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KIRBY ELEMENTS AND QUANTUM INVARIANTS

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Introduction

During the last decade, deep connections between low-dimensional topology and the purely algebraic theory of quantum groups or, more generally, of braided categories were highlighted. In particular, this led to a new class of 3-manifold invariants, called *quantum invariants*, defined in several ways.

The aim of the present paper is to give an, as general as possible, method of constructing quantum invariants of 3-manifolds starting from a ribbon category or a ribbon Hopf algebra. With this formalism, we recover the 3-manifold invariants of Reshetikhin and Turaev [**24**, **26**], of Hennings, Kauffman and Radford [**9**, **10**], and of Lyubashenko [**16**], when these are well defined.

Let $\Bbbk$ be a field and $\mathcal{C}$ be a $\Bbbk$-linear ribbon category (not necessarily semisimple). Under some technical assumption, namely the existence of a coend $A \in \mathrm{Ob}(\mathcal{C})$ of the functor $(X, Y) \in \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \mapsto X^* \otimes Y \in \mathcal{C}$, a scalar $\tau_{\mathcal{C}}(L; \alpha)$ can be associated to any framed link L in S^3 and any morphism $\alpha \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$; see [**16**]. Recall (see [**17**]) that the object A of $\mathcal{C}$ is then a Hopf algebra in the category $\mathcal{C}$.

By a *Kirby element* of $\mathcal{C}$, we shall mean a morphism $\alpha \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$ such that $\tau_{\mathcal{C}}(L; \alpha)$ is invariant under isotopies of L and under 2-handle slides. By the Kirby theorem [**12**], we get that if a Kirby element α of $\mathcal{C}$ is *normalizable*, that is, such that $\tau_{\mathcal{C}}(\bigcirc^{\pm 1}; \alpha) \neq 0$, then $\tau_{\mathcal{C}}(L; \alpha)$ can be normalized to an invariant $\tau_{\mathcal{C}}(M_L; \alpha)$ of 3-manifolds. Here $\bigcirc^{\pm 1}$ is the unknot with framing ± 1 and M_L denotes the 3-manifold obtained from S^3 by surgery along L.

In general, determining all the Kirby elements of $\mathcal{C}$ is quite a difficult problem. In this paper we characterize a class $\mathcal{AK}(\mathcal{C})$ of Kirby elements of $\mathcal{C}$, called the *algebraic Kirby elements of* $\mathcal{C}$, in terms of the structure maps of the categorical Hopf algebra A. This class is sufficiently large to contain the Kirby elements corresponding to the known quantum invariants.

If the categorical Hopf algebra A admits a two-sided integral $\lambda\colon \mathbb{1} \to A$, then λ is an algebraic Kirby element of $\mathcal{C}$ and the corresponding invariant $\tau_{\mathcal{C}}(M; \lambda)$ is the Lyubashenko invariant [**16**].

When $\mathcal{C}$ is semisimple, we give sufficient conditions for being an algebraic Kirby element of $\mathcal{C}$. Moreover, we show that there exist (even in the non-modular case) algebraic Kirby elements of $\mathcal{C}$ corresponding to the Reshetikhin–Turaev invariants [**24**, **26**] computed from finitely semisimple ribbon full subcategories of $\mathcal{C}$. Note that these elements are not in general two-sided integrals.

More generally, when $\mathcal{C}$ is not semisimple, we show that $\mathcal{AK}(\mathcal{C})$ contains Kirby elements leading to the invariants defined from $\mathcal{AK}(\mathcal{B})$, where $\mathcal{B}$ is any finitely semisimple ribbon subcategory of the semisimple quotient of $\mathcal{C}$. Note that, in

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general, there exist algebraic Kirby elements of $\mathcal{C}$ which are not of this last form. This means that the semisimplification process 'misses' some invariants.

Let H be a finite-dimensional ribbon Hopf algebra. Suppose that $\mathcal{C}$ is the category rep_H of finite-dimensional left H-modules. We parameterize the algebraic Kirby elements of rep_H by a subset $\mathcal{AK}(H)$ of H defined in purely algebraic terms. One of the interests of such a description of $\mathcal{AK}(\mathrm{rep}_H)$ is to avoid the representation theory of H (which may be of wild type; see [**2**]).

If H is unimodular, then $1 \in \mathcal{AK}(H)$ and the corresponding invariant $\tau_{(H,1)}$ is the Hennings–Kauffman–Radford invariant [**9**, **10**]. More generally, and even if H is not unimodular, we show that the invariant $\tau_{(H,z)}$ of 3-manifolds corresponding to $z \in \mathcal{AK}(H)$ can be computed by using the Kauffman–Radford algorithm.

If $\mathcal{V}$ is a set of simple left H-modules which makes $(H, \mathcal{V})$ a premodular Hopf algebra, then there exists $z_{\mathcal{V}} \in \mathcal{AK}(H)$ such that $\tau_{(H,z_{\mathcal{V}})}$ is the Reshetikhin–Turaev invariant computed from $(H, \mathcal{V})$, which hence can be computed by using the Kauffman–Radford algorithm.

When H is semisimple and $\Bbbk$ is of characteristic 0, we show that the Hennings–Kauffman–Radford invariant (computed from H) and the Reshetikhin–Turaev invariant (computed from rep_H) are simultaneously well defined and coincide (even in the non-modular case). In the modular case, this was first shown in [**11**].

We explicitly determine the algebraic Kirby elements of a family of non-unimodular ribbon Hopf algebras which contains Sweedler's Hopf algebra.

As an algebraic application, the operators involved in the description of $\mathcal{AK}(\mathrm{rep}_H)$ in algebraic terms allows us to parameterize all the traces on a finite-dimensional ribbon Hopf algebra H. When H is unimodular, we recover the parameterization given in [**9**, **23**].

The paper is organized as follows. In § 1, we review ribbon categories and coends. In § 2, we define and study Kirby elements. We focus, in § 3, on the case of semisimple ribbon categories and, in § 4, on the case of categories of representations of ribbon Hopf algebras. In § 5, we treat an example in detail. Finally, in the Appendix, we study traces on ribbon Hopf algebras.

1. *Ribbon categories and coends*

In this section, we review some basic definitions concerning ribbon categories and coends. Throughout this paper, we let $\Bbbk$ be a field.

1.1. *Ribbon categories*

Let $\mathcal{C}$ be a strict monoidal category with unit object $\mathbb{1}$ (note that every monoidal category is equivalent to a strict monoidal category in a canonical way; see [**18**]). A *left duality* in $\mathcal{C}$ associates to any object $U \in \mathcal{C}$ an object $U^* \in \mathcal{C}$ and two morphisms $\mathrm{ev}_U \colon U^* \otimes U \to \mathbb{1}$ and $\mathrm{coev}_U \colon \mathbb{1} \to U \otimes U^*$ such that

$$(\mathrm{id}_U \otimes \mathrm{ev}_U)(\mathrm{coev}_U \otimes \mathrm{id}_U) = \mathrm{id}_U \quad \text{and} \quad (\mathrm{ev}_U \otimes \mathrm{id}_{U^*})(\mathrm{id}_{U^*} \otimes \mathrm{coev}_U) = \mathrm{id}_{U^*}.$$

We can (and we always do) impose the following conditions:

$$\mathbb{1}^* = \mathbb{1}, \quad \mathrm{ev}_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}} \quad \text{and} \quad \mathrm{coev}_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}.$$

By a *braided category* we shall mean a monoidal category $\mathcal{C}$ with left duality and endowed with a *braiding*, that is, a system $\{c_{U,V} \colon U \otimes V \to V \otimes U\}_{U,V \in \mathcal{C}}$ of isomorphisms, natural in U and V, satisfying

$$c_{U\otimes V,W} = (c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W}), \tag{1.1}$$

$$c_{U,V\otimes W} = (\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W), \tag{1.2}$$

for all objects $U, V, W \in \mathcal{C}$. Note that (1.1) and (1.2) imply that $c_{U,\mathbb{1}} = c_{\mathbb{1},U} = \mathrm{id}_U$.

A *ribbon category* is a braided category $\mathcal{C}$ endowed with a *twist*, that is, a family of natural isomorphisms $\{\theta_U \colon U \to U\}_{U\in\mathcal{C}}$ satisfying

$$(\theta_U \otimes \mathrm{id}_{U^*})\mathrm{coev}_U = (\mathrm{id}_U \otimes \theta_{U^*})\mathrm{coev}_U, \tag{1.3}$$

$$\theta_{U\otimes V} = c_{V,U}\, c_{U,V}(\theta_U \otimes \theta_V), \tag{1.4}$$

for all objects $U, V \in \mathcal{C}$. It follows from (1.4) that $\theta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$.

A ribbon category $\mathcal{C}$ canonically has a right duality by associating to any object $U \in \mathcal{C}$ its left dual $U^* \in \mathcal{C}$ and two morphisms

$$\widetilde{\mathrm{ev}}_U \colon U \otimes U^* \to \mathbb{1} \quad \text{and} \quad \widetilde{\mathrm{coev}}_U \colon \mathbb{1} \to U^* \otimes U$$

defined by

$$\widetilde{\mathrm{ev}}_U = \mathrm{ev}_U c_{U,U^*}(\theta_U \otimes \mathrm{id}_{U^*}) \quad \text{and} \quad \widetilde{\mathrm{coev}}_U = (\mathrm{id}_{U^*} \otimes \theta_U^{-1})(c_{U^*,U})^{-1}\mathrm{coev}_U.$$

Note that we have $\widetilde{\mathrm{ev}}_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$ and $\widetilde{\mathrm{coev}}_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$.

The *dual* morphism $f^* \colon V^* \to U^*$ of a morphism $f \colon U \to V$ in a ribbon category $\mathcal{C}$ is defined by

$$\begin{aligned} f^* &= (\mathrm{ev}_V \otimes \mathrm{id}_{U^*})(\mathrm{id}_{V^*} \otimes f \otimes \mathrm{id}_{U^*})(\mathrm{id}_{V^*} \otimes \mathrm{coev}_U) \\ &= (\mathrm{id}_{U^*} \otimes \widetilde{\mathrm{ev}}_V)(\mathrm{id}_{U^*} \otimes f \otimes \mathrm{id}_{V^*})(\widetilde{\mathrm{coev}}_U \otimes \mathrm{id}_{V^*}). \end{aligned}$$

It is well known that $(\mathrm{id}_U)^* = \mathrm{id}_{U^*}$ and $(fg)^* = g^* f^*$ for composable morphisms f and g. Axiom (1.3) can be shown to be equivalent to $\theta_U^* = \theta_{U^*}$.

Let $\mathcal{C}$ be a ribbon category. Note that $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$ is a monoid, with composition as multiplication, which is commutative. The *quantum trace* of an endomorphism $f \colon U \to U$ of an object $U \in \mathcal{C}$ is defined by

$$\mathrm{tr}_q(f) = \widetilde{\mathrm{ev}}_U(f \otimes \mathrm{id}_{U^*})\mathrm{coev}_U = \mathrm{ev}_U(\mathrm{id}_{U^*} \otimes f)\widetilde{\mathrm{coev}}_U \in \mathrm{End}_{\mathcal{C}}(\mathbb{1}).$$

For any morphisms $u \colon U \to V$ and $v \colon V \to U$ and any endomorphisms f and g, we have

$$\mathrm{tr}_q(uv) = \mathrm{tr}_q(vu), \quad \mathrm{tr}_q(f^*) = \mathrm{tr}_q(f), \quad \text{and} \quad \mathrm{tr}_q(f \otimes g) = \mathrm{tr}_q(f)\,\mathrm{tr}_q(g).$$

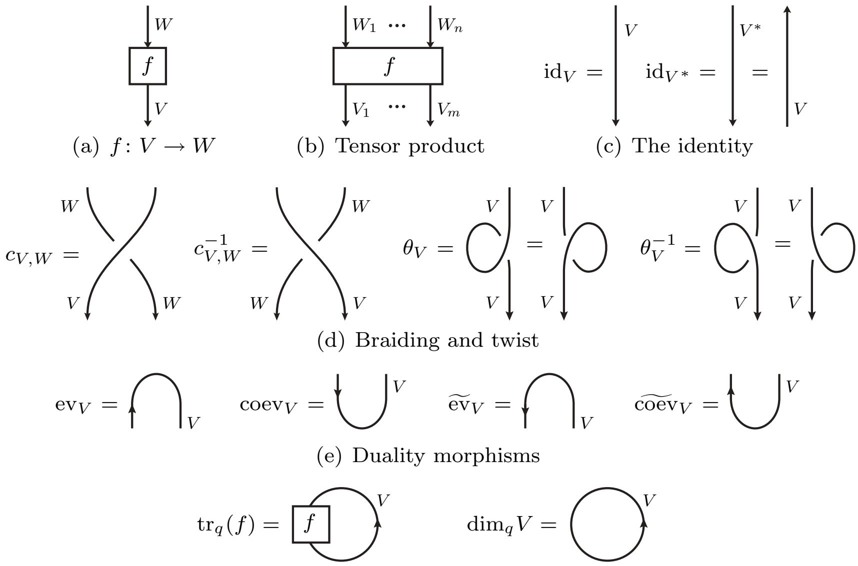
The *quantum dimension* of an object $U \in \mathcal{C}$ is defined by

$$\dim_q(U) = \mathrm{tr}_q(\mathrm{id}_U) = \widetilde{\mathrm{ev}}_U\, \mathrm{coev}_U = \mathrm{ev}_U\, \widetilde{\mathrm{coev}}_U \in \mathrm{End}_{\mathcal{C}}(\mathbb{1}).$$

Isomorphic objects have equal dimensions and $\dim_q(U \otimes V) = \dim_q(U)\dim_q(V)$ for any objects $U, V \in \mathcal{C}$. Note that $\dim_q(\mathbb{1}) = \mathrm{id}_{\mathbb{1}}$.

1.2. $\Bbbk$-*categories*

Let $\Bbbk$ be a field. By a $\Bbbk$-*category*, we shall mean a category for which the sets of morphisms are $\Bbbk$-spaces and the composition is $\Bbbk$-bilinear. By a *monoidal* $\Bbbk$-*category*, we shall mean a $\Bbbk$-category endowed with a monoidal structure whose tensor product is $\Bbbk$-bilinear. Note that if $\mathcal{C}$ is a monoidal $\Bbbk$-category, then $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$ is a commutative $\Bbbk$-algebra (with composition as multiplication). A monoidal $\Bbbk$-category is said to be *pure* if $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \Bbbk$.



(a) $f\colon V \to W$ (b) Tensor product (c) The identity

(d) Braiding and twist

(e) Duality morphisms

(f) Trace and dimension

FIGURE 1. *Plane diagrams of morphisms.*

By a *ribbon* $\Bbbk$*-category*, we shall mean a pure monoidal $\Bbbk$-category endowed with a ribbon structure.

1.3. *Graphical calculus*

Let $\mathcal{C}$ be a ribbon category. Any morphism in $\mathcal{C}$ can be graphically represented by a plane diagram (we use the conventions of [**26**]). This pictorial calculus will allow us to replace algebraic arguments involving commutative diagrams by simple geometric reasoning. This is justified in, for example, [**26**].

A morphism $f\colon V \to W$ in $\mathcal{C}$ is represented by a box with two vertical arrows oriented downwards, as in Figure 1(a). Here V and W should be regarded as 'colors' of the arrows and f should be regarded as a 'color' of the box. More generally, a morphism $f\colon V_1 \otimes \ldots \otimes V_m \to W_1 \otimes \ldots \otimes W_n$ may be represented as in Figure 1(b).

We also use vertical arrows oriented upwards under the convention that the morphism sitting in a box attached to such an arrow involves not the color of the arrow but rather the dual object. The identity endomorphism of an object $V \in \mathcal{C}$ or of its dual V^* will be represented by a vertical arrow as depicted in Figure 1(c). Note that a vertical arrow colored with $\mathbb{1}$ may be deleted from any picture without changing the morphism represented by this picture. The symbol '=' displayed in the figures denotes equality of the corresponding morphisms in $\mathcal{C}$.

The tensor product $f \otimes g$ of two morphisms f and g in $\mathcal{C}$ is represented by placing a picture of f to the left of a picture of g. A picture for the composition $g \circ f$ of two (composable) morphisms g and f is obtained by putting a picture of g on the top of a picture of f and by gluing the corresponding free ends of arrows.

The braiding $c_{V,W}\colon V \otimes W \to W \otimes V$ and its inverse $c^{-1}_{V,W}\colon W \otimes V \to V \otimes W$, the twist $\theta_V\colon V \to V$ and its inverse $\theta_V^{-1}\colon V \to V$, and the duality morphisms $\mathrm{ev}_V\colon V^* \otimes V \to \mathbb{1}$, $\mathrm{coev}_V\colon \mathbb{1} \to V \otimes V^*$, $\widetilde{\mathrm{ev}}_V\colon V \otimes V^* \to \mathbb{1}$, and $\widetilde{\mathrm{coev}}_V\colon \mathbb{1} \to V^* \otimes V$

are represented as in Figures 1(d) and 1(e). The quantum trace of an endomorphism $f\colon V \to V$ in $\mathcal{C}$ and the quantum dimension of an object $V \in \mathcal{C}$ may be depicted as in Figure 1(f).

1.4. *Negligible morphisms*

Let $\mathcal{C}$ be a ribbon $\Bbbk$-category. A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ is said to be *negligible* if $\mathrm{tr}_q(gf) = 0$ for all $g \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$. Denote by $\mathrm{Negl}_{\mathcal{C}}(X,Y)$ the $\Bbbk$-subspace of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ formed by the negligible morphisms.

It is important to note that $\mathrm{Negl}_{\mathcal{C}}$ is a two-sided $\otimes$-ideal of $\mathcal{C}$. This means that the composition (respectively the tensor product) of a negligible morphism with any other morphism is negligible.

Note that a morphism $f\colon \mathbb{1} \to X$ is negligible if and only if $gf = 0$ for all morphisms $g\colon X \to \mathbb{1}$.

1.5. *Dinatural transformations and coends*

Recall that to each category $\mathcal{C}$ is associated its *opposite category* $\mathcal{C}^{\mathrm{op}}$ (by reversing the arrows; see [**18**]).

Let $\mathcal{C}$ and $\mathcal{B}$ be two categories. A *dinatural transformation* between a functor $F\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{B}$ and an object $B \in \mathcal{B}$ is a function d which assigns to each object $X \in \mathcal{C}$ a morphism $d_X\colon F(X,X) \to B$ of $\mathcal{B}$ in such a way that the diagram

$$\begin{tikzcd} F(Y,X) \arrow[r, "F(\mathrm{id}_Y, f)"] \arrow[d, "F(f,\mathrm{id}_X)"'] & F(Y,Y) \arrow[d, "d_Y"] \\ F(X,X) \arrow[r, "d_X"'] & B \end{tikzcd}$$

commutes for every morphism $f\colon X \to Y$ in $\mathcal{C}$.

A *coend* of the functor F is a pair (A,i) consisting of an object A of $\mathcal{B}$ and a dinatural transformation i from F to A which is *universal* among the dinatural transformations from F to a constant, that is, with the property that, to every dinatural transformation d from F to B, there exists a unique morphism $r\colon A \to B$ such that $d_X = r \circ i_X$ for all objects $X \in \mathcal{C}$.

Note that a coend, if it exists, is unique up to unique isomorphism.

For examples of coends, see §§ 3.2 and 4.5.

1.6. *Categorical Hopf algebras from coends*

Let $\mathcal{C}$ be a ribbon category. Consider the functor $F\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ defined by

$$F(X,Y) = X^* \otimes Y \quad \text{and} \quad F(f,g) = f^* \otimes g \tag{1.5}$$

for all objects $X, Y \in \mathcal{C}$ and all morphisms f and g in $\mathcal{C}$.

Suppose that the functor F admits a coend (A,i). Then the object A has a structure of a Hopf algebra in the category $\mathcal{C}$ (see [**17**]). This means that there exist morphisms $m_A\colon A \otimes A \to A$, $\eta_A\colon \mathbb{1} \to A$, $\Delta_A\colon A \to A \otimes A$, $\varepsilon_A\colon A \to \mathbb{1}$, and $S_A\colon A \to A$, which satisfy the same axioms as those of a Hopf algebra except that the usual flip is replaced by the braiding $c_{A,A}\colon A \otimes A \to A \otimes A$. By using the factorization property of the coend (use it twice for the multiplication m_A), we define these structural morphisms as follows:

$$\Delta_A i_X = (i_X \otimes i_X)(\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X)\colon X^* \otimes X \to A \otimes A,$$
$$\varepsilon_A i_X = \mathrm{ev}_X\colon X^* \otimes X \to A \otimes A,$$
$$m_A(i_X \otimes i_Y) = i_{Y\otimes X}(\gamma_{X,Y} \otimes \mathrm{id}_{Y\otimes X})(\mathrm{id}_{X^*} \otimes c_{X,Y^*\otimes Y})\colon X^* \otimes X \otimes Y^* \otimes Y \to A,$$
$$\eta_A = i_{\mathbb{1}}\colon \mathbb{1} = \mathbb{1}^* \otimes \mathbb{1} \to A,$$
$$S_A i_X = (\mathrm{ev}_X \otimes i_{X^*})(\mathrm{id}_{X^*} \otimes c_{X^{**},X} \otimes \mathrm{id}_{X^*})(\mathrm{coev}_{X^*} \otimes c_{X^*,X})\colon X^* \otimes X \to A.$$

Here X and Y are objects of $\mathcal{C}$, and $\gamma_{X,Y}\colon X^*\otimes Y^* \xrightarrow{\simeq} (Y\otimes X)^*$ is the isomorphism defined by $\gamma_{X,Y} = (\mathrm{ev}_X(\mathrm{id}_{X^*} \otimes \mathrm{ev}_Y \otimes \mathrm{id}_X) \otimes \mathrm{id}_{(Y\otimes X)^*})(\mathrm{id}_{X^*\otimes Y^*} \otimes \mathrm{coev}_{Y\otimes X})$.

It can be shown that the antipode S_A is an isomorphism and that $S_A^2 = \theta_A$; see [**17**]. Moreover, as for Hopf algebras, the antipode is anti-(co)multiplicative:

$$S_A m_A = m_A(S_A \otimes S_A)c_{A,A}, \qquad S_A\eta_A = \eta_A,$$
$$\Delta_A S_A = c_{A,A}(S_A \otimes S_A)\Delta_A, \qquad \varepsilon_A S_A = \varepsilon_A.$$

The Hopf algebra A is equipped with a Hopf pairing $\omega_A\colon A \otimes A \to \mathbb{1}$ (see [**16**]) defined by $\omega_A \circ (i_X \otimes i_Y) = \omega_{X,Y}$, where

$$\omega_{X,Y} = (\mathrm{ev}_X \otimes \mathrm{ev}_Y)(\mathrm{id}_{X^*} \otimes c_{Y^*,X}c_{X,Y^*} \otimes \mathrm{id}_Y)\colon X^* \otimes X \otimes Y^* \otimes Y \to A.$$

This pairing is said to be *non-degenerate* if $(\omega_A \otimes \mathrm{id}_{A^*})(\mathrm{id}_A \otimes \mathrm{coev}_A)\colon A \to A^*$ and $(\mathrm{id}_{A^*} \otimes \omega_A)(\widetilde{\mathrm{coev}}_A \otimes \mathrm{id}_A)\colon A \to A^*$ are isomorphisms.

Set

$$\Gamma_l = (\mathrm{id}_A \otimes m_A)(\Delta_A \otimes \mathrm{id}_A)\colon A \otimes A \to A \otimes A, \tag{1.6}$$
$$\Gamma_r = (m_A \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \Delta_A)\colon A \otimes A \to A \otimes A. \tag{1.7}$$

LEMMA 1.1. $\Gamma_l(S_A \otimes S_A)c_{A,A} = c_{A,A}(S_A \otimes S_A)\Gamma_r$.

Proof. By using the anti-(co)multiplicativity of the antipode, we have

$$\begin{aligned}\Gamma_l(S_A \otimes S_A)c_{A,A} &= (\mathrm{id}_A \otimes m_A)(\Delta_A S_A \otimes S_A)c_{A,A}\\ &= (\mathrm{id}_A \otimes m_A)(c_{A,A}(S_A \otimes S_A)\Delta_A \otimes S_A)c_{A,A}\\ &= (S_A \otimes m_A(S_A \otimes S_A))(c_{A,A}\Delta_A \otimes \mathrm{id}_A)c_{A,A}\\ &= (S_A \otimes S_A m_A c^{-1}_{A,A})(c_{A,A}\Delta_A \otimes \mathrm{id}_A)c_{A,A}.\end{aligned}$$

Then, by using (1.1) and (1.2), we get

$$\begin{aligned}\Gamma_l(S_A \otimes S_A)c_{A,A} &= (S_A \otimes S_A m_A c^{-1}_{A,A})c_{A,A\otimes A}(\mathrm{id}_A \otimes c_{A,A}\Delta_A)\\ &= (S_A \otimes S_A m_A)(c_{A,A} \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes c_{A,A})(\mathrm{id}_A \otimes \Delta_A)\\ &= (S_A \otimes S_A m_A)c_{A\otimes A,A}(\mathrm{id}_A \otimes \Delta_A)\\ &= c_{A,A}(S_A \otimes S_A)(m_A \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \Delta_A)\\ &= c_{A,A}(S_A \otimes S_A)\Gamma_r.\end{aligned}$$

□

COROLLARY 1.2. *Suppose that $\mathcal{C}$ is moreover a (monoidal) $\Bbbk$-category. Let $\alpha \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$. If $S_A\alpha - \alpha \in \mathrm{Negl}_{\mathcal{C}}(\mathbb{1}, A)$, then the following assertions are equivalent:*

(a) *$\Gamma_l(\alpha \otimes \alpha) - \alpha \otimes \alpha\colon \mathbb{1} \to A \otimes A$ is negligible;*

(b) *$\Gamma_r(\alpha \otimes \alpha) - \alpha \otimes \alpha\colon \mathbb{1} \to A \otimes A$ is negligible.*

Moreover, if $S_A\alpha = \alpha$, then $\Gamma_l(\alpha \otimes \alpha) = \alpha \otimes \alpha$ if and only if $\Gamma_r(\alpha \otimes \alpha) = \alpha \otimes \alpha$.

