Université de Lille

MASTER 2 (2023) - GROUPES QUANTIQUES ET INVARIANTS QUANTIQUES

TD4 - ALGÈBRES DE HOPF

In what follows, $\Bbbk$ is a field.

Exercice 1. (The restricted dual)
We denote the dual of a $\Bbbk$-vector space V by $V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ and the transpose of a linear map $f\colon V \to W$ by $f^*\colon W^* \to V^*$. Let A be an $\Bbbk$-algebra with product $\mu\colon A \otimes A \to A$ and unit $\eta\colon \Bbbk \to A$. We view $A^* \otimes A^*$ as a subset of $(A \otimes A)^*$ as follows: for any $f, g \in A^*$, we define $f \otimes g \in (A \otimes A)^*$ by setting

$$(f \otimes g)(a \otimes b) = f(a)g(b)$$

for all $a, b \in A$. The *restricted dual* of the algebra A is the subspace of the dual A^* defined by

$$A^\circ = (\mu^*)^{-1}(A^* \otimes A^*),$$

where $\mu^*\colon A^* \to (A \otimes A)^*$ is the transpose of μ.

a. Prove that a form $f \in A^*$ belongs to A° if and only if there is an ideal I of A such that $\dim(A/I) < \infty$ and $f(I) = 0$.

b. Prove that A° is the largest subspace of A^* such that $\mu^*(A^\circ) \subset A^\circ \otimes A^\circ$.

c. Prove that A° is a coalgebra with coproduct and counit

$$\Delta_{A^\circ} = \mu^*\colon A^\circ \to A^\circ \otimes A^\circ \quad \text{and} \quad \varepsilon_{A^\circ} = \eta^*\colon A^\circ \subset A^* \to \Bbbk^* = \Bbbk.$$

d. Assume that A is a Hopf algebra with coproduct $\Delta\colon A \to A \otimes A$, counit $\varepsilon\colon A \to \Bbbk$, and antipode $S\colon A \to A$. Prove that

$$\Delta^*(A^\circ \otimes A^\circ) \subset A^\circ, \qquad \varepsilon^*(\Bbbk^*) \subset A^\circ, \qquad S^*(A^\circ) \subset A^\circ.$$

Deduce that the restricted dual A° of A is a Hopf algebra with product, unit, and antipode defined by

$$\mu_{A^\circ} = \Delta^*\colon A^\circ \otimes A^\circ \to A^\circ, \quad \eta_{A^\circ} = \varepsilon^*\colon \Bbbk = \Bbbk^* \to A^\circ, \quad S_{A^\circ} = S^*\colon A^\circ \to A^\circ.$$

e. Let A be a finite-dimensional Hopf algebra. Then $A^\circ = A^*$, so that A^* is a Hopf algebra. Likewise $A^{**} = (A^*)^*$ is a Hopf algebra. Prove that $A^{**} \simeq A$ as Hopf algebras.

Exercice 2. (Grouplike and primitive elements)
Let A be a Hopf algebra with coproduct Δ and counit ε. An element $g \in A$ is *grouplike* if $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$. An element $a \in A$ is *primitive* if $\Delta(a) = a \otimes 1 + 1 \otimes a$.

a. Prove that the set $G(A)$ of grouplike elements of A is a group (under multiplication).

b. Prove that the grouplike elements of A are linearly independent.

c. Prove that the set $P(A)$ of primitive elements is a Lie subalgebra of A.

d. Let $I = \mathrm{Ker}(\varepsilon)$. Prove that $A = \Bbbk \oplus I$ as vector spaces and that any $a \in I$ satisfies:

$$\Delta(a) = a \otimes 1 + 1 \otimes a \mod I \otimes I.$$

Exercice 3. (PBW basis for quantum groups)
Fix $q \in \Bbbk^*$ not a root of unity. The goal of the exercise is to prove that the set

$$\mathcal{B} := \{F^i K^n E^j \,\big|\, i, j \in \mathbb{N},\, n \in \mathbb{Z}\}$$

is linearly independent; recall from the lecture that $\mathcal{B}$ is a spanning set. The Hopf algebra $U_q(\mathfrak{sl}_2)$ is $\mathbb{Z}$-graded so that $\deg E = 2$, $\deg F = -2$ and $\deg K = 0$. Its coproduct is given by

$$\Delta(E) = K \otimes E + E \otimes 1, \quad \Delta(F) = 1 \otimes F + F \otimes K^{-1}, \quad \Delta(K) = K \otimes K.$$

For $n \in \mathbb{Z}$ let $\pi_n : U_q(\mathfrak{sl}_2) \longrightarrow U_q(\mathfrak{sl}_2)_n$ denote the projection to the degree n component.

a. Prove that the K^n for $n \in \mathbb{Z}$ are grouplike and two-by-two distinct, so that they are linearly independent by the previous exercise. (Hint: apply the vector representation.)

b. Prove that $E^j \neq 0 \neq F^j$ for $j \in \mathbb{N}$. (Hint: apply the irreducible representation of highest weight q^j.)

c. Assume we have a nontrivial linear relation $\sum_{i\in\mathbb{N},n\in\mathbb{Z}} \lambda_{i,n} F^i K^n = 0$.

1. Prove that there exists $i_0 > 0$ and $n_0 \in \mathbb{Z}$ such that $\lambda_{i_0,n_0} \neq 0$ and $\lambda_{i,n} \neq 0$ only if $i < i_0$.

2. Compute $(\pi_0 \otimes \pi_{-2i_0})\Delta(F^i K^n)$ for $i \leq i_0$.

3. Prove that $\sum_{n\in\mathbb{Z}} \lambda_{i_0,n} K^n \otimes F^{i_0} = 0$.

4. Deduce from **a** and **b** the contradiction $\lambda_{i_0,n} = 0$ for all $n \in \mathbb{Z}$.

It follows that the $F^i K^n$ for $i \in \mathbb{N}$ and $n \in \mathbb{Z}$ are linearly independent.

d. Assume we have a nontrivial linear relation $\sum_{i,j\in\mathbb{N},n\in\mathbb{Z}} \lambda_{i,n,j} F^i K^n E^j = 0$.

1. Prove that there exists $j_0 > 0$ and $i_0 \in \mathbb{N}$, $n_0 \in \mathbb{Z}$ such that $\lambda_{i_0,n_0,j_0} \neq 0$ and $\lambda_{i,n,j} \neq 0$ only if $j < j_0$.

2. Compute $(\pi_{2j_0} \otimes \mathrm{Id})\Delta(F^i K^n E^j)$ for $j \leq j_0$.

3. Prove that $\sum_{i\in\mathbb{N},n\in\mathbb{Z}} \lambda_{i,n,j_0} K^n E^{j_0} \otimes F^i K^n = 0$.

4. Deduce from **b** and **c** the contradiction $\lambda_{i,n,j_0} = 0$ for all $i \in \mathbb{N}$ and $n \in \mathbb{Z}$.

Conclude.

Exercice 4. (Simple of Weyl modules)

Let $\mathbb{k} = \mathbb{C}$ and ε be a root of unity of odd order $\ell > 1$. Recall the Weyl module $W(n)$ for $n \in \mathbb{N}$ over the quantum group $\mathbf{U}_\varepsilon$. Let $n + 1 = n_0 + \ell n_1$ be the euclidean division of $n + 1$ by ℓ. Prove that the module $W(n)$ is simple if and only if $n_0 n_1 = 0$.