vee X,\widetilde{\mathrm{ev}}_X,\widetilde{\mathrm{coev}}_X)$ is a right dual of X. Moreover the right dual functor $?^\vee:\mathcal{C}^{\mathrm{op}}\to\mathcal{C}$ defined by this choice of right duals coincides with ${}^\vee ?$ as a strong monoidal functor.

Remark 7.1. Let $\mathcal{C}$ be a sovereign category, with sovereign structure $\phi:1_{\mathcal{C}}\to{}^{\vee\vee}?$. Since $\mathcal{C}$ is autonomous, we have $\phi^{-1}={}^!\phi=\phi^!$ by Lemma 3.4. Explicitly, we have $\phi_X^{-1}={}^\vee(\phi_{X^\vee})=(\phi_{{}^\vee X})^\vee$ and ${}^{\vee\vee}(\phi_{X^{\vee\vee}})=(\phi_{{}^{\vee\vee}X})^{\vee\vee}=\phi_X$ for any object X of $\mathcal{C}$ (up to the canonical isomorphisms of Remark 3.2).

7.2. Sovereign functors

Let $\mathcal{C}$, $\mathcal{D}$ be sovereign categories, with sovereign structures ϕ and ϕ' respectively. Let $F:\mathcal{C}\to\mathcal{D}$ be a strong monoidal functor. Recall (see Section 3.2) that F defines a natural isomorphism $F_1^l(X):F({}^\vee X)\to{}^\vee F(X)$. Hence a natural isomorphism $F_1^{ll}(X)={}^\vee(F_1^l(X)^{-1})F_1^l({}^\vee X):F({}^{\vee\vee}X)\to{}^{\vee\vee}F(X)$. We will say that the functor F is *sovereign* if

$$F_1^{ll}(X)F(\phi_X)=\phi'_{F(X)} \tag{49}$$

for any object X of $\mathcal{C}$.

7.3. Square of the antipode

Let $\mathcal{C}$ be a sovereign category, with sovereign structure $\phi: 1_{\mathcal{C}} \to {}^{\vee\vee}?$, and T be a Hopf monad on $\mathcal{C}$. Define $S^2 \in \mathrm{HOM}(T, T)$ by

$$S_X^2 = \phi_{T(X)}^{-1} s^l_{{}^\vee T(X)} T\big({}^\vee\big(s^l_X\big)\big) T(\phi_X) \tag{50}$$

for any object X of $\mathcal{C}$. We call S^2 the *square of the antipode* of T.

Note that S^2 depends, in general, on the sovereign structure on $\mathcal{C}$.

Example 7.2. Let A be a Hopf algebra in a braided sovereign category $\mathcal{C}$, with braiding τ and sovereign structure ϕ. Then the square of the antipode of the left Hopf monad $A \otimes ?$ on $\mathcal{C}$ (see Example 3.10) is given by $S_X^2 = \phi_A^{-1} \mathcal{U}_A (S_A)^2 \otimes \mathrm{id}_X$ for any object X of $\mathcal{C}$, where S_A is the antipode of A and $\mathcal{U}_A = (\mathrm{ev}_A \tau_{A,{}^\vee A} \otimes \mathrm{id}_{{}^{\vee\vee}A})(\mathrm{id}_A \otimes \mathrm{coev}_{{}^\vee A}): A \to {}^{\vee\vee}A$ is the Drinfeld isomorphism (see Section 8.1). Note that if $\mathcal{C}$ is ribbon with twist θ (see Section 8.1), then $\phi_A = \mathcal{U}_A \theta_A$ and so $S_X^2 = \theta_A^{-1} (S_A)^2 \otimes \mathrm{id}_X$. In particular, if H is a finite-dimensional Hopf algebra over a field $\Bbbk$, then the square of the antipode of the left Hopf monad $H \otimes_\Bbbk ?$ on $\mathrm{vect}(\Bbbk)$ is given by $S_X^2 = (S_H)^2 \otimes \mathrm{id}_X$ for any finite-dimensional $\Bbbk$-vector space X.

Proposition 7.3. *The natural transformation $S^2: T \to T$ is an automorphism of the Hopf monad T (see Section 3.8). Moreover the inverse of S^2, denoted S^{-2}, is given by:*

$$S_X^{-2} = \phi_{T(X)^{\vee\vee}} s^r_{T(X)^\vee} T\big(\big(s^r_X\big)^\vee\big) T\big(\phi^{-1}_{X^{\vee\vee}}\big)$$

for any object X of $\mathcal{C}$ (up to the canonical isomorphisms of Remark 3.2).

Remark 7.4. Recall that, in Section 3.9, we defined an anti-automorphism S of the monoid $(\mathrm{HOM}(1_{\mathcal{C}}, T), *, \eta)$. Nevertheless, the notations are not in conflict since $S^2 f = (S)^2(f)$ and $S^{-2}(f) = (S^{-1})^2(f)$ for every $f \in \mathrm{HOM}(1_{\mathcal{C}}, T)$. In particular $S^2 f$ does not depend on the sovereign structure on $\mathcal{C}$ (unlike S^2).

Proof of Proposition 7.3. Let (M, r) be a T-module. By Theorem 3.8(a), we have ${}^{\vee\vee}(M, r) = ({}^{\vee\vee}M, {}^{\vee\vee}r \Sigma_M)$, where $\Sigma_M = s^l_{{}^\vee T(M)} T({}^\vee(s^l_M))$. Also, if $\varphi: N \to M$ is an isomorphism in $\mathcal{C}$, then $(M, r)^\varphi = (N, \varphi^{-1} r T(\varphi))$ is a T-module. Define a functor $F: T\text{-}\mathcal{C} \to T\text{-}\mathcal{C}$ by

$$(M, r) \mapsto \big({}^{\vee\vee}(M, r)\big)^{\phi_M} = \big(M, \phi_M^{-1} {}^{\vee\vee}r \Sigma_M T(\phi_M)\big) = \big(M, r S_M^2\big), \quad f \mapsto f.$$

By the preliminary remarks, F is well-defined. Since $\phi: 1_{\mathcal{C}} \to {}^{\vee\vee}?$ is monoidal, the functor F is monoidal strict. Also $U_T F = U_T$. Therefore S^2 is a morphism of bimonads by Lemma 2.9, and so of Hopf monads.

Let us show that S^2 is an automorphism. Remark that if (N, s) is a T-module and $\varphi: N \to M$ is an isomorphism in $\mathcal{C}$, then $(N, s)_\varphi = (M, \varphi s T(\varphi^{-1}))$ is a T-module. Also $((M, r)^\varphi)_\varphi = (M, r)$ and $((N, s)_\varphi)^\varphi = (N, s)$ for any T-modules (M, r), (N, s) and any isomorphism $\varphi: N \to M$ in $\mathcal{C}$. Therefore F is an autofunctor of $T\text{-}\mathcal{C}$ with inverse given by $F^{-1}(M, r) = ((M, r)_{\phi_M})^{\vee\vee}$ (up to the canonical isomorphisms of Remark 3.2). Now, by Theorem 3.8(b), $(M, r)^{\vee\vee} = (M^{\vee\vee}, r^{\vee\vee} \Sigma'_M)$, where $\Sigma'_M = s^r_{T(M)^\vee} T((s^r_M)^\vee)$. Therefore:

$$\begin{aligned}F^{-1}(M,r) &= \big({}^{\vee\vee}M, \phi_M r T\big(\phi_M^{-1}\big)\big)^{\vee\vee}\\ &= \big(M, \phi_M^{\vee\vee} r^{\vee\vee} T\big(\phi_M^{-1}\big)^{\vee\vee} \Sigma'_{\vee\vee M}\big)\\ &= \big(M, r\phi_{T(M)}^{\vee\vee} \Sigma'_M T\big(\big(\phi_M^{-1}\big)^{\vee\vee}\big)\big)\\ &= \big(M, r\phi_{T(M)^{\vee\vee}} \Sigma'_M T\big(\phi_{M^{\vee\vee}}^{-1}\big)\big) \quad \text{by Remark 7.1}\\ &= \big(M, rS_M^{-2}\big),\end{aligned}$$

where $S_M^{-2} = \phi_{T(M)^{\vee\vee}} s^r_{T(M)^\vee} T((s^r_M)^\vee) T(\phi^{-1}_{M^{\vee\vee}})$. Again by Lemma 2.9, we get that $S^{-2}: T \to T$ is a morphism of Hopf monads and is an inverse of S^2. $\square$

Lemma 7.5. *Let $\mathcal{C}$ be a sovereign category, with sovereign structure ϕ, and T be a Hopf monad on $\mathcal{C}$. Let $a \in \mathrm{H{\small OM}}(1_\mathcal{C}, T)$ and $a^\sharp \in \mathrm{H{\small OM}}(U_T, U_T)$ as in Lemma 1.3. The following conditions are equivalent*:

(i) $L_a = R_a S^2$, *where L_a and R_a are defined as in* (6);
(ii) $\phi a^\sharp \in \mathrm{H{\small OM}}(U_T, {}^{\vee\vee}?U_T) = \mathrm{H{\small OM}}(U_T, U_T{}^{\vee\vee}?_{T\text{-}\mathcal{C}})$ *lifts to* $\mathrm{H{\small OM}}(1_{T\text{-}\mathcal{C}}, {}^{\vee\vee}?_{T\text{-}\mathcal{C}})$.

Proof. Let (M,r) be a T-module. Recall that $(\phi a^\sharp)_{(M,r)} = \phi_M r a_M$. Also, by Theorem 3.8(a), we have:

$${}^{\vee\vee}(M,r) = \big({}^{\vee\vee}M, {}^{\vee\vee}r s^l_{{}^\vee T(M)} T\big({}^\vee\big(s^l_M\big)\big)\big) = \big({}^{\vee\vee}M, \phi_M r S^2_M T\big(\phi_M^{-1}\big)\big).$$

Therefore $\phi a^\sharp$ lifts to a natural transformation $1_{T\text{-}\mathcal{C}} \to {}^{\vee\vee}?_{T\text{-}\mathcal{C}}$ if and only if, for any T-module (M,r), we have $\phi_M r a_M r = \phi_M r S^2_M T(\phi_M^{-1}) T(\phi_M r a_M)$ or, equivalently, $r\mu_M a_{T(M)} = r\mu_M T(a_M) S^2_M$ since ϕ is an isomorphism and $rT(r) = r\mu_M$. By Lemma 1.3, this last condition is equivalent to $\mu_X a_{T(X)} = \mu_X T(a_X) S^2_X$ for each object X of $\mathcal{C}$, that is, $L_a = R_a S^2$. $\square$

7.4. Sovereign Hopf monads

Let T be a Hopf monad on a sovereign category. A *sovereign element* of T is a grouplike element G (which is $*$-invertible by Lemma 3.21) satisfying:

$$S^2 = \mathrm{ad}_G. \tag{51}$$

Here S^2 is the square of the antipode of T (see Section 7.3) and ad is the adjoint action of T (see Section 1.4).

A *sovereign Hopf monad* is a Hopf monad endowed with a sovereign element.

Proposition 7.6. *Let $\mathcal{C}$ be a sovereign category and T be a Hopf monad on $\mathcal{C}$. Then sovereign elements of T are in bijection with sovereign structures on T-$\mathcal{C}$.*

Proof. Denote $\phi: 1_\mathcal{C} \to {}^{\vee\vee}?$ the sovereign structure of $\mathcal{C}$. Suppose that G is a sovereign element of T. Since $S^2 = \mathrm{ad}_G$ and so $L_G = R_G S^2$, the natural transformation $\phi G^\sharp$ lifts to a natural transformation $\Phi: 1_{T\text{-}\mathcal{C}} \to {}^{\vee\vee}?_{T\text{-}\mathcal{C}}$ by Lemma 7.5. Since ϕ and $G^\sharp$ are monoidal (see Lemma 3.20), so is the lift Φ of $\phi G^\sharp$, which hence defines a sovereign structure on T-$\mathcal{C}$.

Conversely, let $\Phi : 1_{T\text{-}\mathcal{C}} \to {}^{\vee\vee}?_{T\text{-}\mathcal{C}}$ be a sovereign structure on $T\text{-}\mathcal{C}$. Since the natural transformation $\phi^{-1}U_T(\Phi)$ is monoidal, there exists a (unique) grouplike element G of T such that $\phi^{-1}U_T(\Phi) = G^\sharp$ (by Lemma 3.20). Since $\phi G^\sharp = U_T(\Phi)$ lifts to the natural transformation Φ, we have (by Lemma 7.5) that $L_G = R_G S^2$, that is $S^2 = \mathrm{ad}_G$. Hence G is a sovereign element of T. □

7.5. Involutory Hopf monads

A Hopf monad T on a sovereign category $\mathcal{C}$ is *involutory* if it satisfies $S^2 = \mathrm{id}_T$, where S^2 denotes the square of the antipode as defined in Section 7.3. Note that this notion depends on the choice of a sovereign structure on $\mathcal{C}$.

Proposition 7.7. *Let $\mathcal{C}$ be a sovereign category and T a Hopf monad on $\mathcal{C}$. The following conditions are equivalent*:

(i) *T is involutory*;
(ii) *η is a sovereign element of T*;
(iii) *there exists a sovereign structure on the category $T\text{-}\mathcal{C}$ such that the forgetful functor $U_T : T\text{-}\mathcal{C} \to \mathcal{C}$ is sovereign*;
(iv) *we have $s^r_X = \phi^{-1}_{X^\vee} s^l_X T(\phi_{T(X)^\vee})$ for any object X of $\mathcal{C}$ (up to the canonical isomorphisms of Remark* 3.2*), where ϕ is the sovereign structure of $\mathcal{C}$.*

Proof. Clearly (i) implies (ii) since η is grouplike and $\mathrm{ad}_\eta = \mathrm{id}_T$. Assume (ii) and equip $T\text{-}\mathcal{C}$ with the sovereign structure defined by η (see Proposition 7.6). Then the forgetful functor U_T is sovereign. Hence (ii) implies (iii).