Proof. This is an immediate consequence of Lemma 1.1 since $\mathrm{Negl}_{\mathcal{C}}$ is a two-sided $\otimes$-ideal of $\mathcal{C}$. □

2. *Kirby elements of a ribbon category*

In this section, we generalize Lyubashenko's method [**16**] of constructing 3-manifold invariants from ribbon categories.

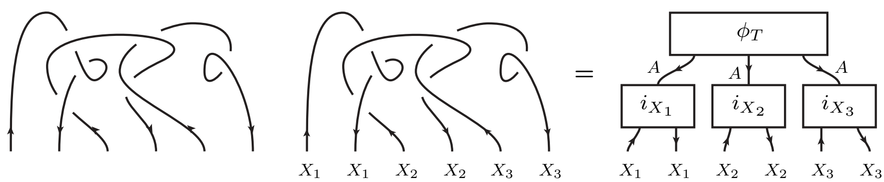
2.1. *Ribbon handles*

Let n be a positive integer. By a *ribbon n-handle* we shall mean an oriented ribbon tangle $T \subset \mathbb{R}^2 \times [0,1]$ with $2n$ bottom endpoints and no top endpoints, consisting of n arc components, without any closed component, such that:
– the kth arc joins the $(2k-1)$th and $2k$th bottom endpoints;
– the kth arc is oriented out of $\mathbb{R}^2 \times [0,1]$ near the $2k$th bottom endpoint.
Diagrams of ribbon handles are drawn with blackboard framing. An example of a ribbon 3-handle is depicted in Figure 2(a).

Let $\mathcal{C}$ be a ribbon category. Suppose that the functor (1.5) admits a coend (A, i). Let T be a ribbon n-handle. For objects $X_1, \dots, X_n \in \mathcal{C}$, let $T_{(X_1,\dots,X_n)}$ be the morphism $X_1^* \otimes X_1 \otimes \dots \otimes X_n^* \otimes X_n \to \mathbb{1}$ in $\mathcal{C}$ graphically represented by a diagram of T where the kth component of T has been colored with the object X_k. Since the braiding and twist of $\mathcal{C}$ are natural and by using the Fubini theorem for coends (see [**18**]), we see that there exists a (unique) morphism $\phi_T \colon A^{\otimes n} \to \mathbb{1}$ such that

$$T_{(X_1,\dots,X_n)} = \phi_T \circ (i_{X_1} \otimes \dots \otimes i_{X_n}) \tag{2.1}$$

for all objects $X_1, \dots, X_n \in \mathcal{C}$. Figure 2(b) is an example for $n = 3$.



(a) A ribbon 3-handle T (b) $T_{(X_1,X_2,X_3)} = \phi_T \circ (i_{X_1} \otimes i_{X_2} \otimes i_{X_3})$

FIGURE 2.

In [**6**], we give a method for computing ϕ_T by using the Hopf algebra A.

2.2. *Kirby elements*

Let $\mathcal{C}$ be a ribbon category such that the functor (1.5) admits a coend (A, i). Let L be a framed link in S^3 with n components. Fix an orientation for L. There always exists a (non-unique) ribbon n-handle T_L such that L is isotopic to $T_L \circ (\cup \otimes \dots \otimes \cup)$, where $\cup$ denote the cup with clockwise orientation; see Figure 3(a). For $\alpha \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$, set

$$\tau_{\mathcal{C}}(L; \alpha) = \phi_{T_L} \circ \alpha^{\otimes n} \in \mathrm{End}_{\mathcal{C}}(\mathbb{1}),$$

where $\phi_{T_L} : A^{\otimes n} \to \mathbb{1}$ is defined as in (2.1).

Definition 2.1. By a *Kirby element* of $\mathcal{C}$, we shall mean a morphism $\alpha \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$ such that, for any framed link L, $\tau_{\mathcal{C}}(L; \alpha)$ is well defined and invariant under isotopies and 2-handle slides of L. A Kirby element α of $\mathcal{C}$ is said to be *normalizable* if $\tau_{\mathcal{C}}(\bigcirc^{\pm 1}; \alpha)$ is invertible in $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$, where $\bigcirc^{\pm 1}$ denotes the unknot with framing ± 1.

Note that the unit $\eta_A \colon \mathbb{1} \to A$ of the categorical Hopf algebra A is a normalizable Kirby element. The invariant of framed links associated with η_A is the trivial one, that is, $\tau_{\mathcal{C}}(L; \eta_A) = 1$ for any framed link L.

In the following, we will denote by $\Theta_\pm \colon A \to \mathbb{1}$ the morphisms defined by

$$\Theta_\pm i_X = \mathrm{ev}_X(\mathrm{id}_{X^*} \otimes \theta_X^{\pm 1}). \tag{2.2}$$

Remark that if α is a Kirby element of $\mathcal{C}$, then $\tau_{\mathcal{C}}(\bigcirc^{\pm 1}; \alpha) = \Theta_\pm \alpha$.

Lemma 2.2. *Let $\alpha \colon \mathbb{1} \to A$. Then $\tau_{\mathcal{C}}(L \sqcup L'; \alpha) = \tau_{\mathcal{C}}(L; \alpha)\, \tau_{\mathcal{C}}(L'; \alpha)$ for any framed link L and L', where $L \sqcup L'$ denotes the disjoint union of L and L'.*

Proof. Let T_L and $T_{L'}$ be ribbon handles such that L and L' are isotopic to $T_L \circ (\cup \otimes \ldots \otimes \cup)$ and $T_{L'} \circ (\cup \otimes \ldots \otimes \cup)$ respectively. Then $T = T_L \otimes T_{L'}$ is a ribbon handle such that the disjoint union $L \sqcup L'$ is isotopic to $T \circ (\cup \otimes \ldots \otimes \cup)$. Therefore $\phi_T = \phi_{T_L} \otimes \phi_{T_{L'}}$ and so $\tau_{\mathcal{C}}(L \sqcup L'; \alpha) = \tau_{\mathcal{C}}(L; \alpha)\, \tau_{\mathcal{C}}(L'; \alpha)$. $\square$

In this paper, all considered 3-manifolds are supposed to be closed, connected, and oriented. Recall (see [**15**]) that every such 3-manifold can be obtained from S^3 by surgery along a framed link $L \subset S^3$. For any framed link L in S^3, we will denote by M_L the 3-manifold obtained from S^3 by surgery along L, by n_L the number of components of L, and by $b_-(L)$ the number of negative eigenvalues of the linking matrix of L.

Normalizable Kirby elements are of special interest due to the following result.

Proposition 2.3. *Let α be a normalizable Kirby element of $\mathcal{C}$. Then*

$$\tau_{\mathcal{C}}(M_L; \alpha) = (\Theta_+ \alpha)^{b_-(L) - n_L}\, (\Theta_- \alpha)^{-b_-(L)}\, \tau_{\mathcal{C}}(L; \alpha)$$

is an invariant of 3-manifolds. Moreover $\tau_{\mathcal{C}}(M \# M'; \alpha) = \tau_{\mathcal{C}}(M; \alpha)\, \tau_{\mathcal{C}}(M'; \alpha)$ for any 3-manifolds M and M'.

Remark 2.4. For any normalizable Kirby element α of $\mathcal{C}$, we have $\tau_{\mathcal{C}}(S^3; \alpha) = 1$ and $\tau_{\mathcal{C}}(S^1 \times S^2; \alpha) = (\Theta_+ \alpha)^{-1}\, \varepsilon_A \alpha$.

Remark 2.5. The invariant of 3-manifolds associated with the unit $\eta_A \colon \mathbb{1} \to A$ of the categorical Hopf algebra A (which is a normalizable Kirby element) is the trivial one, that is, $\tau_{\mathcal{C}}(M; \eta_A) = 1$ for any 3-manifold M.

Proof of Proposition 2.3. The fact that $\tau_{\mathcal{C}}(M_L; \alpha)$ is an invariant of 3-manifolds follows from the Kirby theorem [**12**]. Indeed $\tau_{\mathcal{C}}(L; \alpha)$, $b_-(L)$ and n_L are invariant under 2-handle slides and $\tau_{\mathcal{C}}(\bigcirc^{\pm 1} \sqcup L; \alpha) = (\Theta_\pm \alpha)\, \tau_{\mathcal{C}}(L; \alpha)$ by Lemma 2.2, $b_-(\bigcirc^1 \sqcup L) = b_-(L)$, $b_-(\bigcirc^{-1} \sqcup L) = b_-(L) + 1$, and $n_{\bigcirc^{\pm 1} \sqcup L} = n_L + 1$.

Let L and L' be framed links in S^3. The disjoint union $L \sqcup L'$ is then a framed link in S^3 such that $M_{L \sqcup L'} \simeq M_L \# M_{L'}$. Then the multiplicativity of $\tau_{\mathcal{C}}(M; \alpha)$

with respect to the connected sum of 3-manifolds follows from Lemma 2.2 and the equalities $b_-(L \sqcup L') = b_-(L) + b_-(L')$ and $n_{L\sqcup L'} = n_L + n_{L'}$. $\square$

In general, determining when a morphism $A^{\otimes n} \to \mathbb{1}$ is of the form ϕ_T for some ribbon n-handle T is quite a difficult problem. Hence so is the problem of determining all the (normalizable) Kirby elements of $\mathcal{C}$. In the next section, we characterize a class of (normalizable) Kirby elements of $\mathcal{C}$ by means of the structural morphisms of the categorical Hopf algebra A. This class will be shown to be sufficiently large to contain the Lyubashenko invariant (which is a categorical version of the Hennings–Kauffman–Radford invariant) and the Reshetikhin–Turaev invariant (computed from a semisimple quotient of $\mathcal{C}$) when these are well defined.

2.3. *Algebraic Kirby elements*

Let $\mathcal{C}$ be a ribbon $\Bbbk$-category such that the functor (1.5) admits a coend (A, i). Recall the notion of negligible morphisms (see § 1.4). Set

$$\begin{aligned}\mathcal{AK}(\mathcal{C}) &= \{\alpha \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) \mid S_A\alpha - \alpha \in \mathrm{Negl}_{\mathcal{C}}(\mathbb{1}, A) \text{ and} \\ &\qquad\qquad \Gamma_l(\alpha\otimes\alpha) - \alpha\otimes\alpha \in \mathrm{Negl}_{\mathcal{C}}(\mathbb{1}, A\otimes A)\},\\ \mathcal{AK}(\mathcal{C})^{\mathrm{norm}} &= \{\alpha \in \mathcal{AK}(\mathcal{C}) \mid \Theta_+\alpha \neq 0 \text{ and } \Theta_-\alpha \neq 0\},\end{aligned}$$

where $\Gamma_l\colon A\otimes A \to A\otimes A$ and $\Theta_\pm\colon A \to \mathbb{1}$ are defined in (1.6) and (2.2). Note that, by Corollary 1.2, the morphism Γ_l used in the definition of $\mathcal{AK}(\mathcal{C})$ can be replaced by the morphism Γ_r defined in (1.7).

Remark that the sets $\mathcal{AK}(\mathcal{C})$ and $\mathcal{AK}(\mathcal{C})^{\mathrm{norm}}$ always contain a non-zero element, namely the unit $\eta_A\colon \mathbb{1} \to A$.

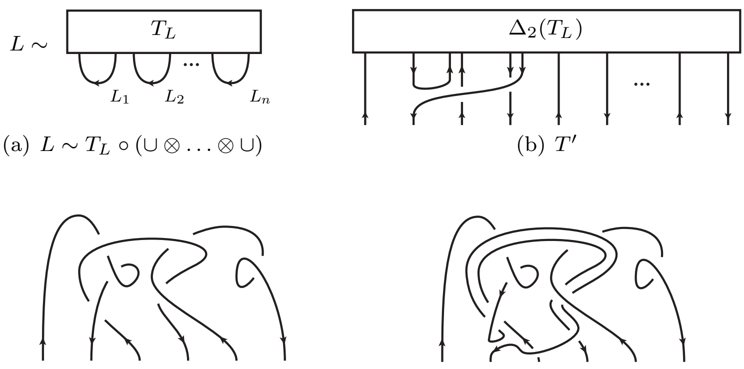
Theorem 2.6. *The elements of $\mathcal{AK}(\mathcal{C})$ are Kirby elements of $\mathcal{C}$. Moreover $\mathcal{AK}(\mathcal{C})^{\mathrm{norm}}$ is made of the elements of $\mathcal{AK}(\mathcal{C})$ which are normalizable.*

Definition 2.7. The elements of $\mathcal{AK}(\mathcal{C})$ are called the *algebraic Kirby elements of $\mathcal{C}$*.

Remark 2.8. It follows from Proposition 2.3 that any normalizable algebraic Kirby element α of $\mathcal{C}$ leads to a 3-manifold invariant $\tau_{\mathcal{C}}(M;\alpha)$ with values in $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \Bbbk$. This invariant is multiplicative with respect to the connected sum. Note that $\tau_{\mathcal{C}}(S^3;\alpha) = 1$ and $\tau_{\mathcal{C}}(S^1\times S^2;\alpha) = (\Theta_+\alpha)^{-1}\,\varepsilon_A\alpha$.

Remark 2.9. Let $\alpha \in \mathcal{AK}(\mathcal{C})^{\mathrm{norm}}$, $n \in \mathrm{Negl}_{\mathcal{C}}(\mathbb{1}, A)$, and $k \in \Bbbk^*$. Since $\mathrm{Negl}_{\mathcal{C}}$ is a two-sided $\otimes$-ideal of $\mathcal{C}$ and $\mathrm{Negl}_{\mathcal{C}}(\mathbb{1},\mathbb{1}) = 0$, we have $k\alpha + n \in \mathcal{AK}(\mathcal{C})^{\mathrm{norm}}$ and $\tau_{\mathcal{C}}(M; k\alpha+n) = \tau_{\mathcal{C}}(M;\alpha)$ for any 3-manifold M.

Remark 2.10. If $\mathcal{C}$ is 3-modular as in [**16**], then A admits a (non-zero) two-sided integral $\lambda \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$, that is, $m_A(\lambda\otimes \mathrm{id}_A) = \lambda\,\varepsilon_A = m_A(\mathrm{id}_A \otimes \lambda)$, and we have $\lambda \in \mathcal{AK}(\mathcal{C})^{\mathrm{norm}}$ and $\tau_{\mathcal{C}}(M;\lambda)$ is the Lyubashenko invariant of 3-manifolds. Note that if $\mathcal{C}$ admits split idempotents, then λ is unique (up to scalar multiples); see [**3**]. Nevertheless, the condition that A possesses a (non-zero) two-sided integral is quite limitative (for example, when $\mathcal{C}$ is the category rep_H of representations of a finite-dimensional Hopf algebra H, this implies that H must be unimodular). In § 5, we give an example of a non-unimodular ribbon Hopf algebra H and of an



(a) $L \sim T_L \circ (\cup \otimes \ldots \otimes \cup)$ (b) T'

(c) T_L (d) T'

FIGURE 3.

element $\alpha \in \mathcal{AK}(\mathrm{rep}_H)$ which is not a two-sided integral and leads to a non-trivial invariant.

REMARK 2.11. In § 3.5 (see Corollary 3.11), we show that $\mathcal{AK}(\mathcal{C})$ contains elements corresponding to the Reshetikhin–Turaev invariants defined using finitely semisimple ribbon full subcategories of the semisimple quotient of $\mathcal{C}$.

Proof of Theorem 2.6. Fix $\alpha \in \mathcal{AK}(\mathcal{C})$. Let $L = L_1 \cup \ldots \cup L_n$ be a framed link. Firstly, since $S_A\alpha - \alpha \in \mathrm{Negl}_{\mathcal{C}}(\mathbb{1}, A)$, then $\tau_{\mathcal{C}}(L;\alpha)$ does not depend on the choice of T_L nor on the orientation of L and is an isotopic invariant of the framed link L. Indeed, this is proved in the case $S_A\alpha = \alpha$ in [**16**, Proposition 5.2.1]. The same arguments work when $S_A\alpha - \alpha \in \mathrm{Negl}_{\mathcal{C}}(\mathbb{1}, A)$ since $\mathrm{Negl}_{\mathcal{C}}$ is a two-sided $\otimes$-ideal of $\mathcal{C}$.

Let us show that $\tau_{\mathcal{C}}(L;\alpha)$ is invariant under 2-handle slides. Choose an orientation for L. Without loss of generality, we can suppose that the component L_1 slides over L_2. Let L_2' be a copy of L_2 (following the framing) and set

$$L' = (L_1 \# L_2') \cup L_2 \cup \ldots \cup L_n.$$

We have to show that $\tau_{\mathcal{C}}(L';\alpha) = \tau_{\mathcal{C}}(L;\alpha)$. Let T_L be a ribbon n-handle such that L is isotopic to $T_L \circ (\cup \otimes \ldots \otimes \cup)$, where the ith cup corresponds to the component L_i; see Figure 3(a). Let $\Delta_2(T_L)$ be the $(2n+2,0)$-tangle obtained by copying the 2nd component of T_L (following the framing) in such a way that the endpoints of the new component are between the 2nd and 3rd bottom endpoints of T_L and between the 4th and 5th bottom endpoints of T_L. A ribbon n-handle T' such that L' is isotopic to $T' \circ (\cup \otimes \ldots \otimes \cup)$, where the ith cup corresponds to the ith component of L', can be constructed from $\Delta_2(T_L)$ as in Figure 3(b). For example, if T_L is the ribbon 3-handle depicted in Figure 3(c), then T' is the ribbon 3-handle of Figure 3(d). By the equalities of Figure 4 where $X_1, \ldots, X_n$ are any objects of $\mathcal{C}$, and by the uniqueness of the factorization through a coend, we get $\phi_{T'} = \phi_{T_L}(\Gamma_l \otimes \mathrm{id}_{A^{\otimes(n-2)}})$. Therefore, since $\Gamma_l(\alpha\otimes\alpha) - \alpha\otimes\alpha \in \mathrm{Negl}_{\mathcal{C}}(\mathbb{1}, A\otimes A)$, we get

$$\tau_{\mathcal{C}}(L';\alpha) = \phi_{T'}\alpha^{\otimes n} = \phi_{T_L}(\mathrm{id}_{A^{\otimes 2}} \otimes \alpha^{\otimes(n-2)})\Gamma_l(\alpha\otimes\alpha) = \phi_{T_L}\alpha^{\otimes n} = \tau_{\mathcal{C}}(L;\alpha).$$

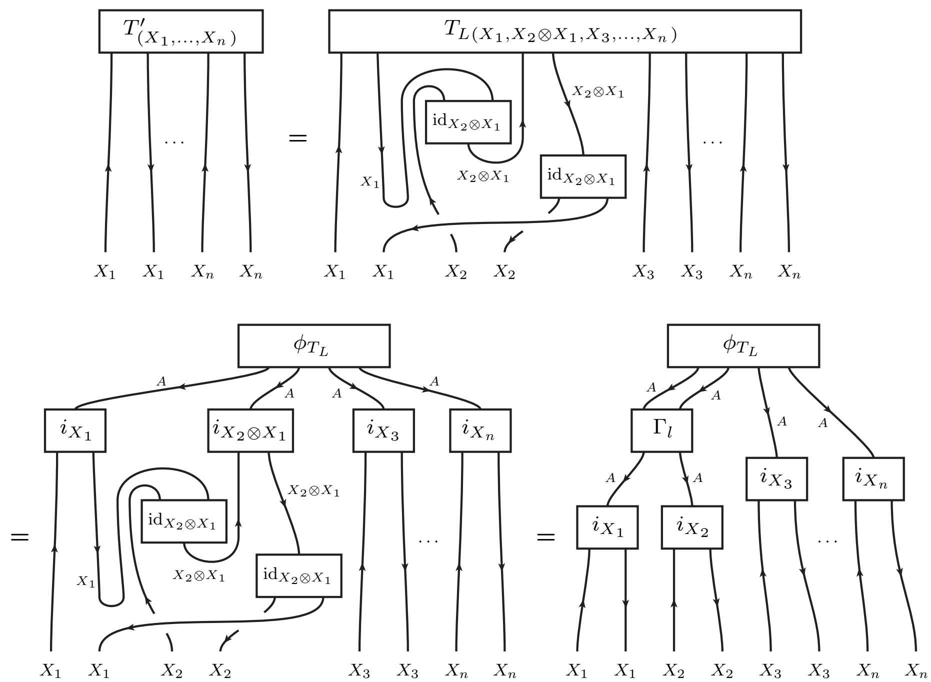


FIGURE 4.

Finally, let $\alpha \in \mathcal{AK}(\mathcal{C})$. Since $\tau_{\mathcal{C}}(\bigcirc^{\pm 1}; \alpha) = \Theta_{\pm}\alpha$ and $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \Bbbk$ is a field, we see that α is normalizable if and only if $\Theta_{\pm}\alpha \neq 0$. $\square$

2.4. *Algebraic Kirby elements via ribbon functors*

Let us see that algebraic Kirby elements can be 'pulled back' via ribbon functors. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be ribbon $\Bbbk$-categories. Suppose that the functor (1.5) for $\mathcal{A}$ admits a coend (A, i) and that the functor (1.5) for $\mathcal{B}$ admits a coend (B, j). Let $\pi\colon \mathcal{A} \to \mathcal{C}$ and $\iota\colon \mathcal{B} \to \mathcal{C}$ be ribbon functors. Since A and B are categorical Hopf algebras and π and ι are ribbon functors, the objects $\pi(A)$ and $\iota(B)$ are Hopf algebras in $\mathcal{C}$ (with structure maps induced by π and ι respectively).

PROPOSITION 2.12. *Suppose that π is surjective, ι is full and faithful, and that there exists a Hopf algebra morphism $\varphi\colon \iota(B) \to \pi(A)$ such that $\pi(i_X) = \varphi \circ \iota(j_Y)$ for all objects $X \in \mathcal{A}$ and $Y \in \mathcal{B}$ with $\pi(X) = \iota(Y)$. Let $\beta \in \mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, B)$ and $\alpha \in \mathrm{Hom}_{\mathcal{A}}(\mathbb{1}, A)$ such that $\pi(\alpha) = \varphi \circ \iota(\beta)$.*

(a) *If $\beta \in \mathcal{AK}(\mathcal{B})$, then $\alpha \in \mathcal{AK}(\mathcal{A})$ and $\tau_{\mathcal{A}}(L; \alpha) = \tau_{\mathcal{B}}(L; \beta)$ for any framed link L.*

(b) *If $\beta \in \mathcal{AK}(\mathcal{B})^{\mathrm{norm}}$, then $\alpha \in \mathcal{AK}(\mathcal{A})^{\mathrm{norm}}$ and $\tau_{\mathcal{A}}(M; \alpha) = \tau_{\mathcal{B}}(M; \beta)$ for any 3-manifold M.*

Proof. Let us prove part (a). Suppose that $\beta \in \mathcal{AK}(\mathcal{B})$. Since the structure maps of $\pi(A)$ and $\iota(B)$ are induced by π and ι from those of A and B respectively, and since $\varphi\colon \iota(B) \to \pi(A)$ is a Hopf algebra morphism, we have

$$\begin{aligned}\pi(S_A\alpha - \alpha) &= (S_{\pi(A)} - \mathrm{id}_{\pi(A)})\pi(\alpha) = (S_{\pi(A)} - \mathrm{id}_{\pi(A)})\varphi\iota(\beta)\\ &= \varphi(S_{\iota(B)} - \mathrm{id}_{\iota(B)})\iota(\beta) = \varphi\iota(S_B\beta - \beta)\end{aligned}$$

and

$$\begin{aligned}\pi(\Gamma_l^A(\alpha\otimes\alpha) - \alpha\otimes\alpha) &= (\Gamma_l^{\pi(A)} - \mathrm{id}_{\pi(A)^{\otimes 2}})(\pi(\alpha)\otimes\pi(\alpha))\\ &= (\Gamma_l^{\pi(A)} - \mathrm{id}_{\pi(A)^{\otimes 2}})\varphi\iota(\beta\otimes\beta)\\ &= \varphi(\Gamma_l^{\iota(B)} - \mathrm{id}_{\iota(B)^{\otimes 2}})\iota(\beta\otimes\beta)\\ &= \varphi\iota(\Gamma_l^B(\beta\otimes\beta) - \beta\otimes\beta).\end{aligned}$$

Now, since $S_B\beta - \beta$ and $\Gamma_l^B(\beta\otimes\beta) - \beta\otimes\beta$ are negligible in $\mathcal{B}$, ι is full, and $\mathrm{tr}_q^{\mathcal{B}} = \mathrm{tr}_q^{\mathcal{C}} \circ \iota$, we get that $\pi(S_A\alpha - \alpha)$ and $\pi(\Gamma_l^A(\alpha\otimes\alpha) - \alpha\otimes\alpha)$ are negligible in $\mathcal{C}$. Hence, since π is surjective and $\mathrm{tr}_q^{\mathcal{C}}\circ\pi = \mathrm{tr}_q^{\mathcal{A}}$, the morphisms $S_A\alpha - \alpha$ and $\Gamma_l^A(\alpha\otimes\alpha) - \alpha\otimes\alpha$ are negligible in $\mathcal{A}$, that is, $\alpha \in \mathcal{AK}(\mathcal{A})$.

