QUANTUM GROUPS

Contents

1. Hopf algebras

Throughout the text, the ground field is $\mathbb{K}$. All vector spaces are defined over $\mathbb{K}$.

1.1. **Recall on tensor product.** Let V_1, V_2 and W be vector spaces. A map $F : V_1 \times V_2 \longrightarrow W$ is called *bilinear* if for $\lambda \in \mathbb{K}$, $v_1, v_1' \in V_1$ and $v_2, v_2' \in V_2$ we have

$$F(\lambda v_1 + v_1', v_2) = \lambda F(v_1, v_2) + F(v_1', v_2), \quad F(v_1, \lambda v_2 + v_2') = \lambda F(v_1, v_2) + F(v_1, v_2').$$

Proposition 1.1. *Given two vector spaces V_1 and V_2, we have a vector space V and a bilinear map $\iota : V_1 \times V_2 \longrightarrow V$ satisfying the following* **universal property***:*

> *if W is a vector space and $F : V_1 \times V_2 \longrightarrow W$ is a bilinear map, there exists a unique linear map $f : V \longrightarrow W$ such that $F = f \circ \iota$.*

If (V', ι') is another pair of a vector space and a bilinear map satisfying the universal property, then we have a unique linear isomorphism $f : V \longrightarrow V'$ such that $\iota' = f \circ \iota$.

Write $V := V_1 \otimes V_2$. For $v_1 \in V_1$ and $v_2 \in V_2$, let $v_1 \otimes v_2 \in V$ denote $\iota(v_1, v_2)$.

If $f : V_1 \longrightarrow V_1'$ and $g : V_2 \longrightarrow V_2'$ are linear maps, then we have a bilinear map $F : V_1 \times V_2 \longrightarrow V_1' \otimes V_2'$ given by $F(v_1, v_2) = f(v_1) \otimes g(v_2)$. From the universal property, we obtain a linear map $\mathcal{F} : V_1 \otimes V_2 \longrightarrow V_1' \otimes V_2'$ such that $\mathcal{F}(v_1 \otimes v_2) = f(v_1) \otimes g(v_2)$. Such a linear map is denoted by $f \otimes g$. Whenever composition is well-defined, we have

$$(f_1 \otimes g_1) \circ (f_2 \otimes g_2) = f_1 \circ f_2 \otimes g_1 \circ g_2, \quad \mathrm{Id}_{V \otimes W} = \mathrm{Id}_V \otimes \mathrm{Id}_W.$$

Example 1.2. The map $F : V_1 \times V_2 \longrightarrow V_2 \otimes V_1$ defined by $F(v_1, v_2) = v_2 \otimes v_1$ is bilinear. It induces a linear map $f : V_1 \otimes V_2 \longrightarrow V_2 \otimes V_1$ such that $f(v_1 \otimes v_2) = v_2 \otimes v_1$. Such a linear map is denoted by σ_{V_1, V_2}, and called *flip map*.

The tensor product of n vector spaces $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ is well-defined for $n \geq 3$. We have natural identifications of tensor products and identities of linear maps

$$(U \otimes V) \otimes W = U \otimes (V \otimes W), \quad V \otimes \mathbb{K} = V = \mathbb{K} \otimes V,$$
$$(f \otimes g) \otimes h = f \otimes (g \otimes h), \quad f \otimes \mathrm{Id}_{\mathbb{K}} = f = \mathrm{Id}_{\mathbb{K}} \otimes f.$$

1.2. **Algebras and representations.** An *algebra* is a vector space A (addition and scalar multiplication) together with a bilinear map $A \times A \longrightarrow A, (a, b) \mapsto ab$ and an element 1 such that

$$(ab)c = a(bc), \quad \lambda a = (\lambda 1)a = a(\lambda 1).$$

This is equivalent to a vector space A together with two linear maps $A \otimes A \longrightarrow A$ and $\mathbb{K} \longrightarrow A$ satisfying associativity and unity. Call 1 the identity element.

A *subalgebra* of A is a subspace B stable under multiplication and containing 1.

An *ideal* of A is a subspace I such that $AI \subset I$ and $IA \subset I$. Given such I, the quotient space A/I is an algebra with multiplication

$$\overline{a}\overline{b} = \overline{ab}.$$

An *algebra homomorphism* $F : A \longrightarrow B$ is a linear map such that

$$F(a_1 a_2) = F(a_1) F(a_2), \quad F(1) = 1.$$

The tensor product of two algebras A and B is an algebra with multiplication

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2.$$

Example 1.3. If V is a vector space, then the vector space End$V$ of all linear endomorphisms of V forms an algebra, with multiplication given by composition. Let W be another vector space. Then the bilinear map $(f,g) \mapsto f \otimes g$ extends to an injective algebra homomorphism $\mathrm{End}V \otimes \mathrm{End}W \longrightarrow \mathrm{End}(V \otimes W)$. It is an isomorphism if V or W is finite-dimensional.

