Master 2 (2023) - Groupes quantiques et invariants quantiques

TD2 - Polynôme d'Alexander

Exercise.
Let V be a free $\Bbbk$-module of rank 2 with basis $\mathcal{B} = (b_1, b_2)$, where $\Bbbk = \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$.
Consider the following basis of $V \otimes V$:

$$\mathcal{B} \otimes \mathcal{B} = (b_1 \otimes b_1, b_1 \otimes b_2, b_2 \otimes b_1, b_2 \otimes b_2).$$

Define the $\Bbbk$-linear morphisms $R\colon V \otimes V \to V \otimes V$ and $h\colon V \to V$ by

$$\mathrm{Mat}_{\mathcal{B}\otimes\mathcal{B}}(R) = \begin{pmatrix} -t^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & t^{\frac{1}{2}} - t^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & t^{\frac{1}{2}} \end{pmatrix} \quad \text{and} \quad \mathrm{Mat}_{\mathcal{B}}(h) = \begin{pmatrix} -t^{\frac{1}{2}} & 0 \\ 0 & t^{\frac{1}{2}} \end{pmatrix}.$$

a. Prove that R and h satisfy

$$(h \otimes h)R = R(h \otimes h),$$
$$\mathrm{tr}_2((\mathrm{id}_V \otimes h)R^{\pm 1}) = \mathrm{id}_V,$$
$$(R^{-1})^{\circlearrowleft}((\mathrm{id}_V \otimes h)R(h^{-1} \otimes \mathrm{id}_V))^{\circlearrowleft} = \mathrm{id}_{V\otimes V^*},$$
$$(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R) = (R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V).$$

b. Consider the isotopy invariant $F = F_{(V,R,h)}$ of oriented tangles.
Prove that for an oriented $(1,1)$-tangle T, there is $c_T \in \Bbbk$ such that

$$F(T) = c_T\, \mathrm{id}_V.$$

c. Prove that $F(L) = 0$ for all oriented link.

d. For an oriented link L, set

$$\Delta_L(t) = c_T \in \Bbbk = \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$$

where T is any oriented $(1,1)$-tangle whose closure is L.
Prove that $\Delta_L(t)$ is a well-defined isotopy invariant of L. It is called the *Alexander polynomial* of L.

e. Prove that the Alexander polynomial satisfies $\Delta_{\mathrm{unknot}}(t) = 1$ and the skein relation:

$$\Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta_{L_0}(t).$$

f. Prove that the Alexander polynomial is multiplicative with respect to the connected sum.