Let $L = L_1 \cup \ldots \cup L_n$ be a framed link in S^3. Let T_L be a ribbon n-handle such that L is isotopic to $T_L \circ (\cup\otimes\ldots\otimes\cup)$, where the ith cup (with clockwise orientation) corresponds to the component L_i. Let $Y_1,\ldots,Y_n$ be any objects of $\mathcal{B}$. Since π is surjective, there exist objects $X_1,\ldots,X_n$ of $\mathcal{A}$ such that $\pi(X_k) = \iota(Y_k)$. Recall that, by assumption, $\pi(i_{X_k}) = \varphi\iota(j_{Y_k})$. Since ι is full and the domain and codomain of the morphism $\pi(\phi_{T_L}^{\mathcal{A}})\varphi^{\otimes n}$ of $\mathcal{C}$ are $\iota(B^{\otimes n})$ and $\mathbb{1} = \iota(\mathbb{1})$ respectively, there exists a morphism $\xi\colon B^{\otimes n} \to \mathbb{1}$ in $\mathcal{B}$ such that $\iota(\xi) = \pi(\phi_{T_L}^{\mathcal{A}})\varphi^{\otimes n}$. Then

$$\begin{aligned}&\iota\big(\phi_{T_L}^{\mathcal{B}}\circ(j_{Y_1}\otimes\ldots\otimes j_{Y_n})\big)\\ &\quad= \iota(T^{\mathcal{B}}_{L(Y_1,\ldots,X_n)}) = T^{\mathcal{C}}_{L(\iota(Y_1),\ldots,\iota(Y_n))} = T^{\mathcal{C}}_{L(\pi(X_1),\ldots,\pi(X_n))} = \pi(T^{\mathcal{A}}_{L(X_1,\ldots,X_n)})\\ &\quad= \pi(\phi_{T_L}^{\mathcal{A}})(\pi(i_{X_1})\otimes\ldots\otimes\pi(i_{X_n})) = \pi(\phi_{T_L}^{\mathcal{A}})(\varphi\iota(j_{Y_1})\otimes\ldots\otimes\varphi\iota(j_{Y_n}))\\ &\quad= \pi(\phi_{T_L}^{\mathcal{A}})\varphi^{\otimes n}\circ\iota(j_{Y_1}\otimes\ldots\otimes j_{Y_n}) = \iota\big(\xi\circ(j_{Y_1}\otimes\ldots\otimes j_{Y_n})\big).\end{aligned}$$

Therefore, since ι is faithful, $\phi_{T_L}^{\mathcal{B}}\circ(j_{Y_1}\otimes\ldots\otimes j_{Y_n}) = \xi\circ(j_{Y_1}\otimes\ldots\otimes j_{Y_n})$ and so, by the uniqueness of the factorization through a coend, we get $\phi_{T_L}^{\mathcal{B}} = \xi$, that is, $\iota(\phi_{T_L}^{\mathcal{B}}) = \pi(\phi_{T_L}^{\mathcal{A}})\varphi^{\otimes n}$. Hence, since the maps $\mathrm{End}_{\mathcal{A}}(\mathbb{1}) = \Bbbk \to \mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \Bbbk$ and $\mathrm{End}_{\mathcal{B}}(\mathbb{1}) = \Bbbk \to \mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \Bbbk$ induced by π and ι respectively are the identity of $\Bbbk$, we have

$$\begin{aligned}\tau_{\mathcal{C}}(L;\alpha) &= \pi(\tau_{\mathcal{C}}(L;\alpha)) = \pi(\phi_{T_L}^{\mathcal{A}}\alpha^{\otimes n}) = \pi(\phi_{T_L}^{\mathcal{A}})\varphi^{\otimes n}\iota(\beta^{\otimes n})\\ &= \iota(\phi_{T_L}^{\mathcal{B}}\beta^{\otimes n}) = \iota(\tau_{\mathcal{B}}(L;\beta)) = \tau_{\mathcal{B}}(L;\beta).\end{aligned}$$

Let us prove part (b). Suppose that $\beta \in \mathcal{AK}(\mathcal{B})^{\mathrm{norm}}$. Let Y be any object of $\mathcal{B}$. Since π is surjective, there exists an object X of $\mathcal{A}$ such that $\pi(X) = \iota(Y)$. Recall that, by assumption, $\pi(i_X) = \varphi\circ\iota(j_Y)$. Since ι is full and the domain and codomain of the morphism $\pi(\Theta_\pm^{\mathcal{A}})\varphi$ of $\mathcal{C}$ are $\iota(B)$ and $\mathbb{1} = \iota(\mathbb{1})$ respectively, there exists a morphism $\varsigma_\pm\colon B \to \mathbb{1}$ in $\mathcal{B}$ such that $\iota(\varsigma_\pm) = \pi(\Theta_\pm^{\mathcal{A}})\varphi$. We have

$$\begin{aligned}\iota(\varsigma_\pm\circ\iota(j_Y)) &= \pi(\Theta_\pm^{\mathcal{A}})\varphi\circ\iota(j_Y) = \pi(\Theta_\pm^{\mathcal{A}} i_X) = \pi(\mathrm{ev}_X^{\mathcal{A}}(\mathrm{id}_{X^*}\otimes\theta_X^{\mathcal{A}\pm 1}))\\ &= \mathrm{ev}^{\mathcal{C}}_{\pi(X)}(\mathrm{id}_{\pi(X)^*}\otimes\theta^{\mathcal{C}\pm 1}_{\pi(X)}) = \iota(\mathrm{ev}_Y^{\mathcal{B}})(\iota(\mathrm{id}_{Y^*})\otimes\iota(\theta_Y^{\mathcal{B}})^{\pm 1})\\ &= \iota(\mathrm{ev}_Y^{\mathcal{B}}(\mathrm{id}_{Y^*}\otimes\theta_Y^{\mathcal{B}\pm 1})) = \iota(\Theta_\pm^{\mathcal{B}}\circ j_Y).\end{aligned}$$

Therefore, since ι is faithful, $\Theta_\pm^{\mathcal{B}}\circ j_Y = \varsigma_\pm\circ\iota(j_Y)$ and so, by the uniqueness of the factorization through a coend, we get $\Theta_\pm^{\mathcal{B}} = \varsigma_\pm$, that is, $\iota(\Theta_\pm^{\mathcal{B}}) = \pi(\Theta_\pm^{\mathcal{A}})\varphi$. Then

$\Theta^{\mathcal{C}}_{\pm}\alpha = \pi(\Theta^{\mathcal{A}}_{\pm}\alpha) = \pi(\Theta^{\mathcal{A}}_{\pm})\varphi\iota(\beta) = \iota(\Theta^{\mathcal{B}}_{\pm}\beta) = \Theta^{\mathcal{B}}_{\pm}\beta$. Hence, since $\Theta^{\mathcal{B}}_{\pm}\beta \neq 0$, we get $\alpha \in \mathcal{AK}(\mathcal{A})^{\mathrm{norm}}$ and $\tau_{\mathcal{A}}(M;\alpha) = \tau_{\mathcal{B}}(M;\beta)$ for any 3-manifold M. □

3. *The case of semisimple ribbon categories*

In this section, we focus on the case of semisimple categories $\mathcal{B}$. We give sufficient conditions for belonging to $\mathcal{AK}(\mathcal{B})$. In particular, we show that there exist (even in the non-modular case) elements of $\mathcal{AK}(\mathcal{B})$ corresponding to Reshetikhin–Turaev invariants computed with finitely semisimple ribbon subcategories of $\mathcal{B}$. Moreover, we study Kirby elements coming from semisimplification of ribbon categories.

3.1. *Semisimple categories*

Recall that a category $\mathcal{B}$ *admits* (*finite*) *direct sums* if, for any finite set of objects $X_1, \dots, X_n$ of $\mathcal{B}$, there exist an object X and morphisms $p_i\colon X \to X_i$ such that, for any object Y and morphisms $f_i\colon Y \to X_i$, there is a unique morphism $f\colon Y \to X$ with $p_i \circ f = f_i$ for all i. The object X is then unique up to isomorphism. We write $X = \bigoplus_i X_i$ and $f = \bigoplus_i f_i$.

A $\Bbbk$-category is *abelian* if it admits (finite) direct sums, every morphism has a kernel and a cokernel, every monomorphism is the kernel of its cokernel, every epimorphism is the cokernel of its kernel, and every morphism is expressible as the composite of an epimorphism followed by a monomorphism. In particular, an abelian category admits a null object (which is unique up to isomorphism). Note that a morphism of an abelian $\Bbbk$-category which is both a monomorphism and an epimorphism is an isomorphism.

Let $\mathcal{B}$ be an abelian $\Bbbk$-category. A non-null object U of $\mathcal{B}$ is said to be *simple* if every non-zero monomorphism $V \to U$ is an isomorphism, and every non-zero epimorphism $U \to V$ is an isomorphism. Any non-zero morphism between simple objects is an isomorphism. An object U of $\mathcal{B}$ is *scalar* if $\mathrm{End}_{\mathcal{B}}(V) = \Bbbk$. Note that if $\Bbbk$ is algebraically closed, then every simple object is scalar. An object of $\mathcal{B}$ is *indecomposable* if it cannot be written as a direct sum of two non-null objects. Note that every scalar or simple object is indecomposable.

By a *semisimple* $\Bbbk$*-category*, we shall mean an abelian $\Bbbk$-category for which every object is a (finite) direct sum of simple objects. By a *finitely semisimple* $\Bbbk$*-category*, we shall mean a semisimple $\Bbbk$-category which has finitely many isomorphism classes of simple objects. Note that in a semisimple $\Bbbk$-category, every scalar or indecomposable object is simple.

Let $\mathcal{B}$ be a semisimple ribbon $\Bbbk$-category. If the $\mathrm{Hom}_{\mathcal{B}}(X,Y)$ are all finite-dimensional (this is the case for example when the simple objects are scalar), then any negligible morphism of $\mathcal{B}$ is null (see [**5**]). Therefore, for every pair of objects X, Y of $\mathcal{B}$, the pairing

$$\mathrm{Hom}_{\mathcal{B}}(X,Y) \otimes \mathrm{Hom}_{\mathcal{B}}(Y,X) \to \Bbbk, \quad f \otimes g \mapsto \mathrm{tr}_q(gf) \tag{3.1}$$

is non-degenerate. Note that this implies that the quantum dimension of a scalar object of $\mathcal{B}$ is invertible.

Lemma 3.1. *Let $\mathcal{B}$ be a finitely semisimple ribbon $\Bbbk$-category whose simple objects are scalar. Let Λ be a (finite) set of representatives of isomorphism classes* *of simple objects of $\mathcal{B}$. Fix an object X of $\mathcal{B}$. For any $\lambda \in \Lambda$, set*

$$n_\lambda = \dim_\Bbbk \operatorname{Hom}_\mathcal{B}(\lambda, X) = \dim_\Bbbk \operatorname{Hom}_\mathcal{B}(X, \lambda)$$

and let $\{f_i^\lambda \mid 1 \leqslant i \leqslant n_\lambda\}$ be a basis of $\operatorname{Hom}_\mathcal{B}(\lambda, X)$ and $\{g_i^\lambda \mid 1 \leqslant i \leqslant n_\lambda\}$ be a basis of $\operatorname{Hom}_\mathcal{B}(X, \lambda)$ such that $g_i^\lambda f_j^\lambda = \delta_{i,j} \operatorname{id}_\lambda$ for all $1 \leqslant i, j \leqslant n_\lambda$ (such bases exist since the pairing (3.1) *is non-degenerate). Then*

$$\operatorname{id}_X = \sum_{\lambda \in \Lambda} \sum_{1 \leqslant i \leqslant n_\lambda} f_i^\lambda g_i^\lambda.$$

Proof. Since the category $\mathcal{B}$ is semisimple, the composition induces a $\Bbbk$-linear isomorphism $\bigoplus_{\lambda \in \Lambda} \operatorname{Hom}_\mathcal{B}(X, \lambda) \otimes \operatorname{Hom}_\mathcal{B}(\lambda, X) \to \operatorname{End}_\mathcal{B}(X)$. Therefore, for all $\lambda \in \Lambda$ and $1 \leqslant i, j \leqslant n_\lambda$, there exist $a_{\lambda,i,j} \in \Bbbk$ such that

$$\operatorname{id}_X = \sum_{\lambda \in \Lambda} \sum_{1 \leqslant i,j \leqslant n_\lambda} a_{\lambda,i,j}\, f_i^\lambda g_j^\lambda.$$

Let $\lambda \in \Lambda$ and $1 \leqslant i, j \leqslant n_\lambda$. Then

$$\begin{aligned}\delta_{i,j} \operatorname{id}_\lambda = g_i^\lambda f_j^\lambda = g_i^\lambda \operatorname{id}_X f_j^\lambda &= \sum_{\mu \in \Lambda} \sum_{1 \leqslant k,l \leqslant n_\lambda} a_{\mu,k,l}\, g_i^\lambda f_k^\mu g_l^\mu f_j^\lambda \\ &= \sum_{\mu \in \Lambda} \sum_{1 \leqslant k,l \leqslant n_\lambda} a_{\mu,k,l}\, \delta_{\lambda,\mu}\, \delta_{i,k}\, \delta_{j,l} \operatorname{id}_\mu = a_{\lambda,i,j} \operatorname{id}_\lambda,\end{aligned}$$

and so $a_{\lambda,i,j} = \delta_{i,j}$. Hence $\operatorname{id}_X = \sum_{\lambda \in \Lambda} \sum_{1 \leqslant i \leqslant n_\lambda} f_i^\lambda g_i^\lambda$. □

3.2. *Algebraic Kirby elements of finitely semisimple ribbon categories*

Let $\mathcal{B}$ be a finitely semisimple ribbon $\Bbbk$-category whose simple objects are scalar. Note that the assumptions on $\mathcal{B}$ imply that the $\Bbbk$-spaces $\operatorname{Hom}_\mathcal{B}(X, Y)$ are finite-dimensional. Denote by Λ a (finite) set of representatives of isomorphism classes of simple objects of $\mathcal{B}$. We can suppose that $\mathbb{1} \in \Lambda$. For any $\lambda \in \Lambda$, there exists a unique $\lambda^\vee \in \Lambda$ such that $\lambda^* \simeq \lambda^\vee$. The map ${}^\vee \colon \Lambda \to \Lambda$ is an involution and $\mathbb{1}^\vee = \mathbb{1}$. Recall that $\dim_q(\lambda) \neq 0$ for any $\lambda \in \Lambda$. Set

$$B = \bigoplus_{\lambda \in \Lambda} \lambda^* \otimes \lambda \in \mathcal{B}.$$

In particular, there exist morphisms $p_\lambda \colon B \to \lambda^* \otimes \lambda$ and $q_\lambda \colon \lambda^* \otimes \lambda \to B$ such that $\operatorname{id}_B = \sum_{\lambda \in \Lambda} q_\lambda p_\lambda$ and $p_\lambda q_\mu = \delta_{\lambda,\mu} \operatorname{id}_{\lambda^* \otimes \lambda}$. Let X be an object of $\mathcal{B}$. Since $\mathcal{B}$ is semisimple, we can write $X = \bigoplus_{i \in I} \lambda_i$, where I is a finite set and $\lambda_i \in \Lambda$. We set

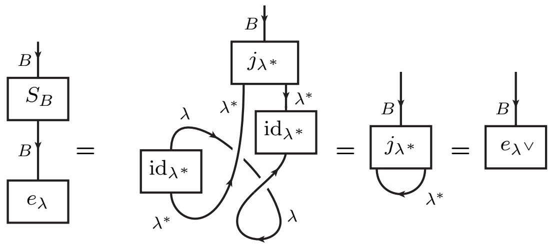
$$j_X = \sum_{i \in I} q_{\lambda_i} \circ (Q_i^* \otimes P_i) \colon X^* \otimes X \to B,$$

where $P_i \colon X \to \lambda_i$ and $Q_i \colon \lambda_i \to X$ are morphisms in $\mathcal{B}$ such that

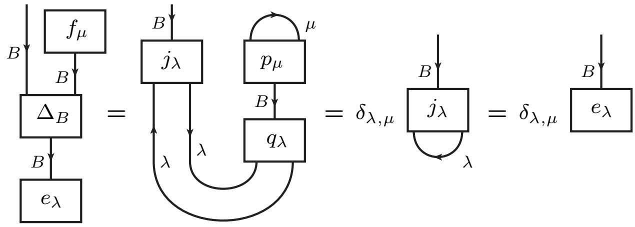
$$\operatorname{id}_X = \sum_{i \in I} Q_i P_i \quad \text{and} \quad P_i Q_j = \delta_{i,j} \operatorname{id}_{\lambda_i}.$$

Note that j_X does not depend on the choice of such morphisms P_i and Q_i. Remark that $j_\lambda = q_\lambda$ for any $\lambda \in \Lambda$. One easily verifies that (B, j) is a coend of the functor (1.5) for $\mathcal{B}$. By § 1.6, the object B is a Hopf algebra in $\mathcal{B}$.

For any $\lambda \in \Lambda$, set $e_\lambda = j_\lambda \widetilde{\operatorname{coev}}_\lambda \colon \mathbb{1} \to B$ and $f_\lambda = \operatorname{ev}_\lambda p_\lambda \colon B \to \mathbb{1}$. Note that we have $f_\lambda e_\mu = \delta_{\lambda,\mu} \dim_q(\lambda)$ for any $\lambda, \mu \in \Lambda$.



(a) $S_B(e_\lambda) = e_{\lambda^\vee}$



(b) $(\mathrm{id}_B \otimes f_\mu)\Delta_B(e_\lambda) = \delta_{\lambda,\mu} e_\lambda$

FIGURE 5.

LEMMA 3.2. (a) *The family* $(e_\lambda)_{\lambda\in\Lambda}$ *is a basis of the* $\Bbbk$*-space* $\mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, B)$.

(b) *The family* $(f_\lambda)_{\lambda\in\Lambda}$ *is a basis of the* $\Bbbk$*-space* $\mathrm{Hom}_{\mathcal{B}}(B, \mathbb{1})$.

(c) *For any* $\lambda, \mu \in \Lambda$, *we have* $S_B(e_\lambda) = e_{\lambda^\vee}$, $\eta_B = e_{\mathbb{1}}$, $\varepsilon_B(e_\lambda) = \dim_q(\lambda)$, $(\mathrm{id}_B \otimes f_\mu)\Delta_B(e_\lambda) = \delta_{\lambda,\mu} e_\lambda$, *and* $m_B(e_\lambda \otimes e_\mu) = \sum_{\nu\in\Lambda} N_{\lambda,\mu}^\nu e_\nu$, *where* $N_{\lambda,\mu}^\nu = \dim_\Bbbk \mathrm{Hom}_{\mathcal{B}}(\lambda \otimes \mu, \nu)$.

Proof. Let us prove part (a). For any $\lambda \in \Lambda$, the $\Bbbk$-space $\mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, \lambda^* \otimes \lambda)$ is one-dimensional (since λ is scalar) with basis $\widetilde{\mathrm{coev}}_\lambda$. Therefore, for any $g \in \mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, B)$, there exists $x_\lambda \in \Bbbk$ such that $p_\lambda g = x_\lambda \widetilde{\mathrm{coev}}_\lambda$, and so $g = \mathrm{id}_B g = \sum_{\lambda\in\Lambda} q_\lambda p_\lambda g = \sum_{\lambda\in\Lambda} x_\lambda e_\lambda$. Hence the family $(e_\lambda)_{\lambda\in\Lambda}$ generates $\mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, B)$. To show that it is free, suppose that $\sum_{\lambda\in\Lambda} x_\lambda e_\lambda = 0$. Then, for any $\mu \in \Lambda$, $0 = \sum_{\lambda\in\Lambda} x_\lambda f_\mu e_\lambda = \dim_q(\mu) x_\mu$ and so $x_\mu = 0$ since $\dim_q(\mu) \neq 0$.

Part (b) can be shown similarly. Let us prove part (c). Let $\lambda, \mu \in \Lambda$. By definition, $\eta_B = j_{\mathbb{1}} = j_{\mathbb{1}} \widetilde{\mathrm{coev}}_{\mathbb{1}} = e_{\mathbb{1}}$ (since $\widetilde{\mathrm{coev}}_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$) and $\varepsilon_B(e_\lambda) = \varepsilon_B j_\lambda \widetilde{\mathrm{coev}}_\lambda = \mathrm{ev}_\lambda \widetilde{\mathrm{coev}}_\lambda = \dim_q(\lambda)$. The equalities $S_B(e_\lambda) = e_{\lambda^\vee}$ and $(\mathrm{id}_B \otimes f_\mu)\Delta_B(e_\lambda) = \delta_{\lambda,\mu} e_\lambda$ are shown in Figures 5(a) and 5(b) respectively. Write $\lambda \otimes \mu = \bigoplus_{i\in I} \lambda_i$. In particular, there exist morphisms $P_i\colon \lambda \otimes \mu \to \lambda_i$ and $Q_i\colon \lambda_i \to \lambda \otimes \mu$ such that $\mathrm{id}_{\lambda\otimes\mu} = \sum_{i\in I} Q_i P_i$ and $P_i Q_j = \delta_{i,j} \mathrm{id}_{\lambda_i}$. Recall that $j_{\lambda\otimes\mu} = \sum_{i\in I} j_{\lambda_i}(Q_i^* \otimes P_i)$. For any $\nu \in \Lambda$, since $\mathrm{Hom}_{\mathcal{B}}(\lambda \otimes \mu, \nu) \cong \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{B}}(\lambda_i, \nu)$, we have

$$N_{\lambda,\mu}^\nu = \dim_\Bbbk \mathrm{Hom}_{\mathcal{B}}(\lambda \otimes \mu, \nu) = \sum_{i\in I} \delta_{\lambda_i,\nu}.$$

Then the equality $m_B(e_\lambda \otimes e_\mu) = \sum_{\nu\in\Lambda} N_{\lambda,\mu}^\nu e_\nu$ is shown in Figure 6. $\square$

Since $\mathcal{B}$ is semisimple with scalar simple objects, the negligible morphisms of $\mathcal{B}$ are null. Therefore a morphism $\alpha \in \mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, B)$ is an algebraic Kirby element of $\mathcal{B}$ if and only if it satisfies $S_B\alpha = \alpha$ and $\Gamma_l(\alpha \otimes \alpha) = \alpha \otimes \alpha$.

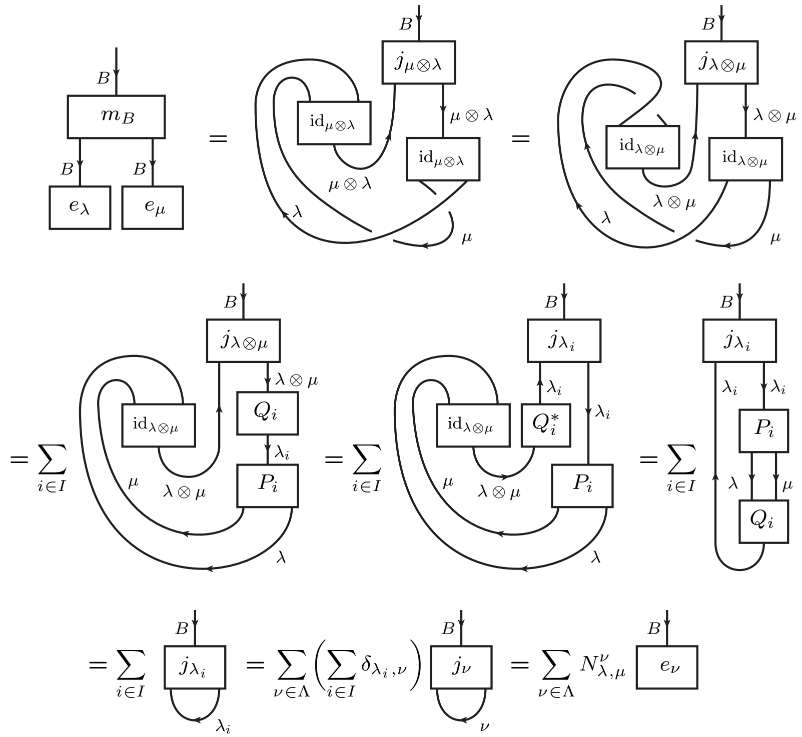


FIGURE 6. $m_B(e_\lambda \otimes e_\mu) = \sum_{\nu\in\Lambda} N_{\lambda,\mu}^{\nu} e_\nu$.

LEMMA 3.3. *Let $\alpha = \sum_{\lambda\in\Lambda} \alpha_\lambda e_\lambda \in \mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, B)$, where $\alpha_\lambda \in \Bbbk$. Suppose that $\alpha \in \mathcal{AK}(\mathcal{B})$. Set $\Lambda_\alpha = \{\lambda \in \Lambda \mid \alpha_\lambda \neq 0\}$. Then $\alpha = \alpha_{\mathbb{1}} \sum_{\lambda\in\Lambda_\alpha} \dim_q(\lambda) e_\lambda$. Moreover we have $\Lambda_\alpha^\vee = \Lambda_\alpha$ and $m_B(\alpha \otimes e_\lambda) = \dim_q(\lambda)\,\alpha = m_B(e_\lambda \otimes \alpha)$ for all $\lambda \in \Lambda_\alpha$.*

Proof. By Lemma 3.2(c), since $S_B(\alpha) = \alpha$, we have $\alpha_\lambda = \alpha_{\lambda^\vee}$ for all $\lambda \in \Lambda$ and so $\Lambda_\alpha^\vee = \Lambda_\alpha$. Let $\mu, \nu \in \Lambda$. Since $\Gamma_r(\alpha \otimes \alpha) = \alpha \otimes \alpha$ and by using Lemma 3.2(c), we have

$$\begin{aligned}
\dim_q(\mu)\dim_q(\nu)\alpha_\nu\alpha_\mu &= (f_\nu \otimes f_\lambda)(\alpha \otimes \alpha) = (f_\nu \otimes f_\mu)\Gamma_r(\alpha \otimes \alpha) \\
&= \sum_{\lambda,\omega\in\Lambda} \alpha_\lambda\alpha_\omega\,(f_\nu \otimes f_\mu)\Gamma_r(e_\lambda \otimes e_\omega) \\
&= \sum_{\lambda,\omega\in\Lambda} \alpha_\lambda\alpha_\omega\, f_\nu m_B(e_\lambda \otimes (\mathrm{id}_B \otimes f_\mu)\Delta_B(e_\omega)) \\
&= \sum_{\lambda\in\Lambda} \alpha_\lambda\alpha_\mu\, f_\nu m_B(e_\lambda \otimes e_\mu) \\
&= \sum_{\lambda,\omega\in\Lambda} \alpha_\lambda\alpha_\mu N_{\lambda,\mu}^{\omega}\, f_\nu e_\omega = \sum_{\lambda\in\Lambda} \alpha_\lambda\alpha_\mu N_{\lambda,\mu}^{\nu} \dim_q(\nu),
\end{aligned}$$

and so

$$\dim_q(\mu)\alpha_\mu\alpha_\nu = \alpha_\mu \sum_{\lambda\in\Lambda} N_{\lambda,\mu}^{\nu}\alpha_\lambda. \tag{3.2}$$

Note that $N_{\lambda,\mu}^{\mathbb{1}} = \dim_{\Bbbk} \mathrm{Hom}_{\mathcal{B}}(\lambda \otimes \mu, \mathbb{1}) = \dim_{\Bbbk} \mathrm{Hom}_{\mathcal{B}}(\lambda, \mu^*) = \delta_{\lambda,\mu^\vee}$ for all $\lambda, \mu \in \Lambda$. Hence (3.2) gives $\dim_q(\mu)\alpha_\mu\alpha_{\mathbb{1}} = \alpha_\mu\alpha_{\mu^\vee} = \alpha_\mu^2$ and so $\alpha_\mu = \alpha_{\mathbb{1}}\dim_q(\mu)$ whenever $\alpha_\mu \neq 0$, that is, $\alpha = \alpha_{\mathbb{1}} \sum_{\lambda\in\Lambda_\alpha} \dim_q(\lambda)e_\lambda$. Finally, for any $\mu \in \Lambda_\alpha$,

$$\begin{aligned} m_B(\alpha \otimes e_\mu) &= \sum_{\lambda\in\Lambda} \alpha_\lambda\, m_B(e_\lambda \otimes e_\mu) = \sum_{\lambda,\nu\in\Lambda} N_{\lambda,\mu}^{\nu}\alpha_\lambda\, e_\nu \\ &= \sum_{\nu\in\Lambda} \dim_q(\mu)\alpha_\nu e_\nu \quad \text{(by (3.2) since } \alpha_\mu \neq 0) \\ &= \dim_q(\mu)\alpha. \end{aligned}$$

Likewise, using $\Gamma_l(\alpha \otimes \alpha) = \alpha \otimes \alpha$, one gets $m_B(e_\mu \otimes \alpha) = \dim_q(\mu)\alpha$. □

By Lemma 3.3, determining $\mathcal{AK}(\mathcal{B})$ resumes to find the subsets E of Λ for which $\sum_{\lambda\in E} \dim_q(\lambda)e_\lambda$ belongs to $\mathcal{AK}(\mathcal{B})$. In the next theorem, we show that among these subsets, there are those corresponding to monoidal subcategories of $\mathcal{B}$.

THEOREM 3.4. *Let $\mathcal{D}$ be a semisimple ribbon full subcategory of $\mathcal{B}$. Let $\Lambda_{\mathcal{D}}$ be a (finite) set of representatives of isomorphism classes of simple objects of $\mathcal{D}$. We can suppose that $\Lambda_{\mathcal{D}} \subset \Lambda$. Then $\sum_{\lambda\in\Lambda_{\mathcal{D}}} \dim_q(\lambda)e_\lambda$ is an algebraic Kirby element of $\mathcal{B}$.*

REMARK 3.5. We do not know if every algebraic Kirby element of $\mathcal{B}$ is of this form (up to scalar multiples). Nevertheless, in Corollary 3.8, we explore some cases where this holds.

REMARK 3.6. In Section 3.3, we verify that $\sum_{\lambda\in\Lambda_{\mathcal{D}}} \dim_q(\lambda)e_\lambda$ leads to the Reshetikhin–Turaev invariant defined using $\mathcal{D}$.

REMARK 3.7. If $\mathcal{B}$ is modular in the sense that the pairing $\omega_B\colon B \otimes B \to \mathbb{1}$ is non-degenerate (see § 1.6) or, equivalently, that the S-matrix is invertible, then $\sum_{\lambda\in\Lambda} \dim_q(\lambda)e_\lambda$ is a two-sided integral of B (see [**11**]) and so belongs to $\mathcal{AK}(\mathcal{B})$. Nevertheless, $\sum_{\lambda\in\Lambda} \dim_q(\lambda)e_\lambda$ is not in general a two-sided integral of B.

Proof of Theorem 3.4. Firstly, since $\Lambda_{\mathcal{D}}^\vee = \Lambda_{\mathcal{D}}$ and by Lemma 3.2(c), we have

$$S_B(\alpha_{\mathcal{D}}) = \sum_{\lambda\in\Lambda_{\mathcal{D}}} \dim_q(\lambda)S_B(e_\lambda) = \sum_{\lambda\in\Lambda_{\mathcal{D}}} \dim_q(\lambda^\vee)e_{\lambda^\vee} = \alpha_{\mathcal{D}}.$$

Secondly, to show $\Gamma_r(\alpha_{\mathcal{D}} \otimes \alpha_{\mathcal{D}}) = \alpha_{\mathcal{D}} \otimes \alpha_{\mathcal{D}}$, it suffices to show $\Gamma_r(\alpha_{\mathcal{D}} \otimes e_\lambda) = \alpha_{\mathcal{D}} \otimes e_\lambda$ for any $\lambda \in \Lambda_{\mathcal{D}}$. Fix $\lambda \in \Lambda_{\mathcal{D}}$. Let $\mu, \nu \in \Lambda_{\mathcal{D}}$ and set $n_{\mu,\nu} = \dim_{\Bbbk} \mathrm{Hom}_{\mathcal{B}}(\nu \otimes \lambda, \mu)$. Since the pairing $g \otimes f \in \mathrm{Hom}_{\mathcal{B}}(\nu \otimes \lambda, \mu) \otimes \mathrm{Hom}_{\mathcal{B}}(\mu, \nu \otimes \lambda) \mapsto \mathrm{tr}_q(gf) \in \Bbbk$ is non-degenerate, there exist a basis $\{f_i^{\mu,\nu} \mid 1 \leqslant i \leqslant n_{\mu,\nu}\}$ of $\mathrm{Hom}_{\mathcal{B}}(\mu, \nu \otimes \lambda)$ and a basis $\{g_i^{\mu,\nu} \mid 1 \leqslant i \leqslant n_{\mu,\nu}\}$ of $\mathrm{Hom}_{\mathcal{B}}(\nu \otimes \lambda, \mu)$ such that $g_j^{\mu,\nu} f_i^{\mu,\nu} = \delta_{i,j}\, \mathrm{id}_\mu$ for all $1 \leqslant i, j \leqslant n_{\mu,\nu}$. By Lemma 3.1, we have

$$\sum_{\mu\in\Lambda} \sum_{1\leqslant i\leqslant n_{\mu,\nu}} f_i^{\mu,\nu} g_i^{\mu,\nu} = \mathrm{id}_{\nu\otimes\lambda}. \tag{3.3}$$

Let $\mu, \nu \in \Lambda_{\mathcal{D}}$. For any $1 \leqslant i \leqslant n_{\mu,\nu}$, set

$$\begin{aligned} F_i^{\nu,\mu} &= (g_i^{\mu,\nu} \otimes \mathrm{id}_{\lambda^*})(\mathrm{id}_\nu \otimes \mathrm{coev}_\lambda)\colon \nu \to \mu \otimes \lambda^*, \\ G_i^{\nu,\mu} &= (\mathrm{id}_\nu \otimes \widetilde{\mathrm{ev}}_\lambda)(f_i^{\mu,\nu} \otimes \mathrm{id}_\lambda)\colon \mu \otimes \lambda^* \to \nu. \end{aligned}$$

One easily checks that $\{F_i^{\nu,\mu} \mid 1 \leqslant i \leqslant n_{\mu,\nu}\}$ is a basis for $\mathrm{Hom}_{\mathcal{B}}(\nu, \mu \otimes \lambda^*)$ and $\{G_i^{\nu,\mu} \mid 1 \leqslant i \leqslant n_{\mu,\nu}\}$ is a basis for $\mathrm{Hom}_{\mathcal{B}}(\mu \otimes \lambda^*, \nu)$. For any $1 \leqslant i, j \leqslant n_{\mu,\nu}$, since $G_j^{\nu,\mu} F_i^{\nu,\mu} \in \mathrm{End}_{\mathcal{B}}(\nu)$ and ν is scalar, we have

$$\begin{aligned}\dim_q(\nu)\, G_j^{\nu,\mu} F_i^{\nu,\mu} &= \mathrm{tr}_q(G_j^{\nu,\mu} F_i^{\nu,\mu})\,\mathrm{id}_\nu = \mathrm{tr}_q(F_i^{\nu,\mu} G_j^{\nu,\mu})\,\mathrm{id}_\nu \\ &= \mathrm{tr}_q\big((g_i^{\mu,\nu} \otimes \mathrm{id}_\lambda)(\mathrm{id}_\nu \otimes \mathrm{coev}_\lambda)(\mathrm{id}_\nu \otimes \widetilde{\mathrm{ev}}_\lambda)(f_j^{\mu,\nu} \otimes \mathrm{id}_\lambda)\big)\,\mathrm{id}_\nu \\ &= \mathrm{tr}_q(g_i^{\mu,\nu} f_j^{\mu,\nu})\,\mathrm{id}_\nu = \mathrm{tr}_q(\delta_{i,j}\,\mathrm{id}_\mu)\,\mathrm{id}_\nu = \delta_{i,j} \dim_q(\mu)\,\mathrm{id}_\nu.\end{aligned}$$

Therefore, by Lemma 3.1,

$$\sum_{\nu \in \Lambda} \sum_{1 \leqslant i \leqslant n_{\mu,\nu}} \dim_q(\nu)\, F_i^{\nu,\mu} G_i^{\nu,\mu} = \dim_q(\mu)\,\mathrm{id}_{\mu \otimes \lambda^*}. \tag{3.4}$$

Finally one gets $\Gamma_r(\alpha_{\mathcal{D}} \otimes e_\lambda) = \alpha_{\mathcal{D}} \otimes e_\lambda$ by the equalities depicted in Figure 7, where $d_\nu = \dim_q(\nu)$. Note that, in Figure 7, the dinaturality of j, the definition of m_B and Δ_B, and equalities (3.3) and (3.4) are used. $\square$

Recall that an object X of $\mathcal{B}$ is invertible if $X^* \otimes X$ is isomorphic to $\mathbb{1}$.

Corollary 3.8. *Suppose that either every simple object of $\mathcal{B}$ is invertible, or the field $\Bbbk = \mathbb{R}$ or $\mathbb{C}$ and the quantum dimensions of the simple objects are positive. Then every algebraic Kirby element of $\mathcal{B}$ is a scalar multiple of $\alpha_{\mathcal{D}} = \sum_{\lambda \in \Lambda_{\mathcal{D}}} \dim_q(\lambda) e_\lambda$, where $\mathcal{D}$ is some semisimple ribbon full subcategory of $\mathcal{B}$ and $\Lambda_{\mathcal{D}} \subset \Lambda$ is a (finite) set of representatives of isomorphism classes of simple objects of $\mathcal{D}$.*

Proof. By Theorem 3.4, each $\alpha_{\mathcal{D}}$ (and so its scalar multiples) is an algebraic Kirby element of $\mathcal{B}$. Conversely, let $\alpha = \sum_{\lambda \in \Lambda} \alpha_\lambda e_\lambda$ be a non-zero algebraic Kirby element of $\mathcal{B}$. By Lemma 3.3, $\alpha_{\mathbb{1}} \neq 0$ and $\alpha_\lambda = \alpha_{\mathbb{1}} \dim_q(\lambda)$ whenever $\alpha_\lambda \neq 0$. Set $\Lambda_\alpha = \{\lambda \in \Lambda \mid \alpha_\lambda \neq 0\}$ and let $\mathcal{D}$ be full subcategory of $\mathcal{B}$ additively generated by Λ_α. Note that Λ_α is then a set of representatives of isomorphism classes of simple objects of $\mathcal{D}$. Firstly, let us show that $\mathcal{D}$ is closed under the tensor product. Fix $\mu, \nu \in \Lambda_\alpha$, and let $\lambda \in \Lambda$ be a direct factor of $\mu \otimes \nu$. We have to show that $\lambda \in \Lambda_\alpha$, that is, $\alpha_\lambda \neq 0$. Equation (3.2) gives

$$\dim_q(\nu)\alpha_\lambda = \sum_{\omega \in \Lambda_\alpha} N_{\omega,\nu}^\lambda \alpha_\omega = \alpha_{\mathbb{1}} \sum_{\omega \in \Lambda_\alpha} N_{\omega,\nu}^\lambda \dim_q(\omega). \tag{3.5}$$

Suppose that every simple object of $\mathcal{B}$ is invertible. Note that this implies that the tensor product of two simple objects is simple. Therefore $\lambda \simeq \mu \otimes \nu$ and $N_{\omega,\nu}^\lambda = \delta_{\omega,\mu}$. We get, from (3.5), $\dim_q(\nu)\alpha_\lambda = \alpha_\mu \neq 0$. Hence $\alpha_\lambda \neq 0$.

Suppose that $\Bbbk = \mathbb{R}$ or $\mathbb{C}$ and the quantum dimensions of the simple objects are positive. Since $\dim_q(\omega) > 0$, $N_{\omega,\nu}^\lambda \geqslant 0$, and $N_{\mu,\nu}^\lambda = \dim_\Bbbk \mathrm{Hom}_{\mathcal{B}}(\mu \otimes \nu, \lambda) \geqslant 1$, we see from (3.5) that $\dim_q(\nu)\alpha_\lambda \geqslant N_{\mu,\nu}^\lambda \dim_q(\mu) > 0$. Hence $\alpha_\lambda \neq 0$.

In all cases, we get that $\mathcal{D}$ is closed under tensor product. Moreover, $\mathcal{D}$ is closed under duality since $\Lambda_\alpha^\vee = \Lambda_\alpha$ by Lemma 3.3. Hence $\mathcal{D}$ is a semisimple ribbon full subcategory of $\mathcal{B}$, and $\alpha = \alpha_{\mathbb{1}} \sum_{\lambda \in \Lambda_\alpha} \dim_q(\lambda) e_\lambda = \alpha_{\mathbb{1}} \alpha_{\mathcal{D}}$. $\square$

3.3. *On the Reshetikhin–Turaev invariant*

Let $\mathcal{B}$ be a finitely semisimple ribbon $\Bbbk$-category whose simple objects are scalar. Let Λ be a set of representatives of isomorphism classes of simple objects of $\mathcal{B}$.

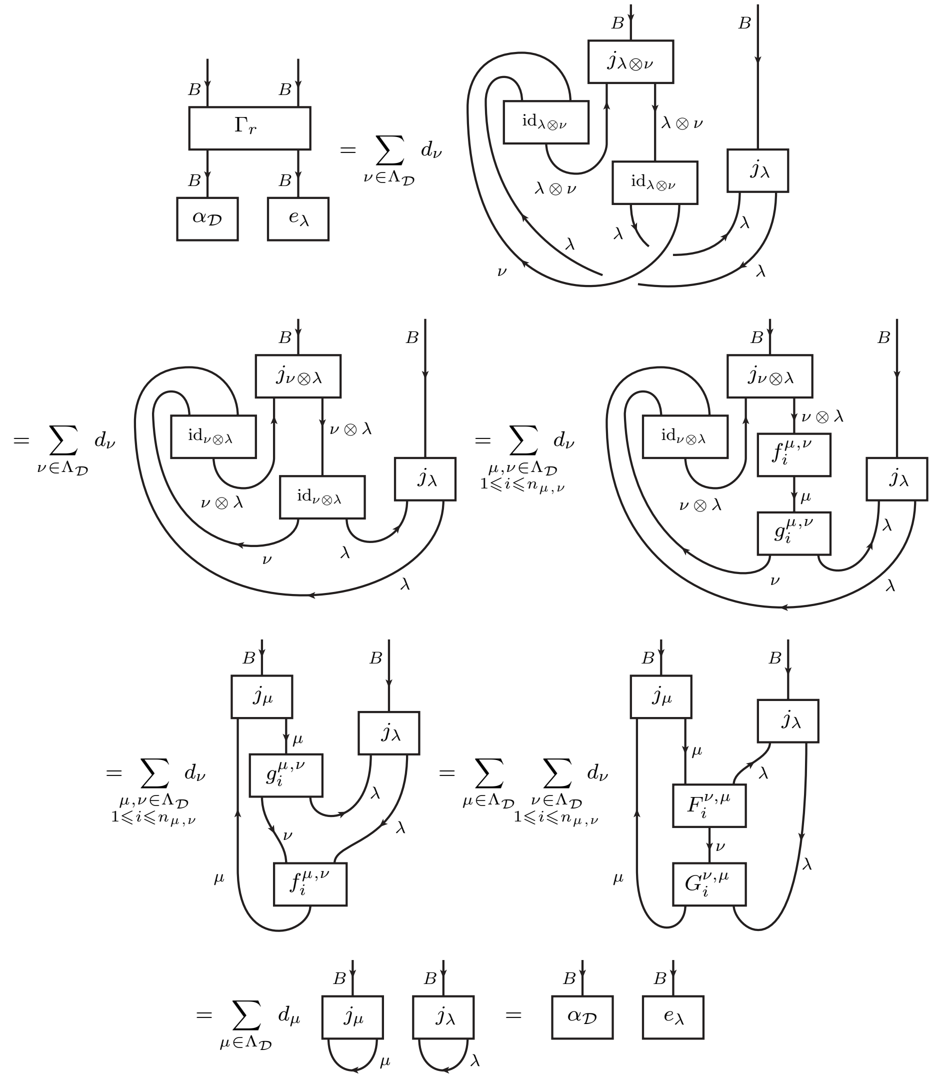


FIGURE 7. $\Gamma_r(\alpha_{\mathcal{D}} \otimes e_\lambda) = \alpha_{\mathcal{D}} \otimes e_\lambda$.

Set $\Delta_\pm = \sum_{\lambda \in \Lambda} v_\lambda^{\pm 1} \dim_q(\lambda)^2 \in \Bbbk$, where $v_\lambda \in \Bbbk$ is the (invertible) scalar defined by $\theta_\lambda = v_\lambda \, \mathrm{id}_\lambda$. Recall (see [**26**, **4**]) that the Reshetikhin–Turaev invariant of 3-manifolds is well defined when $\Delta_+ \neq 0 \neq \Delta_-$. Moreover, if L is a framed link in S^3, it is given by

$$\mathrm{RT}_{\mathcal{B}}(M_L) = \Delta_+^{b_-(L) - n_L} \, \Delta_-^{-b_-(L)} \sum_{c \in \mathrm{Col}(L)} \left(\prod_{j=1}^{n} \dim_q(c(L_j)) \right) F(L, c) \in \Bbbk.$$

Here $\mathrm{Col}(L)$ is the set of functions $c \colon \{L_1, \dots, L_n\} \to \Lambda$, where $L_1, \dots, L_n$ are the components of L, and $F(L, c) \in \mathrm{End}_{\mathcal{B}}(\mathbb{1}) = \Bbbk$ is the morphism represented by a plane diagram of L where the component L_j is colored with the object $c(L_j)$.

By Theorem 3.4, $\alpha_{\mathcal{B}} = \sum_{\lambda \in \Lambda} \dim_q(\lambda) e_\lambda \in \mathcal{AK}(\mathcal{B})$, where e_λ is defined as in § 3.2.

Corollary 3.9. *Suppose that $\Delta_\pm \neq 0$. Then $\alpha_\mathcal{B}$ is a normalizable algebraic Kirby element of $\mathcal{B}$ and $\tau_\mathcal{B}(M;\alpha_\mathcal{B}) = \mathrm{RT}_\mathcal{B}(M)$ for any 3-manifold M.*

Proof. We have

$$\Theta_\pm \alpha_\mathcal{B} = \sum_{\lambda\in\Lambda} \dim_q(\lambda)\, \mathrm{ev}_\lambda(\mathrm{id}_{\lambda^*}\otimes\theta_\lambda^{\pm 1})\widetilde{\mathrm{coev}}_\lambda = \sum_{\lambda\in\Lambda} v_\lambda^{\pm 1}\dim_q(\lambda)^2 = \Delta_\pm \neq 0.$$

Therefore, since $\alpha_\mathcal{B}\in\mathcal{AK}(\mathcal{B})$, one gets $\alpha_\mathcal{B}\in\mathcal{AK}(\mathcal{B})^{\mathrm{norm}}$.

Let $L = L_1\cup\ldots\cup L_n$ be a framed link in S^3. Let T_L be a ribbon n-handle such that L is isotopic to $T_L\circ(\cup\otimes\ldots\otimes\cup)$, where the ith cup corresponds to the component L_i. Then $F(L,c) = \phi_{T_L}\circ(j_{c(L_1)}\,\widetilde{\mathrm{coev}}_{c(L_1)}\otimes\ldots\otimes j_{c(L_n)}\,\widetilde{\mathrm{coev}}_{c(L_n)}) = \phi_{T_L}\circ(e_{c(L_1)}\otimes\ldots\otimes e_{c(L_n)})$ for any $c\in\mathrm{Col}(L)$. Therefore

$$\begin{aligned}
&\sum_{c\in\mathrm{Col}(L)}\Big(\prod_{j=1}^n \dim_q(c(L_j))\Big)F(L,\lambda)\\
&\qquad= \sum_{c\in\mathrm{Col}(L)} \phi_{T_L}\circ\big(\dim_q(c(L_1))\,e_{c(L_1)}\otimes\ldots\otimes\dim_q(c(L_n))e_{c(L_n)}\big)\\
&\qquad= \phi_{T_L}\circ\Big(\sum_{\lambda_1\in\Lambda}\dim_q(\lambda_1)e_{\lambda_1}\otimes\ldots\otimes\sum_{\lambda_n\in\Lambda}\dim_q(\lambda_n)e_{\lambda_n}\Big)\\
&\qquad= \phi_{T_L}\circ(\alpha_\mathcal{B}\otimes\ldots\otimes\alpha_\mathcal{B}).
\end{aligned}$$

Hence

$$\begin{aligned}
\mathrm{RT}_\mathcal{B}(M_L) &= \Delta_+^{b_-(L)-n_L}\,\Delta_-^{-b_-(L)}\sum_{\lambda\in\mathrm{Col}(L)}\Big(\prod_{j=1}^n\dim_q(\lambda(L_j))\Big)F(L,\lambda)\\
&= (\Theta_+\alpha_\mathcal{B})^{b_-(L)-n_L}\,(\Theta_-\alpha_\mathcal{B})^{-b_-(L)}\,\phi_{T_L}\circ(\alpha_\mathcal{B}\otimes\ldots\otimes\alpha_\mathcal{B})\\
&= \tau_\mathcal{B}(M_L;\alpha_\mathcal{B}).
\end{aligned}$$

□

3.4. *Semisimplification of ribbon categories*

Let $\mathcal{C}$ be a ribbon $\Bbbk$-category. For any objects $X,Y\in\mathcal{C}$, recall that $\mathrm{Negl}_\mathcal{C}(X,Y)$ denotes the $\Bbbk$-space of negligible morphisms of $\mathcal{C}$ from X to Y (see § 1.4). Let $\mathcal{C}^s$ be the category whose objects are the same as in $\mathcal{C}$, and whose morphisms are

$$\mathrm{Hom}_{\mathcal{C}^s}(X,Y) = \mathrm{Hom}_\mathcal{C}(X,Y)/\mathrm{Negl}_\mathcal{C}(X,Y)$$

for any objects $X,Y\in\mathcal{C}^s$. The composition, monoidal structure, braiding, twist, and duality of $\mathcal{C}^s$ are induced by those of $\mathcal{C}$.

When $\mathcal{C}$ has finite-dimensional Hom's $\Bbbk$-spaces, the category $\mathcal{C}^s$ is a semisimple ribbon $\Bbbk$-category, called the *semisimplification* of $\mathcal{C}$, and the simple objects of $\mathcal{C}^s$ are the indecomposable objects of $\mathcal{C}$ with non-zero quantum dimension; see [**5**].

Let $\pi\colon\mathcal{C}\to\mathcal{C}^s$ be the functor defined by $\pi(X) = X$ and $\pi(f) = f+\mathrm{Negl}_\mathcal{C}(X,Y)$ for any object X and any morphism $f\colon X\to Y$ in $\mathcal{C}$. This is a surjective ribbon functor. Note that π is bijective on the objects.

3.5. *Algebraic Kirby elements from semisimplification*

Let $\mathcal{C}$ be a ribbon $\Bbbk$-category which admits a coend (A,i) for the functor (1.5) and whose Hom's spaces are finite-dimensional. Denote by $\mathcal{C}^s$ the semisimplification of $\mathcal{C}$ and let $\pi\colon\mathcal{C}\to\mathcal{C}^s$ be its associated surjective ribbon functor (see § 3.4).

Let $\mathcal{B}$ be a finitely semisimple ribbon full subcategory of $\mathcal{C}^s$ whose simple objects are scalar. Let Λ be a (finite) set of representatives of isomorphism classes of simple objects of $\mathcal{B}$ containing $\mathbb{1}$. For any object X of $\mathcal{C}^s$, we denote by $\pi^{-1}(X)$ the (unique) object of $\mathcal{C}$ such that $\pi(\pi^{-1}(X)) = X$.

Let $B = \bigoplus_{\lambda\in\Lambda} \lambda^* \otimes \lambda$. In particular, there exist morphisms $p_\lambda \colon B \to \lambda^* \otimes \lambda$ and $q_\lambda \colon \lambda^* \otimes \lambda \to B$ of $\mathcal{B}$ such that $\mathrm{id}_B = \sum_{\lambda\in\Lambda} q_\lambda p_\lambda$ and $p_\lambda q_\mu = \delta_{\lambda,\mu}\, \mathrm{id}_{\lambda^*\otimes\lambda}$. For any object X of $\mathcal{B}$, we let $j_X \colon X^* \otimes X \to B$ as in § 3.2. Recall that (B, j) is a coend of the functor (1.5) for $\mathcal{B}$, and that $j_\lambda = q_\lambda$ for any $\lambda \in \Lambda$ (see § 3.2).

Since $\mathcal{B}$ is a ribbon full subcategory of $\mathcal{C}^s$ and π is a ribbon functor, the objects B and $\pi(A)$ are Hopf algebras in $\mathcal{C}^s$. Set

$$\varphi = \sum_{\lambda\in\Lambda} \pi(i_{\pi^{-1}(\lambda)}) p_\lambda \in \mathrm{Hom}_{\mathcal{C}^s}(B, \pi(A)). \tag{3.6}$$

LEMMA 3.10. *The map $\varphi \colon B \to \pi(A)$ is a Hopf algebra morphism such that $\pi(i_X) = \varphi j_{\pi(X)}$ for all objects X of $\mathcal{C}$ with $\pi(X) \in \mathcal{B}$.*

Proof. Let X be an object of $\mathcal{C}$ such that $\pi(X) \in \mathcal{B}$. Since $\mathcal{B}$ is semisimple, we can write $\pi(X) = \bigoplus_{k\in K} \lambda_k$, where K is a finite set and $\lambda_k \in \Lambda$. Recall that $j_{\pi(X)} = \sum_{k\in K} q_{\lambda_k}(Q_k^* \otimes P_k)$, where $P_k \colon \pi(X) \to \lambda_k$ and $Q_k \colon \lambda_k \to \pi(X)$ are morphisms in $\mathcal{B}$ such that $\mathrm{id}_{\pi(X)} = \sum_{k\in K} P_k Q_k$ and $P_k Q_l = \delta_{k,l}\, \mathrm{id}_{\lambda_k}$. For any $k \in K$, since π is surjective, there exist morphisms $f_k \colon X \to \pi^{-1}(\lambda_k)$ and $g_k \colon \pi^{-1}(\lambda_k) \to X$ in $\mathcal{C}$ such that $\pi(f_k) = P_k$ and $\pi(g_k) = Q_k$. Then, using the dinaturality of i and since the functor π is ribbon, we have

$$\begin{aligned}
\varphi j_{\pi(X)} &= \sum_{\lambda\in\Lambda}\sum_{k\in K} \pi(i_{\pi^{-1}(\lambda)}) p_\lambda q_{\lambda_k}(Q_k^* \otimes P_k) = \sum_{\lambda\in\Lambda}\sum_{\substack{k\in K\\ \lambda_k=\lambda}} \pi(i_{\pi^{-1}(\lambda)})(Q_k^* \otimes P_k)\\
&= \sum_{\lambda\in\Lambda}\sum_{\substack{k\in K\\ \lambda_k=\lambda}} \pi\big(i_{\pi^{-1}(\lambda)}(g_k^* \otimes f_k)\big) = \sum_{\lambda\in\Lambda}\sum_{\substack{k\in K\\ \lambda_k=\lambda}} \pi\big(i_X(\mathrm{id}_{X^*} \otimes g_k f_k)\big)\\
&= \pi(i_X)\Big(\mathrm{id}_{\pi(X)^*} \otimes \sum_{k\in K} Q_k P_k\Big) = \pi(i_X)(\mathrm{id}_{\pi(X)^*} \otimes \mathrm{id}_{\pi(X)}) = \pi(i_X).
\end{aligned}$$

Let us verify that φ is a Hopf algebra morphism. Let $\lambda, \mu \in \Lambda$. Set $U = \pi^{-1}(\lambda)$ and $V = \pi^{-1}(\mu)$. We have $\varepsilon_{\pi(A)} \varphi j_\lambda = \pi(\varepsilon_A i_U) = \pi(\mathrm{ev}_U) = \mathrm{ev}_\lambda = \varepsilon_B j_\lambda$,

$$\begin{aligned}
\Delta_{\pi(A)} \varphi j_\lambda &= \pi(\Delta_A i_U) = \pi((i_U \otimes i_U)(\mathrm{id}_{U^*} \otimes \mathrm{coev}_U \otimes \mathrm{id}_U))\\
&= (\varphi j_\lambda \otimes \varphi j_\lambda)(\mathrm{id}_{\lambda^*} \otimes \mathrm{coev}_\lambda \otimes \mathrm{id}_\lambda) = (\varphi \otimes \varphi)\Delta_B j_\lambda,
\end{aligned}$$

$$\begin{aligned}
m_{\pi(A)}(\varphi j_\lambda \otimes \varphi j_\mu) &= \pi(m_A(i_U \otimes i_V)) = \pi(i_{V\otimes U}(\gamma_{U,V} \otimes \mathrm{id}_{V\otimes U})(\mathrm{id}_{U^*} \otimes c_{U,V^*\otimes V}))\\
&= \varphi j_{\mu\otimes\lambda}(\gamma_{\lambda,\mu} \otimes \mathrm{id}_{\mu\otimes\lambda})(\mathrm{id}_{\lambda^*} \otimes c_{\lambda,\mu^*\otimes\mu})) = \varphi m_B(j_\lambda \otimes j_\mu),
\end{aligned}$$

and

$$\begin{aligned}
S_{\pi(A)} \varphi j_\lambda &= \pi(S_A i_U) = \pi((\mathrm{ev}_U \otimes i_{U*})(\mathrm{id}_{U^*} \otimes c_{U^{**},U} \otimes \mathrm{id}_{U^*})(\mathrm{coev}_{U^*} \otimes c_{U^*,U}))\\
&= (\mathrm{ev}_\lambda \otimes \varphi j_{\lambda^*})(\mathrm{id}_{\lambda^*} \otimes c_{\lambda^{**},\lambda} \otimes \mathrm{id}_{\lambda^*})(\mathrm{coev}_{\lambda^*} \otimes c_{\lambda^*,\lambda}) = \varphi S_B j_\lambda.
\end{aligned}$$

Therefore, since $\mathrm{id}_B = \sum_{\lambda\in\Lambda} j_\lambda p_\lambda$, we get $\varepsilon_{\pi(A)}\varphi = \varepsilon_B$, $\Delta_{\pi(A)}\varphi = (\varphi \otimes \varphi)\Delta_B$, $m_{\pi(A)}(\varphi \otimes \varphi) = \varphi m_B$, and $S_{\pi(A)}\varphi = \varphi S_B$. Finally, we conclude by remarking that $\varphi\eta_B = \varphi j_{\mathbb{1}} = \pi(i_{\mathbb{1}}) = \pi(\eta_A) = \eta_{\pi(A)}$. □

From Lemma 3.10 and Proposition 2.12, we get the following result.

Corollary 3.11. *Let* $\beta \in \mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, B)$ *and* $\alpha \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$ *with* $\pi(\alpha) = \varphi\beta$.

(a) *If* $\beta \in \mathcal{AK}(\mathcal{B})$, *then* $\alpha \in \mathcal{AK}(\mathcal{C})$ *and* $\tau_{\mathcal{C}}(L;\alpha) = \tau_{\mathcal{B}}(L;\beta)$ *for any framed link* L.

(b) *If* $\beta \in \mathcal{AK}(\mathcal{B})^{\mathrm{norm}}$, *then* $\alpha \in \mathcal{AK}(\mathcal{C})^{\mathrm{norm}}$ *and* $\tau_{\mathcal{C}}(M;\alpha) = \tau_{\mathcal{B}}(M;\beta)$ *for any* 3*-manifold* M.

Remark 3.12. By Corollaries 3.9 and 3.11, we get, in particular, the fact that the Reshetikhin–Turaev invariant defined from a finitely semisimple ribbon full subcategory of the semisimple quotient of a ribbon category $\mathcal{C}$ can be directly defined in $\mathcal{C}$ by picking its corresponding algebraic Kirby element.

Remark 3.13. By Corollary 3.11, we have

$$\bigcup_{\mathcal{B}} \pi^{-1}\big(\varphi_{\mathcal{B}}(\mathcal{AK}(\mathcal{B}))\big) \subset \mathcal{AK}(\mathcal{C}) \quad \text{and} \quad \bigcup_{\mathcal{B}} \pi^{-1}\big(\varphi_{\mathcal{B}}(\mathcal{AK}(\mathcal{B})^{\mathrm{norm}})\big) \subset \mathcal{AK}(\mathcal{C})^{\mathrm{norm}},$$

where $\mathcal{B}$ runs over finitely semisimple ribbon full subcategories of $\mathcal{C}^s$ whose simple objects are scalar, and $\varphi_{\mathcal{B}}$ is the morphism (3.6) corresponding to $\mathcal{B}$. We will see in § 4 that these inclusions may be strict (see Remark 4.11). This means that the semisimplification process 'lacks' some invariants.

4. *The case of categories of representations*

In this section, we focus on the case of the category rep_H of representations of a finite-dimensional ribbon Hopf algebra H. In particular, we describe $\mathcal{AK}(\mathrm{rep}_H)$ in purely algebraic terms. One of the interests of such a description is to avoid the representation theory of H, which may be of wild type. Moreover, we show that the 3-manifolds invariants obtained with these Kirby elements can be computed by using the Kauffman–Radford algorithm (even in the non-unimodular case).

4.1. *Finite-dimensional Hopf algebras*

All considered algebras are supposed to be over the field $\mathbb{k}$. Let H be a finite-dimensional Hopf algebra. Recall that a *left* (respectively *right*) *integral* for H is an element $\Lambda \in H$ such that $x\Lambda = \varepsilon(x)\Lambda$ (respectively $\Lambda x = \varepsilon(x)\Lambda$) for all $x \in H$. A left (respectively right) integral for H^* is then an element $\lambda \in H^*$ such that $x_{(1)}\lambda(x_{(2)}) = \lambda(x)\,1$ (respectively $\lambda(x_{(1)})x_{(2)} = \lambda(x)\,1$) for all $x \in H$. Since H is finite-dimensional, the space of left (respectively right) integrals for H is one-dimensional, and there always exist a non-zero right integral λ for H^* and a non-zero left integral Λ for H such that $\lambda(\Lambda) = \lambda(S(\Lambda)) = 1$; see [**20**, Proposition 3].

By [**23**, Corollary 2], the space H^* endowed with the right H-action defined, for any $f \in H^*$ and $h, x \in H$, by

$$\langle f \cdot h, x\rangle = \langle f, hx\rangle, \tag{4.1}$$

is a free H-module of rank 1 with basis every non-zero right integral λ for H^*. Likewise, H endowed with the right H^*-action $\leftharpoonup$ defined, for any $f \in H^*$ and $x \in H$, by

$$x \leftharpoonup f = f(x_{(1)})x_{(2)}, \tag{4.2}$$

is a free H^*-module of rank 1 with basis $S(\Lambda)$, where Λ is a non-zero left integral for H.

LEMMA 4.1. *Let λ be a right integral for H^* and Λ be a left integral for H such that $\lambda(\Lambda) = \lambda(S(\Lambda)) = 1$. Let $a \in H$ and $f \in H^*$. Then*
(a) $a = \lambda(a\Lambda_{(1)})\, S(\Lambda_{(2)}) = \lambda(S(\Lambda_{(2)})a)\, \Lambda_{(1)}$,
(b) $f = \lambda \cdot a$ *if and only if* $a = (f \otimes S)\Delta(\Lambda)$.

Proof. Let us prove part (a). Since λ is a right integral for H^* and Λ is a left integral for H such that $\lambda(\Lambda) = 1$, we have

$$\begin{aligned}\lambda(a\Lambda_{(1)})\, S(\Lambda_{(2)}) &= \lambda(a_{(1)}\Lambda_{(1)})\, a_{(2)}\Lambda_{(2)}\, S(\Lambda_{(3)})\\ &= \lambda(a_{(1)}\Lambda_{(1)})\, a_{(2)}\varepsilon(\Lambda_{(2)})\\ &= \lambda(a_{(1)}\Lambda)\, a_{(2)} = \lambda(\Lambda)\,\varepsilon(a_{(1)})a_{(2)} = a.\end{aligned}$$

The second equality of part (a) can be proved similarly.

Let us prove part (b). If $f = \lambda \cdot a$ then, by part (a), we have

$$a = \lambda(a\Lambda_{(1)})\, S(\Lambda_{(2)}) = f(\Lambda_{(1)})\, S(\Lambda_{(2)}) = (f \otimes S)\Delta(\Lambda).$$

Conversely, if $a = (f \otimes S)\Delta(\Lambda)$ then, by part (a), we have

$$\lambda(ax) = \lambda(f(\Lambda_{(1)})\, S(\Lambda_{(2)})x) = f(\lambda(S(\Lambda_{(2)})x)\Lambda_{(1)}) = f(x)$$

for all $x \in H$, and so $f = \lambda \cdot a$. □

Recall that an element $h \in H$ is *grouplike* if $\Delta(g) = g\otimes g$ and $\varepsilon(g) = 1$. We denote by $G(H)$ the space of grouplike elements of H. Recall (see [**1**]) that there exist a unique grouplike element g of H such that $x_{(1)}\lambda(x_{(2)}) = \lambda(x)\, g$ for any $x \in H$ and any right integral $\lambda \in H^*$, and a unique grouplike element $\nu \in G(H^*) = \mathrm{Alg}(H, \Bbbk)$ such that $\Lambda x = \nu(x)\Lambda$ for any $x \in H$ and any left integral $\Lambda \in H$. The element $g \in H$ (respectively $\nu \in H^*$) is called the *distinguished grouplike element* of H (respectively of H^*). The Hopf algebra H is said to be *unimodular* if its integrals are two-sided, that is, if $\nu = \varepsilon$.

By [**23**, Theorem 3 and Proposition 3], right integrals for H^* and distinguished grouplike elements of H and H^* are related, for all $x, y \in H$, by

$$\lambda(xy) = \lambda(S^2(y \leftharpoonup \nu)x) = \lambda((S^2(y) \leftharpoonup \nu)x) \quad \text{and} \quad \lambda(S(x)) = \lambda(gx). \tag{4.3}$$

4.2. *Quasitriangular Hopf algebras*

Following [**8**], a Hopf algebra H is *quasitriangular* if it is endowed with an invertible element $R \in H \otimes H$ (the *R-matrix*) such that $R\Delta(x) = \sigma\Delta(x)R$ for any $x \in H$, where $\sigma\colon H \otimes H \to H \otimes H$ denotes the usual flip map, and

$$(\mathrm{id}_H \otimes \Delta)(R) = R_{13}R_{12} \quad \text{and} \quad (\Delta \otimes \mathrm{id}_H)(R) = R_{13}R_{23}.$$

The R-matrix satisfies

$$(\varepsilon \otimes \mathrm{id}_H)(R) = (\mathrm{id}_H \otimes \varepsilon)(R) = 1, \tag{4.4}$$

$$(S \otimes \mathrm{id}_H)(R) = R^{-1} = (\mathrm{id}_H \otimes S^{-1})(R). \tag{4.5}$$

The *Drinfeld element* u associated to R is $u = m(S \otimes \mathrm{id}_H)\sigma(R) \in H$. It is invertible, with $u^{-1} = m(\mathrm{id}_H \otimes S^2)(R_{21})$ and satisfies $S^2(x) = uxu^{-1}$ for all $x \in H$.

Let H be a finite-dimensional quasitriangular Hopf algebra and let $\nu \in H^*$ be the distinguished grouplike element of H^*. Set $h_\nu = (\mathrm{id}_H \otimes \nu)(R) \in H$. One easily verifies that h_ν is grouplike.

4.3. *Ribbon Hopf algebras*

Following [**24**], we say that a *ribbon* Hopf algebra is a quasitriangular Hopf algebra H endowed with a central invertible element $\theta \in H$ (the *twist*) such that $S(\theta) = \theta$ and $\Delta(\theta) = (\theta \otimes \theta)R_{21}R$. Note that the twist satisfies

$$\theta^{-2} = uS(u) = S(u)u \quad \text{and} \quad \varepsilon(\theta_1) = 1.$$

Set $G = \theta u \in H$. Then G is grouplike and satisfies

$$S(u) = G^{-1}uG^{-1} \quad \text{and} \quad S^2(x) = GxG^{-1} \ \text{ for all } x \in H. \tag{4.6}$$

The element G is called the *special grouplike element* of H. By [**21**, Theorem 2] and (4.6), the special grouplike element G of a finite-dimensional ribbon Hopf algebra H is related to the distinguished grouplike element g of H and to h_ν by $g = G^2 h_\nu$. Together with (4.3), this implies that

$$\lambda(S(x)) = \lambda(G^2 h_\nu x) \quad \text{for all } x \in H. \tag{4.7}$$

4.4. *Category of representations of a ribbon Hopf algebra*

Let H be a ribbon Hopf algebra with R-matrix R and twist θ. Denote by rep_H the $\Bbbk$-category of finite-dimensional left H-modules and H-linear homomorphisms. The category rep_H is a monoidal $\Bbbk$-category with tensor product and unit object defined in the usual way using the comultiplication and counit of H. The category rep_H possesses a left duality: for any module $M \in \mathrm{rep}_H$, set $M^* = \mathrm{Hom}_\Bbbk(M, \Bbbk)$, where $h \in H$ acts as the transpose of $x \in M \mapsto S(h)x \in M$. The duality morphism $\mathrm{ev}_M \colon M^* \otimes M \to \mathbb{1} = \Bbbk$ is the evaluation pairing and, if $(e_k)_k$ is a basis of M with dual basis $(e_k^*)_k$, then $\mathrm{coev}(1_\Bbbk) = \sum_k e_k \otimes e_k^*$. The category rep_H is braided: for modules $M, N \in \mathrm{rep}_H$, the braiding $c_{M,N} \colon M \otimes N \to N \otimes M$ is the composition of multiplication by R and the flip map $\sigma_{M,N} \colon M \otimes N \to N \otimes M$. The category rep_H is ribbon: for any module $M \in \mathrm{rep}_H$, the twist $\theta_M \colon M \to M$ is the multiplication by θ. Recall, see § 1.1, that rep_H possesses a right duality $M \in \mathrm{rep}_H \mapsto (M^*, \widetilde{\mathrm{ev}}_M, \widetilde{\mathrm{coev}}_M)$. Finally, the $\Bbbk$-category rep_H is pure, that is, $\mathrm{End}_{\mathrm{rep}_H}(\Bbbk) = \Bbbk$. Hence rep_H is a ribbon $\Bbbk$-category in the sense of § 1.2.

LEMMA 4.2. *Let G be the special grouplike element of H and M be a finite-dimensional left H-module. Then:*

(a) $\widetilde{\mathrm{ev}}_M(m \otimes f) = f(Gm)$ *for any* $f \in M^*$ *and* $m \in M$;
(b) $\widetilde{\mathrm{coev}}_M = (\mathrm{id}_{M^*} \otimes G^{-1}\mathrm{id}_M)\sigma_{M,M^*}\mathrm{coev}_M$.

Proof. Write $R = \sum_i a_i \otimes b_i$. Recall that $u = \sum_i S(b_i)a_i$. Then

$$\begin{aligned}\widetilde{\mathrm{ev}}_M(m \otimes f) &= \mathrm{ev}_M\, c_{M,M^*}(\theta_M \otimes \mathrm{id}_{M^*})(m \otimes f)\\ &= \sum_i \mathrm{ev}_M(b_i f \otimes a_i\theta m) = \sum_i f(S(b_i)a_i\theta m)\\ &= f(u\theta m) = f(Gm).\end{aligned}$$

Let $(e_k)_k$ be a basis of M and $(e_k^*)_k$ be its dual basis. Note that if g is any $\Bbbk$-linear endomorphism of M, then $\sum_k g^*(e_k^*) \otimes e_k = \sum_k e_k^* \otimes g(e_k)$. For any $h \in H$, denote by $\rho(h)$ the $\Bbbk$-linear endomorphism of M defined by $m \in M \mapsto hm \in M$. Recall that $R^{-1} = \sum_i S(a_i) \otimes b_i$ and $u^{-1} = \sum_i b_i S^2(a_i)$. Then

$$\begin{aligned}
\widetilde{\mathrm{coev}}_M(1_\Bbbk) &= (\mathrm{id}_{M^*} \otimes \theta_M^{-1})(c_{M^*,M})^{-1}\mathrm{coev}_M(1_\Bbbk) \\
&= \sum_{k,i} S(a_i)e_k^* \otimes \theta^{-1} b_i \cdot e_k \\
&= \sum_i \big(\mathrm{id}_{M^*} \otimes \rho(\theta^{-1}b_i)\big)\Big(\sum_k e_k^* \otimes \rho(S^2(a_i))(e_k)\Big) \\
&= \Big(\mathrm{id}_{M^*} \otimes \rho\Big(\theta^{-1}\sum_i b_i S^2(a_i)\Big)\Big)\Big(\sum_k e_k^* \otimes e_k\Big) \\
&= \big(\mathrm{id}_{M^*} \otimes \rho(\theta^{-1}u^{-1})\big)\sigma_{M,M^*}\mathrm{coev}_M(1_k) \\
&= (\mathrm{id}_{M^*} \otimes G^{-1}\mathrm{id}_M)\sigma_{M,M^*}\mathrm{coev}_M(1_k).
\end{aligned}$$

This completes the proof of the lemma. □

We immediately deduce from Lemma 4.2 that, in the category rep_H, we have

$$\mathrm{tr}_q(f) = \mathrm{Tr}(Gf) = \mathrm{Tr}(G^{-1}f) \quad \text{and} \quad \dim_q(M) = \mathrm{Tr}(G\,\mathrm{id}_M) = \mathrm{Tr}(G^{-1}\mathrm{id}_M)$$

for all modules $M \in \mathrm{rep}_H$ and all H-linear endomorphisms f of M, where Tr denotes the usual trace of $\Bbbk$-linear endomorphisms.

4.5. *Braided Hopf algebra associated to ribbon Hopf algebras*

Let H be a finite-dimensional ribbon Hopf algebra. The ribbon $\Bbbk$-category rep_H of finite-dimensional left H-modules possesses a coend (A, i) for the functor (1.5). More precisely, $A = H^* = \mathrm{Hom}_\Bbbk(H, \Bbbk)$ as a $\Bbbk$-space and is endowed with the coadjoint left H-action $\triangleright$ given, for any $f \in A$ and $h, x \in H$, by

$$\langle h \triangleright f, x\rangle = \langle f, S(h_{(1)})xh_{(2)}\rangle, \tag{4.8}$$

where $\langle\ ,\ \rangle$ denotes the usual pairing between a $\Bbbk$-space and its dual. Given a module $M \in \mathrm{rep}_H$, the map $i_M\colon M^* \otimes M \to A$ is given by

$$\langle i_M(l \otimes m), x\rangle = \langle l, xm\rangle, \tag{4.9}$$

for all $l \in M^*$, $m \in M$, and $x \in H$.

Lemma 4.3. *If ξ is a dinatural transformation from the functor* (1.5) *for rep_H to a module $Z \in \mathrm{rep}_H$, then the* (*unique*) *morphism $r\colon A \to Z$ such that $\xi_M = ri_M$ for all $M \in \mathrm{rep}_H$ is given by $f \in A = H^* \mapsto r(f) = \xi_H(f \otimes 1_H)$.*

Recall (see § 1.6) that A is a Hopf algebra in rep_H. Using Lemma 4.3, one can describe the structural morphisms of A explicitly in terms of the structure maps of the Hopf algebra H. Nevertheless, it is more convenient to write down its pre-dual structural morphisms: for example, since $A = H^*$ as a $\Bbbk$-space and H is finite-dimensional, the pre-dual of the multiplication $m_A\colon A \otimes A \to A$ of A is a morphism $\Delta^{\mathrm{Bd}}\colon H \to H \otimes H$ such that $(\Delta^{\mathrm{Bd}})^* = m_A$. This yields a $\Bbbk$-space $H^{\mathrm{Bd}} = H$ endowed with a comultiplication $\Delta^{\mathrm{Bd}}\colon H^{\mathrm{Bd}} \to H^{\mathrm{Bd}} \otimes H^{\mathrm{Bd}}$, a counit $\varepsilon^{\mathrm{Bd}}\colon H_1^{\mathrm{Bd}} \to \Bbbk$, a unit $\eta^{\mathrm{Bd}}\colon \Bbbk \to H^{\mathrm{Bd}}$, a multiplication $m^{\mathrm{Bd}}\colon H^{\mathrm{Bd}} \otimes H^{\mathrm{Bd}} \to H^{\mathrm{Bd}}$, and an antipode $S^{\mathrm{Bd}}\colon H^{\mathrm{Bd}} \to H^{\mathrm{Bd}}$. These structure maps, described in the following lemma, satisfy the same axioms as those of a Hopf algebra except that the usual flip maps are replaced by the braiding of rep_H. The space H^{Bd} is called the *braided Hopf algebra associated to* H; see [**19**, **17**].

LEMMA 4.4 (cf. [**16**]). *The braided Hopf algebra H^{Bd} associated to H can be described as follows: $H^{\mathrm{Bd}} = H$ as an algebra, $\varepsilon^{\mathrm{Bd}} = \varepsilon$, and*

$$\Delta^{\mathrm{Bd}}(x) = \sum_i x_{(2)}a_i \otimes S(b_{i(1)})x_{(1)}b_{i(2)} = \sum_i S(a_{i(1)})x_{(1)}a_{i(2)} \otimes S(b_i)x_{(2)},$$
$$S_\alpha^{\mathrm{Bd}}(x) = \sum_i S(a_i)\theta^2 S(x)ub_i = \sum_i S(a_i)S(x)S(u)^{-1}b_i,$$

for any $x \in H$, where $R = \sum_i a_i \otimes b_i$ is the R-matrix, u the Drinfeld element, and θ the twist of H.

4.6. *Algebraic Kirby elements of ribbon Hopf algebras*

Throughout this section, H will denote a finite-dimensional ribbon Hopf algebra with R-matrix $R \in H \otimes H$ and twist $\theta \in H$. Let $u \in H$ be the Drinfeld element of H, $G = \theta u$ be the special grouplike element of H, $\lambda \in H^*$ be a non-zero right integral for H^*, $\Lambda \in H$ be a non-zero left integral for H such that $\lambda(\Lambda) = \lambda(S(\Lambda)) = 1$, $g \in G(H)$ be the distinguished grouplike element of H, $\nu \in G(H^*) = \mathrm{Alg}(H, \Bbbk)$ be the distinguished grouplike element of H^*, and $h_\nu = (\mathrm{id}_H \otimes \nu)(R) \in G(H)$. Let (A, i) be the coend of the functor (1.5) for rep_H, and let H^{Bd} be the braided Hopf algebra associated to H. By § 4.5, $A = H^*$ is endowed with the coadjoint left H-action $\triangleright$ and A is a Hopf algebra in the category rep_H whose structure maps are dual to those of H^{Bd}. Recall that $\cdot$ denotes the right action of H on H^* defined in (4.1) and $\leftharpoonup$ denotes the right H^*-action on H defined in (4.2).

Let $\phi\colon H \to \mathrm{Hom}_\Bbbk(\Bbbk, H^*)$, $T\colon H \to H$, and $\psi\colon H^{\otimes n} \to \mathrm{Hom}_\Bbbk(H^{*\otimes n}, \Bbbk)$ be the maps defined, for $z \in H$, $X \in H^{\otimes n}$, and $F \in H^{*\otimes n}$, by

$$\phi_z(1_\Bbbk) = \lambda \cdot z, \quad T(z) = (S(z) \leftharpoonup \nu)h_\nu, \quad \text{and} \quad \psi_X(F) = \langle F, X\rangle.$$

Denote by $\triangleleft$ be the right action of H on $H^{\otimes n}$ given by

$$(x_1 \otimes \ldots \otimes x_n) \triangleleft h = S(h_{(1)})x_1 h_{(2)} \otimes \ldots \otimes S(h_{(2n-1)})x_n h_{(2n)}$$

for any $h \in H$ and $x_1, \ldots, x_n \in H$. Set

$$L(H) = \{z \in H \mid (x \leftharpoonup \nu)z = zx \text{ for any } x \in H\},$$
$$N(H) = \{z \in L(H) \mid \lambda(za) = 0 \text{ for any } a \in Z(H)\},$$
$$V_n(H) = \{X \in H^{\otimes n} \mid X \triangleleft h = \varepsilon(h)X \text{ for any } h \in H\}.$$

Note that if H is unimodular, then $L(H) = Z(H)$ and $T(z) = S(z)$ for all $z \in H$.

By the definition of $L(H)$ and by (4.3), we have

$$\lambda(zxy) = \lambda(zS^2(y)x) \tag{4.10}$$

for all $z \in L(H)$ and $x, y \in H$.

LEMMA 4.5. *We have the following.*

(a) *The map ϕ is a $\Bbbk$-isomorphism with inverse given by*

$$\alpha \in \mathrm{Hom}_\Bbbk(\Bbbk, H^*) \mapsto \phi^{-1}(\alpha) = (\alpha(1_\Bbbk) \otimes S)\Delta(\Lambda) \in H.$$

(b) *The map ϕ induces a $\Bbbk$-isomorphism between $L(H)$ and* $\mathrm{Hom}_{\mathrm{rep}_H}(\Bbbk, A)$.

(c) *The map* ϕ *induces a* $\Bbbk$*-isomorphism between* $N(H)$ *and* $\mathrm{Negl}_{\mathrm{rep}_H}(\Bbbk, A)$.
(d) *The map* ψ *induces a* $\Bbbk$*-isomorphism between* $V_n(H)$ *and* $\mathrm{Hom}_{\mathrm{rep}_H}(A^{\otimes n}, \Bbbk)$.
(e) $V_1(H) = Z(H)$.

Proof. Let us prove part (a). Since $(H^*, \cdot)$ is a free right H-module of rank 1 with basis λ, ϕ is an isomorphism. The expression of ϕ^{-1} follows from Lemma 4.1(b).

Let us prove part (b). Let $z \in H$. We have to show that $z \in L(H)$ if and only if ϕ_z is H-linear. Suppose first that ϕ_z is H-linear. For all $a, h \in H$,

$$\begin{aligned}\varepsilon(h)\,\lambda(za) &= \langle \varepsilon(S^{-1}(h))\,\phi_z(1_\Bbbk), a\rangle = \langle S^{-1}(h) \triangleright \phi_z(1_\Bbbk), a\rangle \\ &= \lambda(zh_{(2)}aS^{-1}(h_{(1)})) = \lambda((S(h_{(1)}) \leftharpoonup \nu)zh_{(2)}a) \quad \text{by (4.3)},\end{aligned}$$

and so, since $(H^*, \cdot)$ is a free right H-module of rank 1 with basis λ,

$$\varepsilon(h)z = (S(h_{(1)}) \leftharpoonup \nu)zh_{(2)} = \nu^{-1}(h_{(2)})\,S(h_{(1)})zh_{(3)} \tag{4.11}$$

for all $h \in H$. Hence, for any $x \in H$,

$$\begin{aligned}(x \leftharpoonup \nu)z &= \nu(x_{(1)})\,x_{(2)}z = \nu(x_{(1)})\,x_{(2)}\varepsilon(x_{(3)})z \\ &= \nu(x_{(1)})\,\nu^{-1}(x_{(4)})\,x_{(2)}S(x_{(3)})zx_{(5)} \quad \text{(by (4.11))} \\ &= \nu(x_{(1)})\,\nu^{-1}(x_{(3)})\,\varepsilon(x_{(2)})\,zx_{(4)} = \nu(x_{(1)})\,\nu^{-1}(x_{(2)})\,zx_{(3)} \\ &= \varepsilon(x_{(1)})\,zx_{(2)} = zx,\end{aligned}$$

and so $z \in L(H)$. Conversely, suppose that $z \in L(H)$. Then, for any $x, h \in H$,

$$\begin{aligned}\lambda(zS(h_{(1)})xh_{(2)}) &= \lambda(zS^2(h_{(2)})S(h_{(1)})x) \quad \text{(by (4.10))} \\ &= \lambda(zS(h_{(1)}S(h_{(2)}))x) = \varepsilon(h)\,\lambda(zx),\end{aligned}$$

and so ϕ_z is H-linear.

Let us prove part (d). Note that ψ is an isomorphism since H is finite-dimensional. Let $X \in H^{\otimes n}$. For all $h \in H$ and $F \in H^{*\otimes n}$, we have $\psi_X(h \triangleright F) = \langle F, X \triangleleft h\rangle$ and $\varepsilon(h)\,\psi_X(F) = \langle F, \varepsilon(h)\,X\rangle$. Therefore ψ_X is H-linear if and only if $X \in V_n(H)$.

Let us prove part (e). Let $a \in Z(H)$. For all $h \in H$,

$$a \triangleleft h = S(h_{(1)})ah_{(2)} = S(h_{(1)})h_{(2)}a = \varepsilon(h)\,a$$

and so $a \in V_1(H)$. Conversely, let $a \in V_1(H)$. For all $x \in H$,

$$xa = x_{(1)}\varepsilon(x_{(2)})\,a = x_{(1)}(a \triangleleft x_{(2)}) = x_{(1)}S(x_{(2)})ax_{(3)} = \varepsilon(x_{(1)})\,ax_{(2)} = ax,$$

and so $a \in Z(H)$.

Finally, let us prove part (c). Let $z \in L(H)$. Since $\mathrm{End}_{\mathrm{rep}_H}(\Bbbk) = \Bbbk$, we have that ϕ_z is negligible if and only if $\psi_a\phi_z = \lambda(za) = 0$ for all $a \in V_1(H) = Z(H)$, that is, if and only if $z \in N(H)$. $\square$

LEMMA 4.6. *We have the following:*
(a) $L(H)$ *is a commutative algebra with product* $*$ *defined by*

$$\begin{aligned}x * z &= \lambda(xS(z_{(2)}))\,z_{(1)} = \lambda(zS(x_{(2)}))\,x_{(1)} \\ &= \lambda(z_{(1)}S^{-1}(x))\,z_{(2)} = \lambda(x_{(1)}S^{-1}(z))\,x_{(2)}\end{aligned}$$

for any $x, z \in L(H)$, *and with* $S(\Lambda)$ *as unit element;*
(b) *for any* $z \in L(H)$, $S_A\phi_z = \phi_{T(z)}$, *where* S_A *denotes the antipode of the categorical Hopf algebra* A*;*

(c) *T induces on $L(H)$ an involutory algebra automorphism, that is, $T^2(x) = x$, $T(x * z) = T(x) * T(z)$, and $T(S(\Lambda)) = S(\Lambda)$ for all $x, z \in L(H)$.*

Proof. Let us prove part (a). Since A is a Hopf algebra in rep_H, the space $\mathrm{Hom}_{\mathrm{rep}_H}(\Bbbk, A)$ is an algebra for the convolution product $\alpha * \beta = m_A(\alpha \otimes \beta)$ and with unit element η_A. This algebra structure transports to $L(H)$ via the $\Bbbk$-isomorphism $\phi\colon L(H) \to \mathrm{Hom}_{\mathrm{rep}_H}(\Bbbk, A)$. Let $x, z \in L(H)$. Then

$$\begin{aligned} x * z &= \phi^{-1}(\phi_x * \phi_z) = \phi^{-1}(m_A(\phi_x \otimes \phi_z)) \\ &= \langle m_A(\phi_x \otimes \phi_z)(1_\Bbbk), \Lambda_{(1)} \rangle \, S(\Lambda_{(2)}) \quad \text{(by Lemma 4.5(a))} \\ &= \langle \lambda \cdot x \otimes \lambda \cdot z, \Delta^{\mathrm{Bd}}(\Lambda_{(1)}) \rangle \, S(\Lambda_{(2)}). \end{aligned}$$

Write $R = \sum_i a_i \otimes b_i$. By using Lemma 4.4, (4.4) and (4.10), we have

$$\begin{aligned} x * z &= \sum_i \lambda(x\Lambda_{(2)} a_i) \, \lambda(z S(b_{i(1)}) \Lambda_{(1)} b_{i(2)}) \, S(\Lambda_{(3)}) \\ &= \sum_i \lambda(x\Lambda_{(2)} a_i) \, \lambda(z S^2(b_{i(2)}) S(b_{i(1)}) \Lambda_{(1)}) \, S(\Lambda_{(3)}) \\ &= \sum_i \lambda(x\Lambda_{(2)} a_i \varepsilon(b_i)) \, \lambda(z \Lambda_{(1)}) \, S(\Lambda_{(3)}), \\ &= \lambda(x\Lambda_{(2)}) \, \lambda(z\Lambda_{(1)}) \, S(\Lambda_{(3)}). \end{aligned} \tag{4.12}$$

Likewise

$$\begin{aligned} x * z &= \sum_i \lambda(x S(a_{i(1)}) \Lambda_{(1)} a_{i(2)}) \, \lambda(z S(b_i) \Lambda_{(2)}) \, S(\Lambda_{(3)}) \\ &= \sum_i \lambda(x S^2(a_{i(2)}) S(a_{i(1)}) \Lambda_{(1)}) \, \lambda(z S(b_i) \Lambda_{(2)}) \, S(\Lambda_{(3)}) \\ &= \sum_i \lambda(x \Lambda_{(1)}) \, \lambda(z S(\varepsilon(a_i) \, b_i) \Lambda_{(2)}) \, S(\Lambda_{(3)}), \\ &= \lambda(x\Lambda_{(1)}) \, \lambda(z\Lambda_{(2)}) \, S(\Lambda_{(3)}). \end{aligned} \tag{4.13}$$

Now, by Lemma 4.1(a),

$$z = \lambda(z\Lambda_{(1)}) \, S(\Lambda_{(2)}), \tag{4.14}$$

$$x = \lambda(x\Lambda_{(1)}) \, S(\Lambda_{(2)}), \tag{4.15}$$

so that

$$z_{(1)} \otimes S^{-1}(z_{(2)}) = \lambda(z\Lambda_{(1)}) \, S(\Lambda_{(3)}) \otimes \Lambda_{(2)}, \tag{4.16}$$

$$x_{(1)} \otimes S^{-1}(x_{(2)}) = \lambda(x\Lambda_{(1)}) \, S(\Lambda_{(3)}) \otimes \Lambda_{(2)}. \tag{4.17}$$

Hence

$$\begin{aligned} x * z &= \lambda(x\Lambda_{(2)}) \, \lambda(z\Lambda_{(1)}) \, S(\Lambda_{(3)}) \quad \text{(by (4.12))} \\ &= \lambda(x S^{-1}(z_{(2)})) \, z_{(1)} \quad \text{(by (4.16))} \\ &= \lambda(x S(z_{(2)})) \, z_{(1)} \quad \text{(by (4.10))}, \end{aligned}$$

$$\begin{aligned} x * z &= \lambda(z\Lambda_{(1)}) \, \lambda(x\Lambda_{(2)}) \, S(\Lambda_{(3)}) \quad \text{(by (4.12))} \\ &= \lambda(z\Lambda_{(1)}) \, \lambda(x_{(1)}\Lambda_{(2)}) \, x_{(2)} \Lambda_{(3)} S(\Lambda_{(4)}) \\ &= \lambda(x_{(1)} S^{-1}(z)) \, x_{(2)} \quad \text{(by (4.14))}, \end{aligned}$$

and

$$\begin{aligned} x*z &= \lambda(x\Lambda_{(1)})\,\lambda(z\Lambda_{(2)})\,S(\Lambda_{(3)}) \quad \text{(by (4.13))}\\ &= \lambda(zS^{-1}(x_{(2)}))\,x_{(1)} \quad \text{(by (4.17))}\\ &= \lambda(zS(x_{(2)}))\,x_{(1)} \quad \text{(by (4.10))}, \end{aligned}$$

$$\begin{aligned} x*z &= \lambda(x\Lambda_{(1)})\,\lambda(z\Lambda_{(2)})\,S(\Lambda_{(3)}) \quad \text{(by (4.13))}\\ &= \lambda(x\Lambda_{(1)})\,\lambda(z_{(1)}\Lambda_{(2)})\,z_{(2)}\Lambda_{(3)}S(\Lambda_{(4)})\\ &= \lambda(z_{(1)}S^{-1}(x))\,z_{(2)} \quad \text{(by (4.15))}. \end{aligned}$$

Note that these expressions of the product of $L(H)$ show that $L(H)$ is commutative. Moreover, by Lemma 4.5(a), the unit element of $L(H)$ is

$$\phi^{-1}(\eta_A) = (\eta_A(1)\otimes S)\Delta(\Lambda) = (\varepsilon\otimes S)\Delta(\Lambda) = S(\Lambda).$$

Let us prove part (b). Let $z\in L(H)$. Write $R=\sum_i a_i\otimes b_i$. For any $x\in H$,

$$\begin{aligned} \langle S_A\phi_z(1_{\Bbbk}), x\rangle &= \langle \lambda\cdot z, S^{\mathrm{Bd}}(x)\rangle\\ &= \sum_i \lambda(zS(a_i)\theta^2 S(x)ub_i) \quad \text{(by Lemma 4.4)}\\ &= \sum_i \lambda(zS^2(u)S^2(b_i)S(a_i)\theta^2 S(x)) \quad \text{(by (4.10))}\\ &= \lambda(zu^2\theta^2 S(x)) \quad \text{(since } S^2(u)=u \text{ and } (S\otimes S)(R)=R)\\ &= \lambda(zG^2S(x)) = \lambda((G^2 \leftharpoondown \nu)zS(x)) = \nu(G^2)\,\lambda(G^2zS(x))\\ &= \nu(G^2)\,\lambda(G^2h_\nu xS^{-1}(z)G^{-2}) \quad \text{(by (4.7))}\\ &= \nu(G^2)\,\lambda((G^{-2}\leftharpoondown\nu)G^2h_\nu xS^{-1}(x)) \quad \text{(by (4.3))}\\ &= \nu(G^2)\,\nu(G^{-2})\,\lambda(h_\nu xS^{-1}(z))\\ &= \lambda((S(z)\leftharpoondown\nu)h_\nu x) \quad \text{(by (4.3))}\\ &= \lambda(T(z)x) = \langle \phi_{T(z)}(1_{\Bbbk}), x\rangle, \end{aligned}$$

that is, $S_A\phi_z = \phi_{T(z)}$.

Let us prove part (c). Let $z\in L(H)$. Firstly $T(z)\in L(H)$ since $\phi_{T(z)} = S_A\phi_z$ is H-linear. Moreover, since $S_A^2=\theta_A$ and the twist of rep_H is natural and satisfies $\theta_{\Bbbk}=\mathrm{id}_{\Bbbk}$, we have

$$T^2(z) = \phi^{-1}(\phi_{T^2(z)}) = \phi^{-1}(S_A^2\phi_z) = \phi^{-1}(\theta_A\phi_z) = \phi^{-1}(\phi_z\theta_{\Bbbk}) = \phi^{-1}(\phi_z) = z.$$

For any $x,z\in L(H)$, we have

$$\begin{aligned} \phi_{T(x*z)} &= S_A\phi_{x*z} = S_A(\phi_x*\phi_z) = S_Am_A(\phi_x\otimes\phi_z)\\ &= m_A(S_A\otimes S_A)c_{A,A}(\phi_x\otimes\phi_z) = m_A(S_A\otimes S_A)(\phi_z\otimes\phi_x)c_{\Bbbk,\Bbbk}\\ &= m_A(S_A\phi_z\otimes S_A\phi_x) = \phi_{T(z)}*\phi_{T(x)} = \phi_{T(z)*T(x)}, \end{aligned}$$

and so $T(x*z) = T(z)*T(x) = T(x)*T(z)$. Finally, $T(S(\Lambda)) = S(\Lambda)$ since $\phi_{T(S(\Lambda))} = S_A\phi_{S(\Lambda)} = S_A\eta_A = \eta_A = \phi_{S(\Lambda)}$. □

In the next theorem, we describe the sets $\mathcal{AK}(\mathrm{rep}_H)$ and $\mathcal{AK}(\mathrm{rep}_H)^{\mathrm{norm}}$ in algebraic terms. Set

$$\mathcal{AK}(H) = \phi^{-1}(\mathcal{AK}(\mathrm{rep}_H)) \quad\text{and}\quad \mathcal{AK}(H)^{\mathrm{norm}} = \phi^{-1}(\mathcal{AK}(\mathrm{rep}_H)^{\mathrm{norm}}),$$

where $\phi\colon L(H) \to \mathrm{Hom}_{\mathrm{rep}_H}(k, A)$ is as in Lemma 4.5. The elements of $\mathcal{AK}(H)$ are called the *algebraic Kirby elements of* H. Note that $\mathcal{AK}(H)^{\mathrm{norm}} \subset \mathcal{AK}(H)$ and, since $\eta_A \in \mathcal{AK}(\mathrm{rep}_H)^{\mathrm{norm}}$ and by Lemma 4.6, we have $S(\Lambda) \in \mathcal{AK}(H)^{\mathrm{norm}}$.

THEOREM 4.7. *The set $\mathcal{AK}(H)$ is constituted by the element $z \in L(H)$ satisfying:*

(a) *$T(z) - z \in N(H)$, that is, $\lambda(T(z)a) = \lambda(za)$ for all $a \in Z(H)$;*

(b) *$\sum_i \lambda(zx_{i(1)})\lambda(zx_{i(2)}y_i) = \sum_i \lambda(zx_i)\lambda(zy_i)$ for all $X = \sum_i x_i \otimes y_i \in V_2(H)$.*

In particular, $\mathcal{AK}(H)$ contains the elements $z \in L(H)$ satisfying $T(z) = z$ and $\lambda(zx_{(1)})zx_{(2)} = \lambda(zx)z$ for all $x \in H$. Moreover, an element $z \in \mathcal{AK}(H)$ belongs to $\mathcal{AK}(H)^{\mathrm{norm}}$ if and only if $\lambda(z\theta) \neq 0 \neq \lambda(z\theta^{-1})$.

Note that the sets $\mathcal{AK}(H)$ and $\mathcal{AK}(H)^{\mathrm{norm}}$ do not depend on the choice of the non-zero right integral λ for H^*. In § 5, we give an example of the determination of these sets for a family of non-unimodular ribbon Hopf algebras.

Proof. Let $z \in L(H)$. By Lemma 4.6(b), we have $S_A\phi_z = \phi_{T(z)}$. Therefore, using Lemma 4.5(c), we get $S_A\phi_z - \phi_z \in \mathrm{Negl}_{\mathrm{rep}_H}(\Bbbk, A)$ if and only if $T(z) - z \in N(H)$, that is, if and only if $\lambda(T(z)a) = \lambda(za)$ for all $a \in Z(H)$. Note that this last property is, in particular, satisfied when $T(z) = z$.

Since $\mathrm{End}_{\mathrm{rep}_H}(\Bbbk) = \Bbbk$, the morphism $\Gamma_r(\phi_z \otimes \phi_z) - \phi_z \otimes \phi_z\colon \Bbbk \to A \otimes A$ is negligible if and only if $G \circ \big(\Gamma_r(\phi_z \otimes \phi_z) - \phi_z \otimes \phi_z\big) = 0$ for any $G \in \mathrm{Hom}_{\mathrm{rep}_H}(A \otimes A, \Bbbk)$. By Lemma 4.5(d), this is equivalent to $\psi_X(\Gamma_r(\phi_z \otimes \phi_z) - \phi_z \otimes \phi_z) = 0$ for all $X \in V_2(H)$. Now, writing $R = \sum_i a_i \otimes b_i$ and using Lemma 4.4, we have, for any $x, y \in H$,

$$
\begin{aligned}
&\langle \Gamma_r(\phi_z \otimes \phi_z)(1_\Bbbk), x \otimes y\rangle \\
&\qquad = \langle (m_A \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \Delta_A)(\lambda \cdot z \otimes \lambda \cdot z), x \otimes y\rangle \\
&\qquad = \langle (\mathrm{id}_A \otimes \Delta_A)(\lambda \cdot z \otimes \lambda \cdot z), \Delta^{\mathrm{Bd}}(x) \otimes y\rangle \\
&\qquad = \sum_i \langle (\mathrm{id}_A \otimes \Delta_A)(\lambda \cdot z \otimes \lambda \cdot z), S(a_{i(1)})x_{(1)}a_{i(2)} \otimes S(b_i)x_{(2)} \otimes y\rangle \\
&\qquad = \sum_i \langle \lambda \cdot z \otimes \lambda \cdot z, S(a_{i(1)})x_{(1)}a_{i(2)} \otimes S(b_i)x_{(2)}y\rangle \\
&\qquad = \sum_i \lambda(zS(a_{i(1)})x_{(1)}a_{i(2)})\,\lambda(zS(b_i)x_{(2)}y) \\
&\qquad = \sum_i \lambda(zS^2(a_{i(2)})S(a_{i(1)})x_{(1)})\,\lambda(zS(b_i)x_{(2)}y) \quad \text{(by (4.10))} \\
&\qquad = \sum_i \lambda(zx_{(1)})\,\lambda(zS(\varepsilon(a_i)b_i)x_{(2)}y) \\
&\qquad = \lambda(zx_{(1)})\,\lambda(zx_{(2)}y) \quad \text{(by (4.4))}.
\end{aligned}
$$

Therefore the morphism $\Gamma_r(\phi_z \otimes \phi_z) - \phi_z \otimes \phi_z\colon \Bbbk \to A \otimes A$ is negligible if and only if $\lambda(zx_{i(1)})\,\lambda(zx_{i(2)}y_i) = \lambda(zx_i)\,\lambda(zy_i)$ for all $X = \sum x_i \otimes y_i \in V_2(H)$. Note that this last property is, in particular, satisfied when $\lambda(zx_{(1)})\,\lambda(zx_{(2)}y) = \lambda(zx)\,\lambda(zy)$ for all $x, y \in H$, that is (since $(H^*, \cdot)$ is a free right H-module of rank 1 with basis λ), when $\lambda(zx_{(1)})\,zx_{(2)} = \lambda(zx)\,z$ for all $x \in H$. Finally, by using Lemma 4.3, we have

$$\Theta_\pm\phi_z = \mathrm{ev}_H(\mathrm{id}_{H^*} \otimes \theta_H^{\pm 1})(\phi_z(1_\Bbbk) \otimes 1_H) = \mathrm{ev}_H(\lambda \cdot z \otimes \theta^{\pm 1}) = \lambda(z\theta^{\pm 1}). \qquad \square$$

Corollary 4.8. *We have $1 \in \mathcal{AK}(H)$ if and only if H is unimodular.*

Proof. Suppose H is unimodular. Therefore $L(H) = Z(H)$ and $T(z) = S(z)$ for all $z \in Z(H)$. In particular, $1 \in L(H)$ and $T(1) = 1$. Moreover, $\lambda(x_{(1)})x_{(2)} = \lambda(x)1$ for all $x \in H$ (since λ is a right integral for H^*). Hence $1 \in \mathcal{AK}(H)$ by Theorem 4.7.

Conversely, suppose that $1 \in \mathcal{AK}(H)$. In particular, $1 \in L(H)$ and so $z \leftharpoonup \nu = z$ for all $z \in H$. Therefore $\varepsilon(z) = \varepsilon(z \leftharpoonup \nu) = \nu(z_{(1)})\varepsilon(z_{(2)}) = \nu(z)$ for all $z \in H$, that is, H is unimodular. □

4.7. *Algebraic Kirby elements from semisimplification*

Let H be a finite-dimensional ribbon Hopf algebra. Let (A, i) be the coend of the functor (1.5) for rep_H (as in § 4.5). Denote by rep_H^s the semisimplification of rep_H and by π its associated surjective ribbon functor $\mathrm{rep}_H \to \mathrm{rep}_H^s$ (see § 3.4). Let $\phi\colon L(H) \to \mathrm{Hom}_{\mathrm{rep}_H}(\Bbbk, A)$ be as in § 4.6. Set

$$\mathcal{AK}(H)^s = \bigcup_{\mathcal{B}} \phi^{-1}\big(\pi^{-1}\big(\varphi_{\mathcal{B}}(\mathcal{AK}(\mathcal{B}))\big)\big),$$

where $\mathcal{B}$ runs over (equivalence classes of) finitely semisimple ribbon full subcategories of rep_H^s whose simple objects are scalar, and $\varphi_{\mathcal{B}}$ is the morphism (3.6) corresponding to $\mathcal{B}$. By Corollary 3.11, we have $\mathcal{AK}(H)^s \subset \mathcal{AK}(H)$. Note that this inclusion may be strict (see Remark 4.11).

Let $\mathcal{V}$ be a set of representatives of isomorphism classes of indecomposable finite-dimensional left H-modules with non-zero quantum dimension. Note that $\pi(\mathcal{V}) = \{\pi(V) \mid V \in \mathcal{V}\}$ is a set of representatives of isomorphism classes of simple objects of rep_H^s. Let λ be a non-zero right integral for H^*. Since H^* is a free right H-module with basis λ (see § 4.1), there exists a (unique) element $z_V \in H$ such that

$$\lambda(z_V x) = \mathrm{Tr}(G^{-1} x\, \mathrm{id}_V) \tag{4.18}$$

for all $x \in H$, where G is the special grouplike element of H. Recall that $\dim_q(V) = \mathrm{Tr}(G^{-1}\mathrm{id}_V)$ denotes the quantum dimension of V (see Lemma 4.2).

Corollary 4.9. (a) *If $z \in \mathcal{AK}(H)^s$, then $z = k\sum_{V\in\mathcal{W}} \dim_q(V)\, z_V$ for some finite subset $\mathcal{W}$ of $\mathcal{V}$ and some scalar $k \in \Bbbk$.*

(b) *Let $\mathcal{W}$ be a set of representatives of isomorphism classes of simple objects of a finitely semisimple ribbon full subcategory of rep_H^s. We can suppose that $\mathcal{W} \subset \pi(\mathcal{V})$. If the objects of $\mathcal{W}$ are scalar, then*

$$\sum_{V \in \pi^{-1}(\mathcal{W})} \dim_q(V)\, z_V \in \mathcal{AK}(H)^s.$$

Proof. Let $\mathcal{B}$ be finitely semisimple ribbon full subcategory of rep_H^s whose simple objects are scalar, and let (B, j) be the coend of the functor (1.5) for $\mathcal{B}$ (as in § 3.2). We can suppose that there exists a (finite) subset $\mathcal{W}$ of $\mathcal{V}$ such that $\pi(\mathcal{W})$ is a set of representatives of isomorphism classes of simple objects of $\mathcal{B}$. Recall that $B = \bigoplus_{V\in\mathcal{W}} \pi(V)^* \otimes \pi(V)$. In particular, there exist morphisms $p_V\colon B \to \pi(V)^* \otimes \pi(V)$ and $q_V\colon \pi(V)^* \otimes \pi(V) \to B$ of $\mathcal{B}$ such that $\mathrm{id}_B = \sum_{V\in\mathcal{W}} q_V p_V$ and $p_V q_W = \delta_{V,W}\, \mathrm{id}_{\pi(V)^*\otimes\pi(V)}$. Recall that $j_V = q_V$ for any $V \in \mathcal{W}$.

Let $\phi\colon L(H) \to \mathrm{Hom}_{\mathrm{rep}_H}(\Bbbk, A)$ be as in § 4.6. As in (3.6), we set

$$\varphi_{\mathcal{B}} = \sum_{V \in \mathcal{W}} \pi(i_V)p_V \in \mathrm{Hom}_{\mathrm{rep}^s_H}(B, \pi(A)).$$

Let $V \in \mathcal{W}$. Let $(e_i)_i$ be a basis of V with dual basis $(e_i^*)_i$. By Lemma 4.2(b) and (4.9) we have

$$\begin{aligned}\langle i_V\, \widetilde{\mathrm{coev}}_V(1_\Bbbk), x\rangle &= \langle i_V(\mathrm{id}_{V^*} \otimes G^{-1}\mathrm{id}_V)\, \sigma_{V,V^*}\, \mathrm{coev}_V(1_\Bbbk), x\rangle \\ &= \sum_i \langle i_V(e_i^* \otimes G^{-1}e_i), x\rangle = \sum_i \langle e_i^*, xG^{-1}e_i\rangle \\ &= \mathrm{Tr}(xG^{-1}\mathrm{id}_V) = \mathrm{Tr}(G^{-1}x\,\mathrm{id}_V) = \lambda(z_V x) = \langle \phi_{z_V}(1_\Bbbk), x\rangle,\end{aligned}$$

for any $x \in H$, that is, $i_V\, \widetilde{\mathrm{coev}}_V = \phi_{z_\mathcal{V}}$. Moreover,

$$\varphi_{\mathcal{B}} j_{\pi(V)}\, \widetilde{\mathrm{coev}}_{\pi(V)} = \sum_{W \in \mathcal{W}} \pi(i_W)p_W j_{\pi(V)}\, \widetilde{\mathrm{coev}}_{\pi(V)} = \pi(i_V\, \widetilde{\mathrm{coev}}_V) = \pi(\phi_{z_V}).$$

Hence part (a) follows from Lemma 3.3 and Corollary 3.11(a), and part (b) follows from Theorem 3.4 and Corollary 3.11(a). □

Lemma 4.10. *If H is not semisimple, then $\varepsilon(z) = 0$ for any $z \in \mathcal{AK}(H)^s$.*

Remark 4.11. When H is not semisimple, it is possible that $\mathcal{AK}(H)^s \subsetneq \mathcal{AK}(H)$. For example, if H is unimodular but not semisimple, then $1 \in \mathcal{AK}(H)$ (by Corollary 4.8) and $1 \notin \mathcal{AK}(H)^s$ (by Lemma 4.10, since $\varepsilon(1) = 1$).

Proof of Lemma 4.10. Let Λ be a left integral for H such that $\lambda(\Lambda) = 1$. Since H is not semisimple, we have $\varepsilon(\Lambda) = 0$ (by [**1**, Theorem 3.3.2]) and $\Lambda^2 = \varepsilon(\Lambda)\Lambda = 0$. Now, if M is a finite-dimensional left H-module, then $(\Lambda\, \mathrm{id}_M)^2 = \Lambda^2 \mathrm{id}_M = 0$ and so $\mathrm{Tr}(\Lambda\, \mathrm{id}_M) = 0$. Let $z \in \mathcal{AK}(H)^s$. By Corollary 4.9(a), there exist $k \in \Bbbk$ and a finite subset $\mathcal{W}$ of $\mathcal{V}$ such that $z = k \sum_{V \in \mathcal{W}} \dim_q(V)\, z_V$. Then

$$\lambda(z\Lambda) = k \sum_{V \in \mathcal{W}} \dim_q(V)\, \mathrm{Tr}(G^{-1}\Lambda\, \mathrm{id}_V) = k \sum_{V \in \mathcal{W}} \dim_q(V)\varepsilon(G^{-1})\, \mathrm{Tr}(\Lambda\, \mathrm{id}_V) = 0.$$

Hence $\varepsilon(z) = \varepsilon(z)\lambda(\Lambda) = \lambda(\varepsilon(z)\Lambda) = \lambda(z\Lambda) = 0$. □

Recall (see [**7**]) that $\Bbbk$ is a *splitting field* for a $\Bbbk$-algebra A if every simple finite-dimensional left A-module is scalar. Note that this is always the case if $\Bbbk$ is algebraically closed.

Corollary 4.12. *If H is semisimple and $\Bbbk$ is a splitting field for H, then $\mathcal{AK}(H)^s = \mathcal{AK}(H)$ and this set is composed by elements $z \in Z(H)$ satisfying $S(z) = z$ and $\lambda(zx_{(1)})zx_{(2)} = \lambda(zx)z$ for all $x \in H$.*

Proof. Since H is semisimple and finite-dimensional, we have $\mathrm{rep}^s_H = \mathrm{rep}_H$ and that $\mathcal{V}$ is finite. Moreover, since $\Bbbk$ is a splitting field for H, every $V \in \mathcal{V}$ is scalar. Then $\mathcal{AK}(H) = \phi^{-1}(\mathcal{AK}(\mathrm{rep}_H)) \subset \mathcal{AK}(H)^s$ and so $\mathcal{AK}(H)^s = \mathcal{AK}(H)$. Moreover, since H is unimodular (because it is semisimple), we have $L(H) = Z(H)$, $N(H) = 0$, and $T(x) = S(x)$ for all $x \in H$ (see § 4.6). Therefore, by using Theorem 4.7, we get that $z \in \mathcal{AK}(H)$ if and only if $z \in Z(H)$, $S(z) = z$, and $\lambda(zx_{(1)})zx_{(2)} = \lambda(zx)z$ for all $x \in H$. □

Proposition 4.13. *Suppose that H is semisimple and that $\Bbbk$ is a splitting field for H of characteristic* 0. *Then $Z(H) = \bigoplus_{V\in\mathcal{V}} \Bbbk z_V$ and $\sum_{V\in\mathcal{V}} \dim_q(V)\, z_V \in \Bbbk^* 1$.*

Proof. Note that H is cosemisimple since any finite-dimensional semisimple Hopf algebra over a field of characteristic 0 is cosemisimple (see [**14**, Theorem 3.3]). Then $S^2 = \mathrm{id}_H$ by [**13**, Theorem 4] and $\lambda(1) \neq 0$ by [**1**, Theorem 3.3.2].

Note that $\mathcal{V}$ is finite (since H is finite-dimensional). By [**7**, Theorem 25.10], any simple left ideal of H is isomorphic (as a left H-module) to a (unique) element of $\mathcal{V}$. For any $V \in \mathcal{V}$, let $H_V \subset H$ be the sum of all the simple left ideals of H which are isomorphic to V. By [**7**, Theorem 25.15], H_V is a two-sided ideal of H, H_V is a simple $\Bbbk$-algebra (the operations being those induced by H), $H_V H_W = 0$ for $V \neq W \in \mathcal{V}$, $H = \bigoplus_{V\in\mathcal{V}} H_V$, and H is isomorphic (as an algebra) to $\prod_{V\in\mathcal{V}} H_V$. Moreover, if e_V denotes the unit of H_V, then $1 = \sum_{V\in\mathcal{V}} e_V$, $H_V = He_V$, and $e_V e_W = \delta_{V,W}\, e_V$ for all $V, W \in \mathcal{V}$. By [**7**, Theorem 26.4], since H_V is a simple $\Bbbk$-algebra, V is a simple left H_V-module, and $\mathrm{End}_{H_V}(V) = \Bbbk$ (because V is a scalar H-module), we have that H_V is isomorphic (as an algebra) to $\mathrm{End}_\Bbbk(V)$ and that $\dim_\Bbbk(V)$ is the number of simple left ideals appearing in a direct sum decomposition of H_V as such a sum. Then $Z(H_V) = \Bbbk e_V$ (since $Z(\mathrm{End}_\Bbbk(V)) = \Bbbk\, \mathrm{id}_V$) and so $Z(H) = \bigoplus_{V\in\mathcal{V}} \Bbbk e_V$. Moreover, for any $x \in H$, we have

$$\mathrm{Tr}(x\,\mathrm{id}_H) = \sum_{V\in\mathcal{V}} \mathrm{Tr}(x\,\mathrm{id}_{H_V}) = \sum_{V\in\mathcal{V}} \dim_\Bbbk(V)\,\mathrm{Tr}(x\,\mathrm{id}_V). \tag{4.19}$$

The map $x \in H \mapsto \mathrm{Tr}((x\,\mathrm{id}_H)\circ S^2) \in \Bbbk$ is a right integral for H^* (by [**23**, Proposition 2(b)] applied to H^{op}). Therefore, since $S^2 = \mathrm{id}_H$ and, by the uniqueness of integrals, there exists $k \in \Bbbk$ such that $\mathrm{Tr}(x\,\mathrm{id}_H) = k\,\lambda(x)$ for all $x \in H$. Then $k = \dim_\Bbbk(H)/\lambda(1)$ and so, by (4.19), we get, for all $x \in H$,

$$k\,\lambda(x) = \sum_{V\in\mathcal{V}} \dim_\Bbbk(V)\,\mathrm{Tr}(x\,\mathrm{id}_V). \tag{4.20}$$

Let $V \in \mathcal{V}$. By (4.6) and since $S^2 = \mathrm{id}_H$, the special grouplike element G of H is central and so $G^{-1}\mathrm{id}_V$ is H-linear. Therefore, since V is scalar and G is invertible, there exists a (unique) $\gamma_V \in \Bbbk^*$ such that $G^{-1}\mathrm{id}_V = \gamma_V \mathrm{id}_V$. Since H and H^* are semisimple and so unimodular, their special grouplike elements are trivial. Then $G^2 = 1$ (since $g = G^2 h_\nu$) and so $\gamma_V^2 = 1$. Hence, for all $x \in H$,

$$\begin{aligned}\dim_q(V)\,\mathrm{Tr}(G^{-1}x\,\mathrm{id}_V) &= \mathrm{Tr}(G^{-1}\mathrm{id}_V)\,\mathrm{Tr}(xG^{-1}\mathrm{id}_V)\\ &= \gamma_V^2\,\mathrm{Tr}(\mathrm{id}_V)\,\mathrm{Tr}(x\,\mathrm{id}_V) = \dim_\Bbbk(V)\,\mathrm{Tr}(x\,\mathrm{id}_V).\end{aligned} \tag{4.21}$$

For any $x \in H$, we have

$$\begin{aligned}\lambda(\dim_q(V)\,z_V x) &= \dim_q(V)\,\mathrm{Tr}(G^{-1}x\,\mathrm{id}_V) \quad \text{(by (4.18))}\\ &= \dim_\Bbbk(V)\,\mathrm{Tr}(x\,\mathrm{id}_V) \quad \text{(by (4.21))}\\ &= \sum_{W\in\mathcal{V}} \dim_\Bbbk(W)\,\mathrm{Tr}(xe_V\,\mathrm{id}_W) \quad (\text{since } e_V\,\mathrm{id}_W = \delta_{V,W}\,\mathrm{id}_V)\\ &= \lambda(k\,e_V x) \quad \text{(by (4.20))},\end{aligned}$$

and so $\dim_q(V)\,z_V = k\,e_V$ (since H^* is a free right H-module with basis λ).

Finally, since $k = \dim_\Bbbk(H)/\lambda(1) \neq 0$ (because the characteristic of $\Bbbk$ is 0) and $\dim_q(V) \neq 0$ (because rep_H is semisimple, see § 3.1), we get $Z(H) = \bigoplus_{V\in\mathcal{V}} \Bbbk e_V = \bigoplus_{V\in\mathcal{V}} \Bbbk z_V$ and $\sum_{V\in\mathcal{V}} \dim_q(V)\, z_V = k \sum_{V\in\mathcal{V}} e_V = k\,1 \in \Bbbk^* 1$. ◻

4.8. *HKR-type invariants*

Let H be a finite-dimensional ribbon Hopf algebra. We use the notation of § 4.6. By Theorem 2.6 and Proposition 2.3, for any $z \in \mathcal{AK}(H)^{\mathrm{norm}}$,

$$\tau_{(H,z)}(M) = \tau_{\mathrm{rep}_H}(M;\phi_z) \in \Bbbk \tag{4.22}$$

is an invariant of 3-manifolds. Note that the choice of the normalization in the definition of $\tau_{\mathrm{rep}_H}(M;\phi_z)$ (see Proposition 2.3) implies that $\tau_{(H,z)}(M)$ does not depend on the choice of the non-zero right integral λ for H^* used to define $\tau_{\mathrm{rep}_H}(M;\phi_z)$.

By Remark 2.8, for any $z \in \mathcal{AK}(H)^{\mathrm{norm}}$, we have $\tau_{(H,z)}(S^3) = 1$ and $\tau_{(H,z)}$ is multiplicative with respect to the connected sum. Moreover, by Remark 2.9, if $z \in \mathcal{AK}(H)^{\mathrm{norm}}$, $n \in N(H)$, and $k \in \Bbbk^*$, then $kz + n \in \mathcal{AK}(H)^{\mathrm{norm}}$ and, for all 3-manifolds M,

$$\tau_{(H,kz+n)}(M) = \tau_{(H,z)}(M). \tag{4.23}$$

DEFINITION 4.14. An invariant of closed 3-manifolds I with values in $\Bbbk$ is said to be of *HKR-type* if there exist a finite-dimensional ribbon Hopf algebra H (over $\Bbbk$) and $z \in \mathcal{AK}(H)^{\mathrm{norm}}$ such that $I(M) = \tau_{(H,z)}(M)$ for all 3-manifolds M.

In Proposition 4.17, we show that the Reshetikhin–Turaev invariants defined from premodular Hopf algebras (as quantum groups) are of HKR-type.

Let us show that any HKR-type invariant can be computed by using the Kauffman–Radford algorithm (which is given in [**10**] for the case H unimodular and for $z = 1$). Fix a finite-dimensional ribbon Hopf algebra H, a non-zero right integral λ for H^*, and an element $z \in \mathcal{AK}(H)^{\mathrm{norm}}$. Let M be a 3-manifold and $L = L_1 \cup \ldots \cup L_n$ be a framed link in S^3 such that $M \simeq M_L$. Let us recall the Kauffman–Radford algorithm (the algorithm given here corresponds to that of [**10**] when using the ribbon Hopf algebra opposite to H).

(A) Consider a diagram D of L (with blackboard framing). Each crossing of D is decorated with the R-matrix $R = \sum_i a_i \otimes b_i$ as in Figure 8. The diagram obtained after this step is called the *flat diagram of* D. Note that the flat diagram of D is composed by n closed plane curves, each of them arising from a component of L.

FIGURE 8.

(B) On each component of the flat diagram of D, the algebraic decoration is concentrated in an arbitrary point (other than extrema and crossings) according to the rules of Figure 9, where $a, b \in H$.

In that way we get an element $\sum_k v_1^k \otimes \ldots \otimes v_n^k \in H^{\otimes n}$, where v_i^k corresponds to the component of the flat diagram of D arising from L_i; see Figure 10.

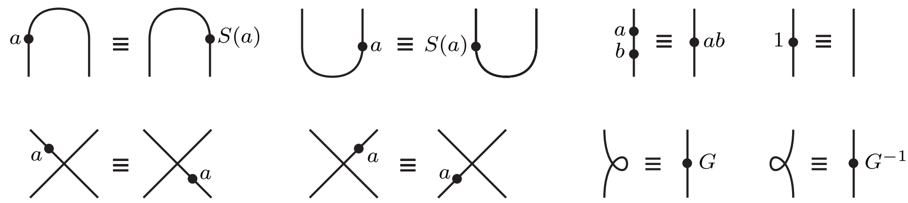


FIGURE 9.

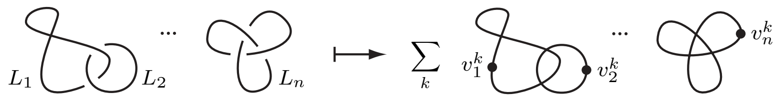


FIGURE 10.

For $1 \leqslant i \leqslant n$, let d_i be the Whitney degree of the flat diagram of L_i obtained by traversing it upwards from the point where the algebraic decorations have been concentrated. The Whitney degree is the total turn of the tangent vector to the curve when one traverses it in the given direction; see Figure 11.

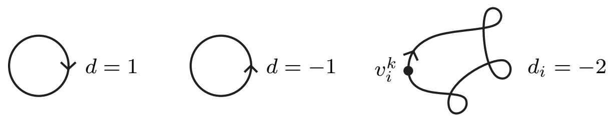


FIGURE 11.

PROPOSITION 4.15. *We have*

$$\tau_{(H,z)}(M) = \lambda(z\theta)^{b_-(L)-n_L}\, \lambda(z\theta^{-1})^{-b_-(L)} \sum_k \lambda(zG^{d_1+1}v_1^k) \dots \lambda(zG^{d_n+1}v_n^k).$$

Proof. Choose an orientation for L. Let T be a ribbon n-handle such that L is isotopic to $T \circ (\cup \otimes \dots \otimes \cup)$, where the ith cup (with clockwise orientation) corresponds to the component L_i. Let D_T be a diagram of T. Note that $D = D_T \circ (\cup \otimes \dots \otimes \cup)$ is a diagram of L. Apply steps (A) and (B) to D_T as in Figure 12. Note that, in this case, $d_i = -1$.

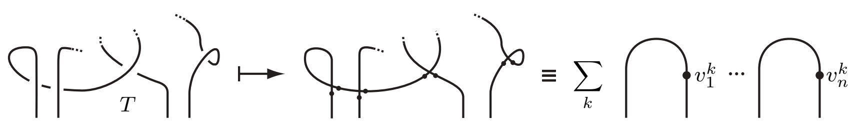


FIGURE 12.

From the definition of the monoidal structure, duality, braiding and twist of rep_H (see § 4.4), it is not difficult to verify that, for any finite-dimensional left

H-modules $M_1, \dots, M_n$,

$$T_{(M_1,\dots,M_n)} = \sum_k \mathrm{ev}_{M_1}(\mathrm{id}_{M_1^*} \otimes v_k^1\, \mathrm{id}_{M_1}) \otimes \dots \otimes \mathrm{ev}_{M_n}(\mathrm{id}_{M_n^*} \otimes v_k^n\, \mathrm{id}_{M_n}).$$

Then, by Lemma 4.3,

$$\begin{aligned}\tau_{\mathrm{rep}_H}(L;\phi_z) = \phi_T \circ \phi_z^{\otimes n} &= \sum_k \mathrm{ev}_H(\phi_z(1_{\Bbbk}) \otimes v_k^1) \otimes \dots \otimes \mathrm{ev}_H(\phi_z(1_{\Bbbk}) \otimes v_k^n)\\ &= \sum_k \lambda(zv_1^k) \dots \lambda(zv_n^k) = \sum_k \lambda(zG^{d_1+1}v_1^k) \dots \lambda(zG^{d_n+1}v_n^k).\end{aligned}$$

Hence the result follows since $\Theta_\pm \phi_z = \lambda(z\theta^{\pm 1})$. □

COROLLARY 4.16. *Suppose that H is unimodular and $\lambda(\theta) \neq 0 \neq \lambda(\theta^{-1})$. Then $1 \in \mathcal{AK}(H)^{\mathrm{norm}}$ and $\tau_{(H,1)}(M)$ is the Hennings–Kauffman–Radford invariant of 3-manifolds defined with the opposite ribbon Hopf algebra H^{op} to H.*

Proof. This is an immediate consequence of Corollary 4.8, Proposition 4.15, and the definition of the Hennings–Kauffman–Radford invariant given, for example, in [**10**]. □

4.9. *Reshetikhin–Turaev from premodular Hopf algebras*

Let $(H, \mathcal{V})$ be a finite-dimensional *premodular* Hopf algebra. This means that (see [**26**]) H is a finite-dimensional ribbon Hopf algebra and $\mathcal{V}$ is a finite set of finite-dimensional pairwise non-isomorphic simple left H-modules such that:

(i) each $V \in \mathcal{V}$ is non-negligible and scalar;
(ii) the trivial left H-module $\Bbbk$ belongs to $\mathcal{V}$;
(iii) for any $V \in \mathcal{V}$, there exists $W \in \mathcal{V}$ such that $V^* \simeq W$;
(iv) for any $V, W \in \mathcal{V}$, $V \otimes W$ splits as a (finite) direct sum of certain modules of $\mathcal{V}$ (possibly with multiplicities) and a negligible H-module.

By a *negligible* H-module we mean a finite-dimensional left H-module N such that $\mathrm{tr}_q(f) = 0$ for any $f \in \mathrm{End}_{\mathrm{rep}_H}(N)$ or, equivalently, such that $\dim_q(N) = 0$.

Consider the semisimplification rep_H^s of rep_H (see § 3.4) and let π be the ribbon functor $\mathrm{rep}_H \to \mathrm{rep}_H^s$ associated to this semisimplification.

Let $\mathcal{B}_\mathcal{V}$ be the full subcategory of rep_H^s whose objects are finite direct sums of objects of $\pi(\mathcal{V}) = \{\pi(V) \mid V \in \mathcal{V}\}$. By the definition of a premodular Hopf algebra, $\mathcal{B}_\mathcal{V}$ is a ribbon full subcategory of rep_H^s. Note that $\mathcal{B}_\mathcal{V}$ is finitely semisimple with scalar simple objects and has $\pi(\mathcal{V})$ as a (finite) set of representatives of isomorphism classes of simple objects. Recall that the Reshetikhin–Turaev invariant $\mathrm{RT}_{\mathcal{B}_\mathcal{V}}(M)$ of 3-manifolds is well defined when $\Delta_\pm^{\mathcal{B}_\mathcal{V}} \neq 0$ (see § 3.3).

Let λ be a non-zero right integral for H^*. For any $V \in \mathcal{V}$, as in (4.18), we let $z_V \in H$ such that $\lambda(z_V x) = \mathrm{Tr}(G^{-1} x\, \mathrm{id}_V)$ for all $x \in H$. Set

$$z_\mathcal{V} = \sum_{V \in \mathcal{V}} \dim_q(V)\, z_V,$$

where $\dim_q(V) = \mathrm{Tr}(G^{-1}\mathrm{id}_V)$. By Corollary 4.9(b), we have $z_\mathcal{V} \in \mathcal{AK}(H)$.

PROPOSITION 4.17. *If $\Delta_\pm^{\mathcal{B}_\mathcal{V}} \neq 0$, then $z_\mathcal{V} \in \mathcal{AK}(H)^{\mathrm{norm}}$ and $\tau_{(H,z_\mathcal{V})}(M) = \mathrm{RT}_{\mathcal{B}_\mathcal{V}}(M)$ for all 3-manifolds M.*

Note that Proposition 4.17 says that the Reshetikhin–Turaev invariant defined from a premodular Hopf algebra is of HKR-type.

Proof. Let (A, i) be the coend of the functor (1.5) for rep_H (as in § 4.5), let (B, j) be the coend of the functor (1.5) for $\mathcal{B}_\mathcal{V}$ (as in § 3.2). Set

$$\alpha_{\mathcal{B}_\mathcal{V}} = \sum_{V \in \mathcal{V}} \dim_q(V) j_{\pi(V)} \widetilde{\mathrm{coev}}_{\pi(V)}.$$

Suppose $\Delta_\pm^{\mathcal{B}_\mathcal{V}} \neq 0$. By Corollary 3.9, we have $\alpha_{\mathcal{B}_\mathcal{V}} \in \mathcal{AK}(\mathcal{B}_\mathcal{V})^{\mathrm{norm}}$ and $\mathrm{RT}_{\mathcal{B}_\mathcal{V}}(M) = \tau_{\mathcal{B}_\mathcal{V}}(M; \alpha_{\mathcal{B}_\mathcal{V}})$ for all 3-manifolds M. Set $\varphi_{\mathcal{B}_\mathcal{V}} : B \to \pi(A)$ as in (3.6). As in the proof of Corollary 4.9, we have $\pi(\phi_{z_\mathcal{V}}) = \varphi_{\mathcal{B}_\mathcal{V}} j_{\pi(V)} \widetilde{\mathrm{coev}}_{\pi(V)}$. Then $\pi(\phi_{z_\mathcal{V}}) = \varphi_{\mathcal{B}_\mathcal{V}} \alpha_{\mathcal{B}_\mathcal{V}}$. Since $\alpha_{\mathcal{B}_\mathcal{V}} \in \mathcal{AK}(\mathcal{B}_\mathcal{V})^{\mathrm{norm}}$ and $\pi(\phi_{z_\mathcal{V}}) = \varphi \alpha_{\mathcal{B}_\mathcal{V}}$, Corollary 3.11(b) gives $\phi_{z_\mathcal{V}} \in \mathcal{AK}(\mathrm{rep}_H)^{\mathrm{norm}}$ and $\tau_{\mathcal{B}_\mathcal{V}}(M; \alpha_{\mathcal{B}_\mathcal{V}}) = \tau_{\mathrm{rep}_H}(M; \phi_{z_\mathcal{V}})$ for all 3-manifolds M. Hence $z_\mathcal{V} \in \mathcal{AK}(H)^{\mathrm{norm}}$ and $\tau_{(H, z_\mathcal{V})}(M) = \mathrm{RT}_{\mathcal{B}_\mathcal{V}}(M)$ for all 3-manifolds M. □

Note that if H is a semisimple finite-dimensional ribbon Hopf algebra, $\Bbbk$ is a splitting field for H, and $\mathcal{V}$ is a set of representatives of isomorphism classes of simple left H-modules, then $(H, \mathcal{V})$ is a premodular Hopf algebra and $\mathcal{B}_\mathcal{V} = \mathrm{rep}_H$.

Corollary 4.18. *Let H be a finite-dimensional semisimple ribbon Hopf algebra. Suppose that the base field $\Bbbk$ is of characteristic* 0 *and is a splitting field for H. Then the Hennings–Kauffman–Radford invariant of* 3*-manifolds computed with H^{op} and the Reshetikhin–Turaev invariant of* 3*-manifolds computed with rep_H are simultaneously well defined (that is, $\Delta_\pm^{\mathrm{rep}_H} \neq 0$ if and only if $1 \in \mathcal{AK}(H)^{\mathrm{norm}}$). Moreover, if they are well defined, then they coincide, that is, $\tau_{(H,1)}(M) = \mathrm{RT}_{\mathrm{rep}_H}(M)$ for any* 3*-manifold M.*

Remark 4.19. The conclusions of Corollary 4.18 may no longer be true when H is not semisimple (see Remark 4.11). Moreover, in the modular case (in the sense of Remark 3.7), Corollary 4.18 was first shown in [**11**].

Proof of Corollary 4.18. By Proposition 4.17, the Reshetikhin–Turaev invariant of 3-manifolds computed from rep_H is well defined if

$$z_\mathcal{V} = \sum_{V \in \mathcal{V}} \dim_q(V)\, z_V \in \mathcal{AK}(H)^{\mathrm{norm}}$$

and is equal to $\tau_{(H, z_\mathcal{V})}$. By Corollary 4.16, the Hennings–Kauffman–Radford invariant of 3-manifolds computed with H^{op} is well defined if $1 \in \mathcal{AK}(H)^{\mathrm{norm}}$ and is equal to $\tau_{(H,1)}$. Now, by Proposition 4.13, $\sum_{V \in \mathcal{V}} \dim_q(V)\, z_V = k\, 1$ for some $k \in \Bbbk^*$. We conclude by using (4.23). □

5. *A non-unimodular example*

Let us examine the case of a family of non-unimodular ribbon Hopf algebras, defined by Radford [**22**], which includes Sweedler's Hopf algebra.

Let n be an odd positive integer and $\Bbbk$ be a field whose characteristic does not divide $2n$. Let H_n be the $\Bbbk$-algebra generated by a and x with the following relations:

$$a^{2n} = 1, \quad x^2 = 0, \quad ax = -xa.$$

The algebra H_n is a Hopf algebra for the following structure maps:

$$\begin{aligned} \Delta(a) &= a \otimes a, & \varepsilon(a) &= 1, & S(a) &= a^{-1}, \\ \Delta(x) &= x \otimes a^n + 1 \otimes x, & \varepsilon(x) &= 0, & S(x) &= a^n x. \end{aligned}$$

The set $\mathcal{B} = \{a^l x^m \mid 0 \leqslant l < 2n,\ 0 \leqslant m \leqslant 1\}$ is a basis for H_n. The dual basis of $\mathcal{B}$ is $\{\overline{a^k x^r} \mid 0 \leqslant k < 2n,\ 0 \leqslant r \leqslant 1\}$, where $\overline{a^k x^r}(a^l x^m) = \delta_{l,k}\,\delta_{m,r}$. Set

$$\Lambda = (1 + a + a^2 + \ldots + a^{2n-1})x \quad \text{and} \quad \lambda = \overline{a^n x}.$$

Then Λ is a left integral for H_n and λ is a right integral for H_n^* such that $\lambda(\Lambda) = \lambda(S(\Lambda)) = 1$. The distinguished grouplike element of H_n is $g = a^n \in G(H_n)$ and the distinguished grouplike element $\nu \in G(H_n^*) = \mathrm{Alg}(H_n, \Bbbk)$ of H_n^* is given by $\nu(a) = -1$ and $\nu(x) = 0$.

Suppose that $\Bbbk$ has a primitive $2n$-root of unity ω. Let s be an odd integer with $1 \leqslant s < 2n$ and let $\beta \in \Bbbk$. Then

$$R_{\omega,s,\beta} = \frac{1}{2n} \sum_{0 \leqslant i,l < 2n} w^{-il}\, a^i \otimes a^{sl} + \frac{\beta}{2n} \sum_{0 \leqslant i,l < 2n} w^{-il}\, a^i x \otimes a^{sl+n} x$$

is an R-matrix for H_n and $h_\nu = (\mathrm{id}_{H_n} \otimes \nu)(R_{\omega,s,\beta}) = a^n$.

Let $\chi\colon \Bbbk^* \to \Bbbk[a]$ be the algebra map defined by $\chi(\alpha) = \sum_{0 \leqslant l < 2n} \alpha^{l^2} e_l$ for all $\alpha \in \Bbbk^*$, where $e_l = (1/2n) \sum_{0 \leqslant i < 2n} \omega^{-il} a^i$. Note that $(e_l)_{0 \leqslant l < 2n}$ is a basis of $\Bbbk[a]$. The quasitriangular Hopf algebra $(H_n, R_{\omega,s,\beta})$ is ribbon with twist $\theta = a^n \chi(\omega^s)$. The special grouplike element of H_n is then $G = a^n$.

Let $T\colon H_n \to H_n$, and let $L(H_n)$, $N(H_n)$, $V_2(H_n)$, $\mathcal{AK}(H_n)$ and $\mathcal{AK}(H_n)^{\mathrm{norm}}$ be as in § 4.6. It is not difficult to verify that

$$\begin{gathered} T(a^k) = (-1)^k a^{n-k} \quad \text{and} \quad T(a^k x) = a^{-k} x \quad \text{for all } 0 \leqslant k < 2n, \\ L(H_n) = \Bbbk[a]x, \qquad Z(H_n) = \Bbbk[a^2], \qquad N(H_n) = \Bbbk[a^2]x, \\ V_2(H_n) = \bigoplus_{0 \leqslant p,q < n} \Bbbk(a^{2p} \otimes a^{2q}) \oplus \bigoplus_{0 \leqslant k,l < 2n} \Bbbk(a^k x \otimes a^l x). \end{gathered}$$

For any divisor d of n, set

$$z_d = \sum_{k=0}^{n/d-1} a^{2dk+n} x.$$

Lemma 5.1. *We have*

$$\mathcal{AK}(H_n) = \bigcup_{d|n} \big(\Bbbk z_d \oplus \Bbbk[a^2]x\big) \quad \textit{and} \quad \mathcal{AK}(H_n)^{\mathrm{norm}} = \bigcup_{d|n} \big(\Bbbk^* z_d \oplus \Bbbk[a^2]x\big).$$

Proof. Let $z \in L(H_n)$. Since $a^{2n} = 1$, we can write $z = \sum_{k \in \mathbb{Z}/2n\mathbb{Z}} \alpha_k a^k x$ for some function $\alpha\colon \mathbb{Z}/2n\mathbb{Z} \to \Bbbk$. Using Theorem 4.7, we have $z \in \mathcal{AK}(H_n)$ if and only if $\alpha_{-k} = \alpha_k$ and $\alpha_k \alpha_{k+l-n} = \alpha_k \alpha_l$ for all k, l odd. Set

$$w = \sum_{k \in \mathbb{Z}/n\mathbb{Z}} \alpha_{2k+n+1} a^{2k+n+1} x.$$

Then $z - w = \sum_{k \in \mathbb{Z}/n\mathbb{Z}} \gamma_k a^{2k+n} x$ where $\gamma\colon \mathbb{Z}/n\mathbb{Z} \to \Bbbk$ is defined by $\gamma_k = \alpha_{2k+n}$. Since n is odd, we have $w \in \Bbbk[a^2]x = N(H_n)$. Then $z \in \mathcal{AK}(H_n)$ if and only if $z - w \in \mathcal{AK}(H_n)$, and so if and only if $\gamma_{-k} = \gamma_k$ and $\gamma_k \gamma_{k+l} = \gamma_k \gamma_l$ for all $k, l \in \mathbb{Z}/n\mathbb{Z}$.

Suppose that $z \in \mathcal{AK}(H_n)$ and $z \neq 0$. We get $\gamma_k\gamma_0 = \gamma_k\gamma_{-k} = \gamma_k^2$ for all k. In particular, $\gamma_0 \neq 0$ and $\gamma_k = \gamma_0$ whenever $\gamma_k \neq 0$. Set $d = \min\{1 \leqslant k \leqslant n \mid \gamma_k \neq 0\}$ (recall that $\gamma_n = \gamma_0 \neq 0$). Note that $\gamma_k = 0$ for all $1 \leqslant k < d$ and, by the above, $\gamma_{k+d} = \gamma_k$ for all k. The integer d divides n. Indeed, let r be such that $rd \leqslant n < rd + d$. Then $0 \leqslant n - rd \leqslant n$ and $\gamma_{n-rd} = \gamma_n = \gamma_0 \neq 0$. Therefore, by definition of d, we get $n - rd = 0$ and so $d \mid n$. Hence $z = \gamma_0 z_d + w$ with $w \in \Bbbk[a^2]x$.

Conversely, one easily verifies that $z_d \in \mathcal{AK}(H_n)$ and so $\Bbbk z_d \oplus \Bbbk[a^2]x \subset \mathcal{AK}(H_n)$.

Let d divide n. For any $\alpha \in \Bbbk$ and $w \in \Bbbk[a^2]x$, we have

$$\lambda((\alpha z_d + w)\theta^{\pm 1}) = \frac{\alpha}{2d} \sum_{k=0}^{2d-1} \left(\omega^{n/d}\right)^{\pm s(n/d)k^2 + nk}.$$

The sum of the right-hand sum of this equality is a Gauss sum which is non-zero if and only if the enhancement $k \in \mathbb{Z}/2d\mathbb{Z} \mapsto \psi(k) = \pm s(n/d)k^2 + nk \in \mathbb{Z}/2d\mathbb{Z}$ is *tame*, that is, $\psi(x) = 0$ for any $x \in \mathbb{Z}/2d\mathbb{Z}$ such that $\psi(x+y) = \psi(x) + \psi(y)$ for all $y \in \mathbb{Z}/2d\mathbb{Z}$; see [**25**]. Since n and s are odd, it is not difficult to verify that ψ is tame. Therefore $\lambda((\alpha z_d + w)\theta^{\pm 1}) \neq 0$ if and only if $\alpha \neq 0$. Hence

$$\Bbbk^* z_d \oplus \Bbbk[a^2]x \subset \mathcal{AK}(H_n)^{\mathrm{norm}}. \qquad \square$$

In conclusion, by Lemma 5.1 and (4.23), the ribbon Hopf algebra H_n leads to $D(n)$ HKR-type invariants of 3-manifolds, where $D(n)$ denotes the number of positive divisors of n, which are $\tau_{(H_n, z_d)}$ with $1 \leqslant d \leqslant n$ and $d \mid n$.

Note that H_n is not unimodular (and so is not semisimple) since $\nu \neq \varepsilon$. Therefore $1 \notin \mathcal{AK}(H_n)$ (by Corollary 4.8), that is, the Hennings–Kauffman–Radford invariant is not defined for H_n. Moreover, the categorical Hopf algebra $A = H_n^*$ of rep_{H_n} does not possess any non-zero two-sided integral (since H_n is not unimodular), and so the Lyubashenko invariant of 3-manifolds is not defined for rep_{H_n}.

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Appendix. *Traces on ribbon Hopf algebras*

Recall that a *trace* on a Hopf algebra H is a form $t \in H^*$ such that $t(xy) = t(yx)$ and $t(S(x)) = t(x)$ for all $x, y \in H$.

Let H be a finite-dimensional ribbon Hopf algebra. Let $\lambda \in H^*$ be a non-zero right integral for H^*, $\nu \in G(H^*) = \mathrm{Alg}(H, \Bbbk)$ be the distinguished grouplike element of H^*, and G be the special grouplike element of H. Recall that $\cdot$ denotes the right action of H on H^* defined in (4.1), that $\leftharpoonup$ denotes the right H^*-action on H defined in (4.2), that $L(H)$ denotes the $\Bbbk$-subspace of H constituted by the elements $z \in H$ satisfying $(x \leftharpoonup \nu)z = zx$ for all $x \in H$, and that T denotes the $\Bbbk$-endomorphism of H defined by $z \mapsto T(z) = (S(z) \leftharpoonup \nu)h_\nu$, where $h_\nu = (\mathrm{id}_H \otimes \nu)(R) \in G(H)$.

The next proposition gives an algebraic description of the space of traces on H.

Proposition A.1. *The space $\{z \in L(H) \mid T(z) = z\}$ is $\Bbbk$-isomorphic to the space of traces on H via the map $z \mapsto \lambda \cdot (zG)$.*

If H is unimodular, then $L(H) = Z(H)$ and $T = S$, and so we recover the parameterization of traces on H given in [**23**, **9**].

Proof. Let $z \in L(H)$ such that $T(z) = z$. Set $t = \lambda \cdot (zG) \in H^*$. By using (4.10) and (4.6), we have $t(xy) = \lambda(zGxy) = \lambda(zS^2(y)Gx) = \lambda(zGyx) = t(yx)$ for any $x, y \in H$. Moreover, for any $x \in H$,

$$\begin{aligned} t(S(x)) &= \lambda(zGS(x)) = \lambda(G^2 h_\nu x G^{-1} S^{-1}(z)) \quad \text{(by (4.7))} \\ &= \lambda((S(z) \leftharpoonup \nu) G^2 h_\nu x G^{-1}) \quad \text{(by (4.3))} \\ &= \lambda(z h_\nu^{-1} G^2 h_\nu x G^{-1}) \quad (\text{since } T(z) = z) \\ &= \lambda(z S^2(G^{-1}) h_\nu^{-1} G^2 h_\nu x) \quad \text{(by (4.10))} \\ &= \lambda(z G^{-1} h_\nu^{-1} G^2 h_\nu x) \\ &= \lambda(z G S^{-4}(h_\nu^{-1}) h_\nu x) \quad \text{(by (4.6))} \\ &= \lambda(z G h_\nu^{-1} h_\nu x) = \lambda(zGx) = t(x). \end{aligned}$$

Conversely, let $t \in H^*$ be a trace on H. Since $(H^*, \cdot)$ is a free right H-module of rank 1 with basis λ and G is invertible, there exists a (unique) $z \in H$ such that $t = \lambda \cdot (zG)$. Let $x \in H$. For all $y \in H$,

$$\begin{aligned} \lambda(zxGy) &= \lambda(zGS^{-2}(x)y) \quad \text{(by (4.6))} \\ &= t(S^{-2}(x)y) = t(yS^{-2}(x)) = \lambda(zGyS^{-2}(x)) \\ &= \lambda((x \leftharpoonup \nu) zGy) \quad \text{(by (4.3))}. \end{aligned}$$

Therefore, since $(H^*, \cdot)$ is a free right H-module of rank 1 with basis λ, we have $(x \leftharpoonup \nu)zG = zxG$ and so $(x \leftharpoonup \nu)z = zx$. Hence $z \in L(H)$. Now, for all $x \in H$,

$$\begin{aligned} \lambda(zGx) &= t(x) = t(S(x)) = \lambda(zGS(x)) \\ &= \lambda(G^2 h_\nu x G^{-1} S^{-1}(z)) \quad \text{(by (4.7))} \\ &= \lambda(G^2 h_\nu x S^{-1}(zG)) \\ &= \lambda(G^2 h_\nu x S^{-1}((G \leftharpoonup \nu) z)) \quad (\text{since } z \in L(H)) \\ &= \lambda((S((G \leftharpoonup \nu) z) \leftharpoonup \nu) G^2 h_\nu x) \quad \text{(by (4.3))}. \end{aligned}$$

Therefore, since $(H^*, \cdot)$ is a free right H-module of rank 1 with basis λ,

$$\begin{aligned} zG &= (S((G \leftharpoonup \nu) z) \leftharpoonup \nu) G^2 h_\nu \\ &= (S(z) \leftharpoonup \nu)(S(G \leftharpoonup \nu) \leftharpoonup \nu) G S^2(h_\nu) G \quad \text{(by (4.6))} \\ &= (S(z) \leftharpoonup \nu) \nu(G) \nu(G^{-1}) G^{-1} G h_\nu G = (S(z) \leftharpoonup \nu) h_\nu G = T(z) G \end{aligned}$$

and so $T(z) = z$. □

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