Proposition 1.4. *Given a vector space V, we have an algebra A and a linear map $\iota : V \longrightarrow B$ satisfying the* **universal property***:*

> *if B is an algebra and $F : V \longrightarrow B$ is a linear map, then there exists a unique algebra homomorphism $f : A \longrightarrow B$ such that $F = f \circ \iota$.*

If (A', ι') is another pair of an algebra and a linear map satisfying the universal property, then there exists a unique algebra isomorphism $f : A \longrightarrow A'$ such that $\iota' = f \circ \iota$.

Call A the tensor algebra of V and denote it by $T(V)$. Let X be a basis of V and Let R be a subset of $T(V)$. Call the quotient algebra $T(V)/\langle R\rangle$ the algebra with generators X and relations R. Here $\langle R\rangle$ denotes the ideal of $T(V)$ generated by R.

Example 1.5. Let V be a finite-dimensional vector space with basis $(e_1, e_2, \cdots, e_n)$. Then $\mathrm{End}V \cong T(W)/\langle R\rangle$ where

$$W := \bigoplus_{1\leq i,j\leq n} \mathbb{K}e_{ij}, \quad R = \{e_{ij}\otimes e_{kl} - \delta_{jk}e_{il} \in W \oplus W^{\otimes 2} \mid 1 \leq i,j,k,l \leq n\} \cup \{\sum_{i=1}^{n} e_{ii} - 1\}.$$

Definition 1.6. A *representation* of an algebra A is a vector space V equipped with an algebra homomorphism $\rho : A \longrightarrow \mathrm{End}V$. Call V an *A-module*.

Submodules, quotient modules, irreducible modules are defined in the obvious way.

Given two A-modules V and W, the tensor product is naturally a module over $A \otimes A$ with structural map

$$A \otimes A \longrightarrow \mathrm{End}V \otimes \mathrm{End}W \hookrightarrow \mathrm{End}(V \otimes W).$$

To make it an A-module, we would like to have an algebra homomorphism $A \longrightarrow A \otimes A$.

1.3. **Monoid algebra.** By a *monoid* we mean a set M endowed with a binary operation $M \times M \longrightarrow M, a \otimes b \mapsto ab$ and an element 1 such that: $(ab)c = a(bc)$ and $1a = a = a1$. An algebra is a monoid with the binary operation given by multiplication.

To a monoid M we attach its monoid algebra $\mathbb{K}[M] := T(V)/\langle R\rangle$ where

$$V := \bigoplus_{a\in M} \mathbb{K}e_a, \quad R := \{e_a \otimes e_b - e_{ab} \in V \oplus V^{\otimes 2} \mid a,b \in M\} \cup \{e_1 - 1\}.$$

(i) *Universal property*: If A is an algebra and $F : M \longrightarrow A$ is a monoid homomorphism, then there exists a unique algebra homomorphism $f : \mathbb{K}[M] \longrightarrow A$ such that $F = f \circ \iota$. Here $\iota : M \longrightarrow \mathbb{K}[M]$ is the map $\iota(a) = e_a$.

(ii) *Explicit realization*: The algebra $\mathbb{K}[M]$ is realized on the vector space $\oplus_{a\in M}\mathbb{K}e_a$. Its multiplication is induced by $e_a e_b = e_{ab}$.

Example 1.7. If M is the monoid $\mathbb{Z}^n$ of n-tuples of integers, then $\mathbb{K}[M]$ is the Laurent polynomial algebra $\mathbb{K}[t_1^{\pm 1}, t_2^{\pm 1}, \cdots, t_n^{\pm 1}]$.

1.4. **Universal enveloping algebra.** A *Lie algebra* is a vector space $\mathfrak{g}$ equipped with a bilinear map $\mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}$, $(x, y) \mapsto [x, y]$, called *Lie bracket*, such that

$$[x, x] = 0 = [[x, y], z] + [[y, z], x] + [[z, x], y].$$

An algebra is viewed as a Lie algebra with Lie bracket $[a, b] := ab - ba$.

