MASTER 2 (2023) - GROUPES QUANTIQUES ET INVARIANTS QUANTIQUES

HOMEWORK - MEDNYKH'S IDENTITY

Exercice 1.
Let G be a finite group. Consider the $\mathbb{C}$-vector space $V = \mathbb{C}[G]$ spanned by G.
Define $a \in V \otimes V \otimes V$ and $\mu \in (V \otimes V)^*$ by

$$a = \frac{1}{|G|^2} \sum_{\substack{g,h,k \in G \\ ghk=1}} g \otimes h \otimes k \qquad \text{and} \qquad \mu(g \otimes h) = |G|\, \delta_{gh,1}.$$

Prove that (V, a, μ) is state sum triple, that is,

$$\tau_3(a) = a, \quad \mu\tau_2 = \mu, \quad \mu_{34}(a \otimes a) = \tau_4(\mu_{34}(a \otimes a)), \quad \mu_{19}\mu_{34}\mu_{67}(a \otimes a \otimes a) = a.$$

Here $\otimes = \otimes_{\Bbbk}$, the map τ_n is the $\Bbbk$-linear automorphism of $V^{\otimes n}$ defined by

$$\tau_n(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = x_2 \otimes \cdots \otimes x_n \otimes x_1,$$

and μ_{ij} denotes the contraction of the i-th and j-th component using μ.

Exercice 2.
Let G be a finite group. Denote by Z_G the topological invariant of closed oriented surfaces associated with the state sum triple (V, a, μ) of Exercise 1. The goal of the exercise is to prove that

$$Z_G(\Sigma) = |G|^{\chi(\Sigma)-1} \left|\mathrm{Hom}(\pi_1(\Sigma, *), G)\right|$$

for all oriented closed connected surface Σ and $* \in \Sigma$, where $\chi(\Sigma)$ is the Euler characteristic of Σ. To this aim, consider a triangulation $\mathcal{T}$ of Σ such that $*$ is a vertex of $\mathcal{T}$. Let $\mathcal{O}$ be the set of oriented edges of $\mathcal{T}$. By a *G-state* of $\mathcal{T}$, we mean a map $c\colon \mathcal{O} \to G$ such that for all $e \in \mathcal{O}$ and all triangles Δ of $\mathcal{T}$,

$$c(-e) = c(e)^{-1} \quad \text{and} \quad c(e_1^{\Delta})c(e_2^{\Delta})c(e_3^{\Delta}) = 1,$$

where $e_1^{\Delta}, e_2^{\Delta}, e_3^{\Delta}$ are the three edges adjacent to Δ oriented and cyclically ordered by the orientation of Δ induced by that of Σ. Denote by $S_G(\mathcal{T})$ the set of G-states of $\mathcal{T}$. Let $\mathcal{V}$ be the set of vertices of $\mathcal{T}$. The *gauge group* of $\mathcal{T}$ is the set $\mathcal{G}_*$ of maps $\phi\colon \mathcal{V} \to G$ such that $\phi(*) = 1$, endowed with the product defined by

$$(\phi\phi')(v) = \phi(v)\phi'(v)$$

for all $\phi, \phi' \in \mathcal{G}_*$ and all $v \in \mathcal{V}$.

a. Prove that

$$Z_G(\Sigma) = |G|^{n_1(\mathcal{T})-2n_2(\mathcal{T})} \left|S_G(\mathcal{T})\right|$$

where $n_i(\mathcal{T})$ denotes the numbers of i-cells of $\mathcal{T}$.

b. Prove that $\mathcal{G}_*$ acts freely on the left on $S_G(\mathcal{T})$ by

$$(\phi \cdot c)(e) = \phi(v_e^{\mathrm{in}})c(e)\phi(v_e^{\mathrm{out}})^{-1}$$

for all $\phi \in \mathcal{G}_*$ and $e \in \mathcal{O}$, where $v_e^{\mathrm{in}}, v_e^{\mathrm{out}} \in \mathcal{V}$ are the incoming and outgoing vertices of e, respectively.

c. Using that any loop in Σ based at $*$ is homotopic to a finite sequence of oriented edges of $\mathcal{T}$, construct a map $\Gamma\colon S_G(\mathcal{T}) \to \mathrm{Hom}(\pi_1(\Sigma, *), G)$.

d. Prove that Γ is $\mathcal{G}_*$-equivariant and induces a bijection

$$S_G(\mathcal{T})/\mathcal{G}_* \cong \mathrm{Hom}(\pi_1(\Sigma, *), G).$$

e. Conclude.

Exercice 3. (Mednykh's identity)

a. Let $(A, m, u, \Delta, \varepsilon)$ be a semisimple commutative Frobenius $\mathbb{C}$-algebra and let $e_1, \ldots, e_n$ be its primitive idempotents. Consider the 2-dimensional TQFT Z_A associated to A. Prove that for any closed connected oriented surface Σ of genus g,

$$Z_A(\Sigma) = \sum_{i=1}^{n} \varepsilon(e_i)^{1-g}.$$

b. Let G be a finite group. Denote by $\mathrm{Irr}(G)$ the set of isomorphic classes of irreducible complex representations of G. Prove that for any closed connected orientable surface Σ,

$$\left|\mathrm{Hom}(\pi_1(\Sigma), G)\right| = |G|^{1-\chi(\Sigma)} \sum_{V \in \mathrm{Irr}(G)} \dim(V)^{\chi(\Sigma)}.$$