Let us prove that (iii) implies (iv). By Theorem 3.8, we have preferred choices of left and right duals of (M, r), namely ${}^\vee(M, r) = ({}^\vee M, s^l_M T({}^\vee r))$ and $(M, r)^\vee = (M^\vee, s^r_M T(r^\vee))$. With this choice of duals, $(U_T)^l_1(M, r) = \mathrm{id}_{{}^\vee M}$. Let Φ be a sovereign structure on $T\text{-}\mathcal{C}$ such that U_T is sovereign. We have $U_T(\Phi) = \phi_{U_T}$ by (49). Let (M, r) be a T-module. We have $\Phi_{(M,r)^\vee} s^r_M T(r^\vee) = s^l_M T({}^\vee r) T(\Phi_{(M,r)^\vee})$ since $\Phi_{(M,r)^\vee}$ is T-linear, and so $\phi_{M^\vee} s^r_M T(r^\vee) = s^l_M T(\phi_{T(M)^\vee}) T(r^\vee)$. Hence (iv) by Lemma 1.3.

Finally, let us prove that (iv) implies (i). For any object X of $\mathcal{C}$, we have:

$$\begin{aligned} S^2_X &= \phi^{-1}_{T(X)} s^l_{{}^\vee T(X)} T\big({}^\vee\big(s^l_X\big)\big) T(\phi_X) \\ &= \phi^{-1}_{T(X)} \phi_{T(X)} s^r_{{}^\vee T(X)} T\big(\phi^{-1}_{T({}^\vee T(X))^\vee}\big) T\big({}^\vee\big(s^l_X\big)\big) T(\phi_X) \\ &= s^r_{{}^\vee T(X)} T\big(\big(s^l_X\big)^\vee\big) = \mathrm{id}_{T(X)} \quad \text{by Proposition 3.11.} \end{aligned}$$

Hence T is involutory. □

8. Quasi-triangular and ribbon Hopf monads

In this section, we define R-matrices and twists for a Hopf monad. They encode the facts that the category of modules over the Hopf monad is braided or ribbon. We first review some well-known properties of braided and ribbon categories.

8.1. Braided categories, twists, and ribbon categories

Recall that a *braiding* on a monoidal category $\mathcal{C}$ is a natural isomorphism $\tau \in \mathrm{Hom}(\otimes, \otimes^{\mathrm{op}})$ such that:

$$\tau_{X,Y\otimes Z} = (\mathrm{id}_Y \otimes \tau_{X,Z})(\tau_{X,Y} \otimes \mathrm{id}_Z); \tag{52}$$

$$\tau_{X\otimes Y,Z} = (\tau_{X,Z} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes \tau_{Y,Z}); \tag{53}$$

for all objects X, Y, Z of $\mathcal{C}$. A *braided* category is a monoidal category endowed with a braiding. If τ is a braiding on $\mathcal{C}$, then so is its *mirror* $\overline{\tau}$ defined by $\overline{\tau}_{X,Y} = \tau^{-1}_{Y,X}$.

If $\mathcal{C}$ is a braided category, with braiding τ, and if (X, Y, e, h) is a duality in $\mathcal{C}$, then $(Y, X, e\tau_{X,Y}, \tau^{-1}_{Y,X}h)$ is a duality too. In particular, a braided category which is left (respectively right) autonomous is also right (respectively left) autonomous, and so is autonomous.

Let $\mathcal{C}$ be a braided autonomous category. Let $\mathcal{U}: 1_{\mathcal{C}} \to {}^{\vee\vee}?$ be the natural transformation defined, for any object X of $\mathcal{C}$, by:

$$\mathcal{U}_X = (\mathrm{ev}_X\, \tau_{X,{}^\vee X} \otimes \mathrm{id}_{{}^{\vee\vee}X})(\mathrm{id}_X \otimes \mathrm{coev}_{{}^\vee X}). \tag{54}$$

Lemma 8.1. *The natural transformation $\mathcal{U}$ enjoys the following properties*:

(a) $\mathcal{U}_{X\otimes Y} = (\mathcal{U}_X \otimes \mathcal{U}_Y)\tau^{-1}_{X,Y}\tau^{-1}_{Y,X}$ *for all objects X, Y of $\mathcal{C}$*;
(b) $\mathcal{U}_{\mathbb{1}} = ({}^{\vee\vee}?)_0 : \mathbb{1} \xrightarrow{\sim} {}^{\vee\vee}\mathbb{1}$;
(c) *$\mathcal{U}$ is an isomorphism and, for any object X of $\mathcal{C}$,*

$$\mathcal{U}_X^{-1} = (\mathrm{ev}_{{}^\vee X} \otimes \mathrm{id}_X)\big(\mathrm{id}_{{}^{\vee\vee}X} \otimes \tau^{-1}_{{}^\vee X,X}\, \mathrm{coev}_X\big).$$

We will refer to $\mathcal{U}$ as the *Drinfeld isomorphism* of $\mathcal{C}$.

Remark 8.2. The Drinfeld isomorphism $\mathcal{U}$ is monoidal (and so is a sovereign structure on $\mathcal{C}$) if and only if the braiding τ is a *symmetry*, that is, $\overline{\tau} = \tau$.

Recall that a *twist* on a braided category $\mathcal{C}$, with braiding τ, is a natural isomorphism $\Theta \in \mathrm{Hom}(1_{\mathcal{C}}, 1_{\mathcal{C}})$ satisfying:

$$\Theta_{X\otimes Y} = (\Theta_X \otimes \Theta_Y)\tau_{Y,X}\tau_{X,Y} \tag{55}$$

for all objects X, Y of $\mathcal{C}$. If $\mathcal{C}$ is autonomous, then a twist Θ on $\mathcal{C}$ is said to be *self-dual* if it satisfies

$${}^\vee ?\Theta = \Theta^\vee ? \quad (\text{or, equivalently, } ?^\vee\Theta = \Theta ?^\vee). \tag{56}$$

A *ribbon category* is a braided autonomous category endowed with a self-dual twist.

The following proposition establishes a correspondence (via the Drinfeld isomorphism) between the notions of sovereign structure and twist in the context of an autonomous braided category.

Proposition 8.3 *(Deligne). Let $\mathcal{C}$ be an autonomous braided category, and denote $\mathcal{U}$ its Drinfeld isomorphism. The assignment $\Theta \mapsto \mathcal{U}\Theta$ defines a bijection between twists on $\mathcal{C}$ and sovereign structures on $\mathcal{C}$.*

8.2. Quasi-triangular bimonads

Let T be a monad on a monoidal category $\mathcal{C}$. Recall (see Section 1.2) that a natural transformation $R \in \mathrm{HOM}(\otimes, T \otimes^{\mathrm{op}} T) = \mathrm{HOM}(\otimes, \otimes^{\mathrm{op}} \circ T^{\times 2})$ is $*$-invertible if there exists a (necessarily unique) natural transformation $R^{*-1} \in \mathrm{HOM}(\otimes^{\mathrm{op}}, T \otimes T) = \mathrm{HOM}(\otimes^{\mathrm{op}}, \otimes \circ T^{\times 2})$ such that $R^{*-1} * R = \eta \otimes \eta$ and $R * R^{*-1} = \eta \otimes^{\mathrm{op}} \eta$, where $*$ is the convolution product as defined in (4).

An R-*matrix* for a bimonad (T, μ, η) on a monoidal category $\mathcal{C}$ is a $*$-invertible natural transformation $R \in \mathrm{HOM}(\otimes, T \otimes^{\mathrm{op}} T)$ such that:

$$(\mu_Y \otimes \mu_X) R_{T(X),T(Y)} T_2(X, Y) = (\mu_Y \otimes \mu_X) T_2\big(T(Y), T(X)\big) T(R_{X,Y}); \tag{57}$$

$$\begin{aligned}&\big(\mathrm{id}_{T(Z)} \otimes T_2(X, Y)\big) R_{X \otimes Y, Z}\\ &\quad = (\mu_Z \otimes \mathrm{id}_{T(X) \otimes T(Y)})(R_{X,T(Z)} \otimes \mathrm{id}_{T(Y)})(\mathrm{id}_X \otimes R_{Y,Z});\end{aligned} \tag{58}$$

$$\begin{aligned}&\big(T_2(Y, Z) \otimes \mathrm{id}_{T(X)}\big) R_{X, Y \otimes Z}\\ &\quad = (\mathrm{id}_{T(Y) \otimes T(Z)} \otimes \mu_X)(\mathrm{id}_{T(Y)} \otimes R_{T(X),Z})(R_{X,Y} \otimes \mathrm{id}_Z);\end{aligned} \tag{59}$$

for all objects X, Y, Z of $\mathcal{C}$. A *quasitriangular bimonad* is a bimonad equipped with an R-matrix.

Example 8.4. Let H be a bialgebra over a field $\Bbbk$. Let $r = \sum_i a_i \otimes b_i \in H \otimes_\Bbbk H$. For any $\Bbbk$-vector spaces X and Y, set:

$$R_{X,Y}(x \otimes y) = \sum_i b_i \otimes y \otimes a_i \otimes x \in H \otimes_\Bbbk Y \otimes_\Bbbk H \otimes_\Bbbk X.$$

Then R is an R-matrix for the bimonad $H \otimes_\Bbbk ?$ on $\mathrm{Vect}(\Bbbk)$ if and only if r is an R-matrix for H (in the usual sense).

Theorem 8.5. *Let T be a bimonad on a monoidal category $\mathcal{C}$. Any* R*-matrix R for T yields a braiding τ on T-$\mathcal{C}$ as follows*:

$$\tau_{(M,r),(N,s)} = (s \otimes t) R_{M,N} : (M, r) \otimes (N, s) \to (N, s) \otimes (M, r)$$

for any T-modules (M, r) and (N, s). This assignment gives a bijection between R*-matrices for T and braidings on T-$\mathcal{C}$.*

Proof. Let $R \in \mathrm{HOM}(\otimes, T \otimes^{\mathrm{op}} T)$ and set $\tau = R^\sharp$, where the canonical bijection $?^\sharp$: $\mathrm{HOM}(\otimes, T \otimes^{\mathrm{op}} T) \to \mathrm{HOM}(U_T \otimes U_T, U_T \otimes^{\mathrm{op}} U_T)$ of Lemma 1.4 is given by $f^\sharp_{(M,r),(N,s)} = (s \otimes r) f_{M,N}$ for all T-modules (M, r) and (N, s). In this correspondence, τ is an isomorphism if and only if R is $*$-invertible, and τ is T-linear in each variable (and so lifts to an element of $\mathrm{HOM}(\otimes_{T\text{-}\mathcal{C}}, \otimes^{\mathrm{op}}_{T\text{-}\mathcal{C}})$) if and only if R satisfies (57). Moreover, τ satisfies (52) and (53) if and only if R satisfies (58) and (59). Hence the bijection between R-matrices and braidings. $\square$

Corollary 8.6. *If R is an* R*-matrix for a bimonad T, then $R_{21}^{*-1} = R^{*-1}\sigma_{\mathcal{C},\mathcal{C}}$ is also an* R*-matrix for T. Moreover, if τ is the braiding on T-$\mathcal{C}$ associated with R, then its mirror $\overline{\tau}$ is the braiding on T-$\mathcal{C}$ associated with R_{21}^{*-1}.*

Proof. Let R, R' be two R-matrices for T and let τ, τ' be their associated braidings on T-$\mathcal{C}$ (see Theorem 8.5). Given two T-modules (M,r) and (N,s), we have

$$\begin{aligned}
\tau_{(N,s),(M,r)}\tau'_{(M,r),(N,s)} &= (r\otimes s)R_{N,M}(s\otimes r)R'_{M,N}\\
&= \big(rT(r)\otimes sT(s)\big)R_{T(N),T(M)}R'_{M,N}\\
&= (r\mu_r\otimes s\mu_s)(R_{2,1})_{T(M),T(N)}R'_{M,N} \quad \text{by (3)}\\
&= (r\otimes s)(R_{2,1}*R')_{M,N} = (R_{2,1}*R')^\sharp_{(N,s),(M,r)}.
\end{aligned}$$

As a result, by Lemma 1.4, $\tau' = \overline{\tau}$ if and only if $R' = R_{2,1}^{*-1}$. $\square$

Corollary 8.7. *Let T be a quasitriangular bimonad on a monoidal category $\mathcal{C}$. Then its* R*-matrix R verifies $(\mathrm{id}\otimes T_0)R_{\mathbb{1},X} = \eta_X = (T_0\otimes\mathrm{id})R_{X,\mathbb{1}}$ as well as the following Yang–Baxter equation:*

$$\begin{aligned}
&(\mu_Z\otimes\mu_Y\otimes\mu_X)(R_{T(Y),T(Z)}\otimes\mathrm{id}_{T^2(X)})(\mathrm{id}_{T(Y)}\otimes R_{T(X),Z})(R_{X,Y}\otimes\mathrm{id}_Z)\\
&\quad = (\mu_Z\otimes\mu_Y\otimes\mu_X)(\mathrm{id}_{T^2(Z)}\otimes R_{T(X),T(Y)})(R_{X,T(Z)}\otimes\mathrm{id}_{T(Y)})(\mathrm{id}_X\otimes R_{Y,Z}).
\end{aligned}$$

Moreover, if $\mathcal{C}$ is left autonomous and T has a left antipode s^l, then

$$\begin{aligned}
R_{X,Y}^{*-1} &= \big(\mathrm{id}_{T(X)\otimes T(Y)}\otimes \mathrm{ev}_X\big(s^l_X\otimes\mathrm{id}_X\big)\big)\\
&\quad\circ(\mathrm{id}_{T(X)}\otimes R_{{}^\vee T(X),Y}\otimes\mathrm{id}_X)(\mathrm{coev}_{T(X)}\otimes\mathrm{id}_{Y\otimes X});\\
{}^\vee R_{X,Y} &= \big(s^l_Y\otimes s^l_X\big)R_{{}^\vee T(X),{}^\vee T(Y)}.
\end{aligned}$$

Likewise, if $\mathcal{C}$ is right autonomous and T has a right antipode s^r, then

$$\begin{aligned}
R_{X,Y}^{*-1} &= \big(\widetilde{\mathrm{ev}}_Y\big(\mathrm{id}_Y\otimes s^r_Y\big)\otimes\mathrm{id}_{T(X)\otimes T(Y)}\big)\\
&\quad\circ(\mathrm{id}_Y\otimes R_{X,T(Y)^\vee}\otimes\mathrm{id}_{T(Y)})(\mathrm{id}_{Y\otimes X}\otimes\widetilde{\mathrm{coev}}_{T(Y)});\\
R^\vee_{X,Y} &= \big(s^r_Y\otimes s^r_X\big)R_{T(X)^\vee,T(Y)^\vee}.
\end{aligned}$$

Proof. The corollary results, by standard application of Lemma 1.4, from the facts that a braiding τ satisfies $\tau_{X,\mathbb{1}} = \mathrm{id}_X = \tau_{\mathbb{1},X}$, the Yang–Baxter equation, $\tau^{-1}_{X,Y} = (\mathrm{id}\otimes\mathrm{ev}_X)(\mathrm{id}\otimes\tau_{{}^\vee X,Y}\otimes\mathrm{id})(\mathrm{coev}_X\otimes\mathrm{id})$ and ${}^\vee(\tau_{X,Y}) = \tau_{{}^\vee X,{}^\vee Y}$ when $\mathcal{C}$ is left autonomous, and $\tau^{-1}_{X,Y} = (\widetilde{\mathrm{ev}}_Y\otimes\mathrm{id})(\mathrm{id}\otimes\tau_{X,Y^\vee}\otimes\mathrm{id})(\mathrm{id}\otimes\widetilde{\mathrm{coev}}_Y)$ and $(\tau_{X,Y})^\vee = \tau_{X^\vee,Y^\vee}$ when $\mathcal{C}$ is right autonomous. $\square$

Corollary 8.8. *Let T be a quasitriangular bimonad on an autonomous category $\mathcal{C}$. If T is a left or right Hopf monad, then T is a Hopf monad.*

Proof. Suppose that T has a left antipode. Since $T\text{-}\mathcal{C}$ is left autonomous (by Theorem 3.8(a)) and braided (by Theorem 8.5), $T\text{-}\mathcal{C}$ is also right autonomous. Hence T has a right antipode (by Theorem 3.8(b)). □

8.3. Drinfeld elements

In this section, T is a quasitriangular Hopf monad on a sovereign category $\mathcal{C}$ (see Section 7.1). Let $\phi : 1_{\mathcal{C}} \to {}^{\vee\vee}?$ be the sovereign structure of $\mathcal{C}$ and R be the R-matrix of T.

The *Drinfeld element* of T is the natural transformation $u \in \mathrm{Hom}(1_{\mathcal{C}}, T)$ defined, for any object X of $\mathcal{C}$, by:

$$u_X = \big(\mathrm{ev}_{T(X)}\big(s^l_{T(X)} \otimes \mathrm{id}_{T(X)}\big) R_{X,{}^\vee T^2(X)} \otimes \mu_X \phi^{-1}_{T^2(X)}\big)\big(\mathrm{id}_X \otimes \mathrm{coev}_{{}^\vee T^2(X)}\big). \tag{60}$$

Example 8.9. Let H be a finite-dimensional quasitriangular Hopf algebra over a field $\Bbbk$. Recall that $H \otimes_{\Bbbk} ?$ is a quasitriangular Hopf monad on vect($\Bbbk$) (see Example 8.4). Then the Drinfeld element u of $H \otimes_{\Bbbk} ?$ is given by $u_X(x) = d \otimes x$, where d is the (usual) Drinfeld element of H. Recall that $d = \sum_i S(b_i)a_i$, where $r = \sum_i a_i \otimes b_i \in H \otimes_{\Bbbk} H$ is the R-matrix of H.

Lemma 8.10. *We have $U_T(\mathcal{U}) = \phi u^\sharp$, where $U_T : T\text{-}\mathcal{C} \to \mathcal{C}$ is the forgetful functor, $?^\sharp : \mathrm{Hom}(1_{\mathcal{C}}, T) \to \mathrm{Hom}(U_T, U_T)$ is the canonical bijection of Lemma* 1.3*, and $\mathcal{U}$ is the Drinfeld isomorphism of $T\text{-}\mathcal{C}$* (*see Section* 8.1).

Proof. Let (M, r) be a T-module. By Theorems 3.8 and 8.5, we have

$$\begin{aligned}
U_T(\mathcal{U}_{(M,r)}) &= U_T\big((\mathrm{ev}_{(M,r)}\, \tau_{(M,r),{}^\vee(M,r)} \otimes \mathrm{id}_{{}^{\vee\vee}(M,r)})(\mathrm{id}_{(M,r)} \otimes \mathrm{coev}_{{}^\vee(M,r)})\big) \\
&= \big(\mathrm{ev}_M\big(s^l_M T\big({}^\vee r\big) \otimes r\big) R_{M,{}^\vee M} \otimes \mathrm{id}_{{}^{\vee\vee}M}\big)(\mathrm{id}_M \otimes \mathrm{coev}_{{}^\vee M}) \\
&= \big(\mathrm{ev}_{T(M)}\big(s^l_M T\big({}^\vee\big(rT(r)\big)\big) \otimes \mathrm{id}_{T(M)}\big) R_{M,{}^\vee M} \otimes \mathrm{id}_{{}^{\vee\vee}M}\big)(\mathrm{id}_M \otimes \mathrm{coev}_{{}^\vee M}) \\
&= \big(\mathrm{ev}_{T(M)}\big(s^l_M T\big({}^\vee(r\mu_M)\big) \otimes \mathrm{id}_{T(M)}\big) R_{M,{}^\vee M} \otimes \mathrm{id}_{{}^{\vee\vee}M}\big)(\mathrm{id}_M \otimes \mathrm{coev}_{{}^\vee M}) \\
&= \big(\mathrm{ev}_{T(M)}\big(s^l_M \otimes \mathrm{id}_{T(M)}\big) R_{M,{}^\vee T^2(M)} \otimes {}^{\vee\vee}(r\mu_M)\big)\big(\mathrm{id}_M \otimes \mathrm{coev}_{{}^\vee T^2(M)}\big).
\end{aligned}$$

Since $\phi_M^{-1\,\vee\vee}(r\mu_M) = r\mu_M \phi^{-1}_{T^2(M)}$, we get $U_T(\mathcal{U}_{(M,r)}) = \phi_M r u_M = \phi_M u^\sharp_{(M,r)}$. □

Proposition 8.11. *The Drinfeld element u of T satisfies*:

(a) $T_2 u_\otimes = (u \otimes u) * R^{*-1} * R^{*-1}_{21}$, *where* $(T_2 u_\otimes)_{X,Y} = T_2(X,Y) u_{X \otimes Y}$;
(b) $T_0 u_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$;
(c) *u is $*$-invertible and, for any object X of $\mathcal{C}$,*

$$u^{*-1}_X = (\mathrm{ev}_{{}^\vee X} \otimes \mathrm{id}_{T(X)})\big(\phi_X \otimes \big(s^l_X \otimes \mu_X\big) R_{T(X),{}^\vee T(X)}\, \mathrm{coev}_{T(X)}\big);$$

(d) $S^2 = \mathrm{ad}_u$, *where S^2 and ad_u are as in* (7) *and* (50) *respectively.*

Proof. Denote τ the braiding of $T\text{-}\mathcal{C}$ induced by R. Let $\mathcal{U}$ be the Drinfeld isomorphism of $T\text{-}\mathcal{C}$ (see Section 8.1) and $?^\sharp : \mathrm{Hom}(1_{\mathcal{C}}, T) \to \mathrm{Hom}(U_T, U_T)$ be the canonical bijection of Lemma 1.3. Recall that $U_T(\mathcal{U}) = \phi u^\sharp$ by Lemma 8.10.

Let us prove Part (a). Let (M,r) and (N,s) be T-modules. By Lemma 8.1(a),

$$\mathcal{U}_{(M,r)\otimes(N,s)} = (\mathcal{U}_{(M,r)} \otimes \mathcal{U}_{(N,s)})\tau^{-1}_{(M,r),(N,s)}\tau^{-1}_{(N,s),(M,r)}.$$

Evaluating with U_T, and since $U_T(\mathcal{U})_{(M,r)} = \phi_M u^\sharp_{(M,r)} = \phi_M r u_M$, we get:

$$\phi_{M\otimes N}(r\otimes s)T_2(M,N)u_{M\otimes N} = (\phi_M r u_M \otimes \phi_N s u_N)(r\otimes s)R^{*-1}_{M,N}(s\otimes r)R^{*-1}_{N,M}.$$

Therefore, since ϕ is a monoidal natural isomorphism,

$$\begin{aligned}
&(r\otimes s)T_2(M,N)u_{M\otimes N}\\
&\quad= (ru_M\otimes su_N)\big(rT(r)\otimes sT(s)\big)R^{*-1}_{T(M),T(N)}R^{*-1}_{N,M}\\
&\quad= \big(rT(r)u_{T(M)}\otimes sT(s)u_{T(N)}\big)(\mu_M\otimes\mu_N)R^{*-1}_{T(M),T(N)}R^{*-1}_{N,M} \quad \text{by (3)}\\
&\quad= (r\mu_M u_{T(M)}\otimes s\mu_N u_{T(N)})(\mu_M\otimes\mu_N)R^{*-1}_{T(M),T(N)}R^{*-1}_{N,M} \quad \text{by (3)}\\
&\quad= (r\otimes s)\big((u\otimes u)*R^{*-1}*R^{*-1}_{21}\big)_{M,N}.
\end{aligned}$$

Hence Part (a) by Lemma 1.4.

Let us prove Part (b). We have:

$$\begin{aligned}
T_0u_{\mathbb{1}} &= T_0u^\sharp_{(T(\mathbb{1}),\mu_{\mathbb{1}})}\eta_{\mathbb{1}} \quad \text{by Lemma 1.3}\\
&= T_0\phi^{-1}_{T(\mathbb{1})}U_T(\mathcal{U}_{(T(\mathbb{1}),\mu_{\mathbb{1}})})\eta_{\mathbb{1}}\\
&= \phi_{\mathbb{1}}^{-1\vee\vee}T_0U_T(\mathcal{U}_{(T(\mathbb{1}),\mu_{\mathbb{1}})})\eta_{\mathbb{1}}\\
&= \phi_{\mathbb{1}}^{-1}U_T(\mathcal{U}_{(\mathbb{1},T_0)})T_0\eta_{\mathbb{1}} \quad \text{since } T_0 \text{ is } T\text{-linear by (16)}\\
&= \phi_{\mathbb{1}}^{-1}U_T(\mathcal{U}_{(\mathbb{1},T_0)}) \quad \text{by (18)}\\
&= \mathrm{id}_{\mathbb{1}} \quad \text{by Lemma 8.1(b).}
\end{aligned}$$

Let us prove Part (c). Set $u' = (U_T(\mathcal{U}^{-1})\phi)^\flat \in \mathrm{H{\small OM}}(1_{\mathcal{C}},T)$, that is,

$$u'_X = (\mathrm{ev}_{^\vee X}\otimes\mathrm{id}_{T(X)})\big(\phi_X\otimes\big(s^l_X\otimes\mu_X\big)R_{T(X),T(X)^*}\,\mathrm{coev}_{T(X)}\big).$$

Then $u'^\sharp u^\sharp = U_T(\mathcal{U}^{-1})\phi\phi^{-1}U_T(\mathcal{U}) = \mathrm{id}_{U_T}$ and $u^\sharp u'^\sharp = \phi^{-1}U_T(\mathcal{U})U_T(\mathcal{U}^{-1})\phi = \mathrm{id}_{U_T}$. Therefore $u' * u = \eta = u * u'$ by Lemma 1.3, that is, u is $*$-invertible with inverse u'.

Finally, let us prove Part (d). The natural transformation $\phi u^\sharp \in \mathrm{H{\small OM}}(U_T, {}^{\vee\vee}?U_T)$ lifts to the natural transformation $\mathcal{U}\in\mathrm{H{\small OM}}(1_{T\text{-}\mathcal{C}}, {}^{\vee\vee}?_{T\text{-}\mathcal{C}})$ by Lemma 8.10. Therefore $L_u = R_uS^2$ by Lemma 7.5, and so $S^2 = \mathrm{ad}_u$ since u is $*$-invertible. □

8.4. Ribbon Hopf monads

Let T be a monad on a monoidal category $\mathcal{C}$. Recall (see Section 1.2) that $\theta \in \mathrm{HOM}(1_{\mathcal{C}}T)$ is $*$-invertible if there exists a (necessarily unique) natural transformation $\theta^{*-1} \in \mathrm{HOM}(1_{\mathcal{C}}, T)$ such that $\theta^{*-1} * \theta = \eta = \theta * \theta^{*-1}$, where $*$ is the convolution product as defined in (5). Recall also (see Section 1.3) that $\theta \in \mathrm{HOM}(1_{\mathcal{C}}T)$ is central if $\mu_X \theta_{T(X)} = \mu_X T(\theta_X)$ for each object X of $\mathcal{C}$.

A *twist* for a quasitriangular bimonad T on a monoidal category $\mathcal{C}$ is a central and $*$-invertible natural transformation $\theta : 1_{\mathcal{C}} \to T$ such that:

$$T_2\theta_{\otimes} = (\theta \otimes \theta) * R_{21} * R, \tag{61}$$

where R is the R-matrix of T and $R_{21} = R\sigma_{\mathcal{C},\mathcal{C}}$. Explicitly, (61) means that

$$T_2(X,Y)\theta_{X\otimes Y} = (\mu_X\theta_{T(X)}\mu_X \otimes \mu_Y\theta_{T(Y)}\mu_Y)R_{T(Y),T(X)}R_{X,Y}$$

for all objects X, Y of $\mathcal{C}$.

A twist of a quasitriangular Hopf monad on an autonomous category is said to be *self-dual* if it satisfies:

$$S(\theta) = \theta, \tag{62}$$

where $S : \mathrm{HOM}(1_{\mathcal{C}}T) \to \mathrm{HOM}(1_{\mathcal{C}}T)$ is the map defined in (37). Explicitly, (62) means that ${}^\vee\theta_X = s^l_X \theta_{{}^\vee T(X)}$ (or, equivalently, $\theta_X^\vee = s^r_X \theta_{T(X)^\vee}$) for every object X of $\mathcal{C}$.

A *ribbon Hopf monad* is a quasitriangular Hopf monad on an autonomous category endowed with a self-dual twist.

Example 8.12. Let H be a finite-dimensional quasitriangular Hopf algebra over a field $\Bbbk$. Then $H \otimes_{\Bbbk} ?$ is a quasitriangular monad on $\mathrm{vect}(\Bbbk)$, see Example 8.4. Let $v \in H$ and set $\theta_X(x) = v \otimes x$ for any finite-dimensional $\Bbbk$-vector space X and $x \in X$. Then θ is self-dual twist for $H \otimes_{\Bbbk} ?$ if and only if v is a ribbon element for H.

Theorem 8.13. *Let T be a quasitriangular Hopf monad on an autonomous category $\mathcal{C}$. Any twist θ for T yields a twist Θ on T-$\mathcal{C}$ as follows:*

$$\Theta_{(M,r)} = r\theta_M : (M,r) \to (M,r)$$

for any T-module (M,r). This assignment gives a bijection between twists for T and twists on T-$\mathcal{C}$. Moreover, in this correspondence, θ is self-dual (and so T is ribbon) if and only if Θ is self-dual (and so T-$\mathcal{C}$ is ribbon).

Proof. Let $\theta \in \mathrm{HOM}(1_{\mathcal{C}}, T)$ and set $\Theta = \theta^\sharp$, where $?^\sharp$ is the canonical bijection $?^\sharp$: $\mathrm{HOM}(1_{\mathcal{C}}, T) \to \mathrm{HOM}(U_T, U_T)$ given by $f^\sharp_{(M,r)} = rf_M$ for any T-module (M,r). In this correspondence, Θ is an isomorphism if and only if θ is $*$-invertible, and Θ is T-linear (and so lifts to a natural transformation $1_{T\text{-}\mathcal{C}} \to 1_{T\text{-}\mathcal{C}}$) if and only if θ is central (by Lemma 1.5). Moreover Θ satisfies (55) if and only if θ satisfies (61). Hence the bijection between twists for T and twists on T-$\mathcal{C}$. Finally, Θ satisfies (56) if and only if θ satisfies (62) by definition of S. □

8.5. Ribbon and sovereign Hopf monads

By Theorem 8.13, given a ribbon Hopf monad T on a autonomous category $\mathcal{C}$, the category T-$\mathcal{C}$ of T-modules is ribbon and so sovereign. However $\mathcal{C}$ itself is not necessarily sovereign. If $\mathcal{C}$ is sovereign, then the sovereign structure on T-$\mathcal{C}$ is encoded by a sovereign element of T (see Section 7.4). In this case, we recover the usual relations between the Drinfeld element and the twist.

Theorem 8.14. *Let T be a quasitriangular Hopf monad on a sovereign category $\mathcal{C}$. Let u be the Drinfeld element of T. Then the map $\theta \mapsto G = u * \theta$ defines a bijection between twists of T and sovereign elements of T. In this correspondence, a twist θ is self-dual (and so T is ribbon) if and only if the sovereign element $G = u * \theta$ satisfies $S(u) = G^{*-1} * u * G^{*-1}$.*

Proof. Let ϕ be the sovereign structure on $\mathcal{C}$, $\mathcal{U}$ be the Drinfeld isomorphism of T-$\mathcal{C}$ (see Section 8.1), $?^\sharp : \mathrm{HOM}(1_{\mathcal{C}}, T) \to \mathrm{HOM}(U_T, U_T)$ be the canonical bijection of Lemma 1.3, and $?^\flat$ be the inverse of $?^\sharp$. Recall that $U_T(\mathcal{U}) = \phi u^\sharp$ by Lemma 8.10. By Proposition 8.3, the assignment $\Theta \mapsto \mathcal{U}\Theta$ defines a bijection between twists on T-$\mathcal{C}$ and sovereign structures on T-$\mathcal{C}$. By Theorem 8.13, twists Θ on T-$\mathcal{C}$ are in bijection with twists θ for T. By Proposition 7.6, sovereign structures on T-$\mathcal{C}$ are in bijection with sovereign elements G of T. Hence a bijection between twists θ for T and sovereign elements G of T, which is given by:

$$\theta \mapsto G = \left(\phi^{-1} U_T(\mathcal{U}\Theta)\right)^\flat = \left(u^\sharp \theta^\sharp\right)^\flat = u * \theta.$$

Via this correspondence, we have $S(\theta) = \theta$ if and only if $S(u^{*-1} * G) = u^{*-1} * G$ or, equivalently (see Lemmas 3.18 and 3.21), $S(u) = G^{*-1} * u * G^{*-1}$. $\square$

Corollary 8.15. *Let T be a ribbon Hopf monad on a sovereign category $\mathcal{C}$, with twist θ and Drinfeld element u. Then $\theta^{*-2} = u * S(u) = S(u) * u$.*

Proof. Since $G = u * \theta$ is grouplike by Theorem 8.14, we have $G^{*-1} = S(G)$ by Lemma 3.21. Now $S(G) = S(u * \theta) = S(\theta) * S(u) = \theta * S(u)$ by Lemma 3.18 and (62). Therefore $\theta^{*-1} * u^{*-1} = \theta * S(u)$ and so $\theta^{*-2} = S(u) * u$. Likewise, since we also have $G = \theta * u$ (because θ is central), we get $\theta^{*-2} = u * S(u)$. $\square$

Corollary 8.16. *Let T be a quasitriangular Hopf monad on a sovereign category $\mathcal{C}$. Let u be the Drinfeld element of T. Suppose that T is involutory (see Section 7.5). Then u^{*-1} is a twist for T, which is self-dual if and only if $S(u) = u$.*

Proof. Results directly from Theorem 8.14 since, when T is involutory, the unit η of T is a sovereign element of T (by Proposition 7.7). $\square$

9. Examples and applications

In this section, we give other examples of Hopf monads, so as to illustrate the generality of the notion.

9.1. Tannaka reconstruction

Fiber functors are an interesting source of examples of Hopf monads.

Let $\Bbbk$ be a field. Given a $\Bbbk$-algebra B, we denote ${}_B\mathrm{Mod}_B$ the category of B-bimodules, ${}_B\mathrm{mod}_B$ the category of finitely generated B-bimodules, ${}_B\mathrm{proj}_B$ the category of finitely generated projective B-bimodules, and ${}_B\mathrm{mod}$ the category of finitely generated left B-modules.

A *tensor category over* $\Bbbk$ is an autonomous category endowed with a structure of $\Bbbk$-linear abelian category such that $\otimes$ is bilinear and $\mathrm{End}(\mathbb{1}) = \Bbbk$. Let $\mathcal{C}$ be a tensor category over $\Bbbk$ and B be a $\Bbbk$-algebra. A *B-fiber functor for* $\mathcal{C}$ is a $\Bbbk$-linear exact strong monoidal functor $\mathcal{C} \to {}_B\mathrm{Mod}_B$. A B-fiber functor takes values in ${}_B\mathrm{proj}_B$ (because it preserves duality) and it is faithful if B is non-trivial (that is, $B \neq 0$).

We say that a tensor category $\mathcal{C}$ is *bounded* if there exists a k-linear equivalence of categories $\Xi : \mathcal{C} \to {}_E\mathrm{mod}$ for some finite-dimensional $\Bbbk$-algebra E. By Proposition 2.14 of [5] due to O. Gabber, this is equivalent to the more intrinsic following conditions:

- in $\mathcal{C}$, all objects have finite length and Hom spaces are finite-dimensional;
- $\mathcal{C}$ admits a generator, i.e., an object X such that any object is a subquotient of $X^{\oplus n}$ for some integer n.