To a Lie algebra $\mathfrak{g}$ we attach its *universal enveloping algebra* $U(\mathfrak{g}) := T(\mathfrak{g})/\langle R \rangle$ where

$$R := \{x \otimes y - y \otimes x - [x, y] \in \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2} \mid x, y \in \mathfrak{g}\}.$$

(i) *Universal property*: If A is an algebra and $F : \mathfrak{g} \longrightarrow A$ is a Lie algebra homomorphism, then there exists a unique algebra homomorphism $f : U(\mathfrak{g}) \longrightarrow A$ such that $F = f \circ \iota$. Here $\iota : \mathfrak{g} \longrightarrow U(\mathfrak{g})$ is the composition $\mathfrak{g} \longrightarrow T(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$.
(ii) Let $(\mathcal{B}, \prec)$ be an ordered basis of $\mathfrak{g}$. Then the defining ideal of $U(\mathfrak{g})$ is generated by $x \otimes y - y \otimes x - [x, y]$ for $x \prec y$ in $\mathcal{B}$.
(iii) *Poincaré–Birkhoff–Witt*: The ordered monomials in the $x + \langle R \rangle$ for $x \in \mathcal{B}$ with respect to the ordering of $\mathcal{B}$ form a basis of the quotient space $U(\mathfrak{g})$ of $T(\mathfrak{g})$.

Example 1.8. $\mathfrak{g} = \mathfrak{sl}_2$ is the Lie algebra of two-by-two traceless matrices. It has a basis $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and its Lie bracket is

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

$U(\mathfrak{sl}_2)$ is the algebra with generators e, f, h and relations

$$ef - fe = h, \quad he - eh = 2e, \quad hf - fh = -2f.$$

1.5. **Coalgebras and convolution product.** A *coalgebra* is a vector space C together with two linear maps $\Delta : C \longrightarrow C \otimes C$ and $\epsilon : C \longrightarrow \mathbb{K}$ such that

$$(\Delta \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \Delta)\Delta, \quad (\epsilon \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \epsilon)\Delta = \mathrm{Id}.$$

C is called *co-commutative* if $\sigma\Delta = \Delta$. A sub-coalgebra is a subspace D of C such that $\Delta(D) \subset D \otimes D$. A co-ideal is a subspace I of C such that $\epsilon(I) = \{0\}$ and $\Delta(I) \subset I \otimes C + C \otimes I$. The quotient space C/I is naturally a coalgebra.

A coalgebra homomorphism from a coalgebra (C, Δ, ϵ) to a coalgebra (C', Δ', ϵ') is a linear map $f : C \longrightarrow C'$ such that

$$(f \otimes f)\Delta = \Delta' f, \quad \epsilon = \epsilon' f.$$

The tensor product $C \otimes C'$ is a coalgebra with coproduct $(\mathrm{Id} \otimes \sigma \otimes \mathrm{Id})(\Delta \otimes \Delta')$ and counit $\epsilon \otimes \epsilon'$.

Lemma 1.9. *Let (A, m, η) be an algebra and (C, Δ, ϵ) be a coalgebra. For f and g two linear maps from C to A, define their* **convolution product** *$f * g$ to be the linear map $m(f \otimes g)\Delta : C \longrightarrow A$. Namely, for $x \in C$ we have*

$$(f * g)(x) = \sum_i f(a_i)g(b_i) \quad \text{if } \Delta(x) = \sum_i a_i \otimes b_i.$$

Then the vector space $\mathrm{Hom}(C, A)$ *equipped with the convolution product is an algebra whose identity element is $\eta\epsilon$.*

As a consequence, the linear dual of a coalgebra is naturally an algebra.

Example 1.10. (i) Let C be the vector space with basis $(e_i : 1 \leq i \leq n)$. Then C is a coalgebra by setting $\Delta(e_i) = e_i \otimes e_i$ and $\epsilon(e_i) = 1$. Its dual is identified the vector space of maps from $\{1, 2, \cdots, n\}$ to $\mathbb{K}$. The convolution product of two such maps f and g is simply the usual product: $(f * g)(i) = f(i)g(i)$ for $1 \leq i \leq n$.

(ii) Let C be the vector space with basis $(e_{ij} : 1 \leq i, j \leq n)$. It is a coalgebra:

$$\Delta(e_{ij}) = \sum_{k=1}^{n} e_{ik} \otimes e_{kj}, \quad \epsilon(e_{ij}) = \delta_{ij}.$$

Its dual is the algebra $\mathrm{Mat}_{n\times n}(\mathbb{K})$ of $n \times n$ matrices.

1.6. **Bialgebras.**

Lemma 1.11. *Let B be vector space with is equipped with an algebra structure (B, m, η) and a coalgebra structure (B, