Université de Lille

Master 2 (2023) - Groupes quantiques et invariants quantiques

TD3 - Algèbres de Frobenius

By a *state sum triple*, we mean a triple (V, a, μ), where V is a finite-dimensional vector space over a field $\mathbb{k}$, $a \in V \otimes V \otimes V$, and $\mu \in (V \otimes V)^*$, such that

$$\tau_3(a) = a, \quad \mu\tau_2 = \mu, \quad \mu_{34}(a \otimes a) = \tau_4(\mu_{34}(a \otimes a)), \quad \mu_{19}\mu_{34}\mu_{67}(a \otimes a \otimes a) = a.$$

Here $\otimes = \otimes_{\mathbb{k}}$, the map τ_n is the $\mathbb{k}$-linear automorphism of $V^{\otimes n}$ defined by

$$\tau_n(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = x_2 \otimes \cdots \otimes x_n \otimes x_1,$$

and μ_{ij} denotes the contraction of the i-th and j-th component using μ.

Exercice 1. (Product and coproduct of a state sum triple)
Let (V, a, μ) be a state sum triple. Set

$$\Delta = (\mu \otimes \mathrm{id}_{V^{\otimes 2}})(\mathrm{id}_V \otimes a)\colon V \to V \otimes V \quad \text{and} \quad m = (\mu \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \Delta)\colon V \otimes V \to V.$$

a. Prove that Δ is coassociative, m is associative, and

$$(m \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \Delta) = \Delta m = (\mathrm{id}_V \otimes m)(\Delta \otimes \mathrm{id}_V).$$

b. Prove that the $\mathbb{k}$-linear map $\pi = m\Delta\colon V \to V$ satisfies

$$m = \pi m = m(\mathrm{id}_V \otimes \pi) = m(\pi \otimes \mathrm{id}_V)$$

and is a projector (i.e., $\pi^2 = \pi$).

c. Assume that μ is non-degenerate. Prove that

$$\mathrm{Im}(\pi) = \mathrm{Im}(m) = \{(\mathrm{id}_V \otimes \phi)(a) \,|\, \phi \in (V \otimes V)^*\}$$

and

$$(V, m) \text{ has a unit} \iff \pi = \mathrm{id}_V \iff V = \{(\mathrm{id}_V \otimes \phi)(a) \,|\, \phi \in (V \otimes V)^*\}.$$

Exercice 2.
Let (V, a, μ) be a state sum triple. Consider the kernel $\mathrm{Ker}(\mu) = \{x \in V \,|\, \mu(x \otimes \cdot) = 0\}$ of μ and the canonical projection map

$$p\colon V \to \overline{V} = V/\mathrm{Ker}(\mu).$$

a. Prove that there is a unique $\overline{\mu} \in (\overline{V} \otimes \overline{V})^*$ such that $\mu = \overline{\mu}(p \otimes p)$.
b. Prove that $\overline{\mu}$ is non-degenerate.
c. Prove that $(\overline{V}, \overline{a}, \overline{\mu})$ is a state sum triple, where $\overline{a} = (p \otimes p \otimes p)(a)$.

Exercice 3.
Let (V, a, μ) be a state sum triple. Set

$$\widetilde{V} = \{(\mathrm{id}_V \otimes \phi)(a) \,|\, \phi \in (V \otimes V)^*\}.$$

a. Prove that $\widetilde{V}$ is the smallest vector subspace of V such that $a \in \widetilde{V} \otimes \widetilde{V} \otimes \widetilde{V}$.
b. Prove that $(\widetilde{V}, \widetilde{a}, \widetilde{\mu})$ is a state sum triple, where $\widetilde{a} = a$ and $\widetilde{\mu} = \mu_{|\widetilde{V} \otimes \widetilde{V}}$.
c. Prove that if μ is non-degenerate, then so is $\widetilde{\mu}$.

Exercice 4.
Let (V, a, μ) be a state sum triple such that $V = \{(\mathrm{id}_V \otimes \phi)(a) \,|\, \phi \in (V \otimes V)^*\}$ and μ is non-degenerate. Consider the center A of the $\mathbb{k}$-algebra (V, m). Prove that $\mu_{|A \otimes A}$ is non-degenerate.

Exercice 5.

Let (V, a, μ) be a state sum triple. The goal of this exercise is to determine the Frobenius algebra associated to the 2-dimensional TQFT $Z_{(V,a,\mu)}$. We say that two state sum triples are *equivalent* if they define isomorphic TQFTs.

a. Let $(\overline{V}, \overline{a}, \overline{\mu})$ be the state sum triple of Exercise 2.
Prove that the state sum triples $(\overline{V}, \overline{a}, \overline{\mu})$ et (V, a, μ) are equivalent.

b. Let $(\widetilde{V}, \widetilde{a}, \widetilde{\mu})$ be the state sum triple of Exercise 3.
Prove that the state sum triples $(\widetilde{V}, \widetilde{a}, \widetilde{\mu})$ et (V, a, μ) are equivalent.

c. Assume that $V = \{(\mathrm{id}_V \otimes \phi)(a) \,|\, \phi \in (V \otimes V)^*\}$ and μ is non-degenerate.
Prove that the Frobenius algebra associated to the 2-dimensional TQFT $Z_{(V,a,\mu)}$ is isomorphic to the center of the $\mathbb{k}$-algebra (V, m), where

$$m = (\mu \otimes \mathrm{id}_V \otimes \mu)(\mathrm{id}_V \otimes a \otimes \mathrm{id}_V) \colon V \otimes V \to V.$$

d. Conclude by noticing that the state sum triple $(\widetilde{\overline{V}}, \widetilde{\overline{a}}, \widetilde{\overline{\mu}})$ satisfies the assumptions of question c.