Theorem 9.1. *Let $\mathcal{C}$ be a bounded tensor category over a field $\Bbbk$, B be a non-trivial finite-dimensional $\Bbbk$-algebra, and ω be a B-fiber functor for $\mathcal{C}$. Then the functor ω, viewed as a functor $\mathcal{C} \to {}_B\mathrm{mod}_B$, admits a left adjoint F. The endofunctor $T = \omega F$ is a bimonad on ${}_B\mathrm{mod}_B$ and induces, by restriction, a Hopf monad T_0 on ${}_B\mathrm{proj}_B$. The categories T_0-${}_B\mathrm{proj}_B$ and T-${}_B\mathrm{mod}_B$ are isomorphic (as $\Bbbk$-linear monoidal categories). Furthermore, the canonical functor $\mathcal{C} \to T\text{-}{}_B\mathrm{mod}_B \cong T_0\text{-}{}_B\mathrm{proj}_B$ is a $\Bbbk$-linear strong monoidal equivalence.*

Proof. Let R and E be two finite-dimensional $\Bbbk$-algebras. Let $G : {}_E\mathrm{mod} \to {}_R\mathrm{mod}$ be a $\Bbbk$-linear exact functor. Being right exact, G is of the form ${}_RM \otimes_E ?$ for some R–E-bimodule M. Since G is left exact, M is flat, and it is also of finite type, hence projective. Let ${}^\vee M$ be the E–R-bimodule $\mathrm{Hom}_R({}_RM_E, {}_RR_R)$. Then $F = {}_E{}^\vee M \otimes_R ?$ is left adjoint to G. So $T = GF$ is a monad on ${}_R\mathrm{mod}$. Moreover, the canonical functor $K : {}_E\mathrm{mod} \to T\text{-}{}_R\mathrm{mod}$ is an equivalence if G is faithful (that is, if M is faithfully flat as an E-module).

Via a $\Bbbk$-linear equivalence $\Xi : \mathcal{C} \to {}_E\mathrm{mod}$, this applies to $\omega : \mathcal{C} \to {}_B\mathrm{mod}_B$ (with $R = B \otimes_\Bbbk B^{\mathrm{op}}$), and shows that ω has a left adjoint F. Hence $T = \omega F$ is a bimonad on ${}_B\mathrm{mod}_B$ by Theorem 3.14. The canonical functor $K : \mathcal{C} \to T\text{-}{}_B\mathrm{mod}_B$ is a $\Bbbk$-linear strong monoidal equivalence. Now ω, and so T, takes values in ${}_B\mathrm{proj}_B$. Denote $\omega_0 : \mathcal{C} \to {}_B\mathrm{proj}_B$ and $F_0 : {}_B\mathrm{proj}_B \to \mathcal{C}$ the restrictions of ω and F. Then F_0 is left adjoint to ω_0 and $T_0 = F_0\omega_0$ is a Hopf monad on ${}_B\mathrm{proj}_B$ (by Theorem 3.14). Lastly, consider a T-module (N, r), where N is an object of ${}_B\mathrm{mod}_B$. Since K is an equivalence, (N, r) is isomorphic to $K(Y)$, for some Y in $\mathcal{C}$, and so $N \simeq \omega(Y)$. In particular N is in ${}_B\mathrm{proj}_B$. Therefore we have $T\text{-}{}_B\mathrm{mod}_B = T_0\text{-}{}_B\mathrm{proj}_B$, hence the theorem. □

Corollary 9.2. *Let $\mathcal{C}$ be a semisimple tensor category over a field $\Bbbk$. Assume the set Λ of isomorphy classes of simple objects of $\mathcal{C}$ is finite and* $\mathrm{End}(V) = \Bbbk$ *for each simple object V of $\mathcal{C}$. Let B be the $\Bbbk$-algebra $\Bbbk^\Lambda$. Then there exist a $\Bbbk$-linear Hopf monad T on ${}_B\mathrm{mod}_B$ and a $\Bbbk$-linear strong monoidal equivalence $\mathcal{C} \to T\text{-}{}_B\mathrm{mod}_B$.*

Proof. By Theorem 4.1 of [6], we have a canonical B-fiber functor $\mathcal{C} \to {}_B\mathrm{mod}_B$. Hence the corollary by Theorem 9.1, noticing that ${}_B\mathrm{mod}_B = {}_B\mathrm{proj}_B$ because B is semisimple. □

9.2. *Double of Hopf monads*

Let $\mathcal{C}$ be an autonomous category and T be a Hopf monad on $\mathcal{C}$. Assume that the coend

$$Z_T(X) = \int^{Y \in \mathcal{C}} {}^{\vee}T(Y) \otimes X \otimes Y$$

exists for every object X of $\mathcal{C}$. Then Z_T is a Hopf monad on $\mathcal{C}$ and the composition $D_T = Z_T \circ T$ has an explicit structure of a quasitriangular Hopf monad such that

$$\mathcal{Z}(T\text{-}\mathcal{C}) \cong D_T\text{-}\mathcal{C}$$

as braided categories, where $\mathcal{Z}(T\text{-}\mathcal{C})$ is the center of $T\text{-}\mathcal{C}$. The Hopf monad D_T is called the *double* of T. See [3] for details.

References

[1] Y. Bespalov, T. Kerler, V. Lyubashenko, V. Turaev, Integrals for braided Hopf algebras, J. Pure Appl. Algebra 148 (2) (2000) 113–164.

[2] A. Bruguières, A. Virelizier, Hopf diagrams and quantum invariants, Algebr. Geom. Topol. 5 (2005) 1677–1710, (electronic).

[3] A. Bruguières, A. Virelizier, The double of a Hopf monad, 2007, in preparation.

[4] J. Barrett, B. Westbury, Invariants of piecewise-linear 3-manifolds, Trans. Amer. Math. Soc. 348 (10) (1996) 3997–4022.

[5] P. Deligne, Catégories tannakiennes, in: Grothendieck Festschrift, vol. II, in: Progr. Math., vol. 87, Birkhäuser, Boston, 1990, pp. 111–195.

[6] T. Hayashi, A canonical Tannaka duality for finite semisimple tensor categories, arXiv: math.QA/9904073, 2003.

[7] T. Kerler, V. Lyubashenko, Non-Semisimple Topological Quantum Field Theories for 3-Manifolds with Corners, Lecture Notes in Math., vol. 1765, Springer-Verlag, Berlin, 2001.

[8] V. Lyubashenko, Invariants of 3-manifolds and projective representations of mapping class groups via quantum groups at roots of unity, Comm. Math. Phys. 172 (3) (1995) 467–516.

[9] S. Mac Lane, Categories for the Working Mathematician, second ed., Springer-Verlag, New York, 1998.

[10] P. McCrudden, Opmonoidal monads, Theory Appl. Categ. 10 (19) (2002) 469–485 (electronic).

[11] I. Moerdijk, Monads on tensor categories, J. Pure Appl. Algebra 168 (2–3) (2002) 189–208.

[12] N. Reshetikhin, V. Turaev, Invariants of 3-manifolds via link polynomials and quantum groups, Invent. Math. 103 (3) (1991) 547–597.

[13] K. Szlachányi, Adjointable monoidal functors and quantum groupoids, in: Hopf Algebras in Noncommutative Geometry and Physics, in: Lect. Notes Pure Appl. Math., vol. 239, 2005, pp. 291–307.

[14] V. Turaev, Quantum Invariants of Knots and 3-Manifolds, Walter de Gruyter & Co., Berlin, 1994.

[15] V. Turaev, O. Viro, State sum invariants of 3-manifolds and quantum $6j$-symbols, Topology 31 (4) (1992) 865–902.

[16] A. Virelizier, Kirby elements and quantum invariants, Proc. London Math. Soc. 93 (2) (2006) 474–514.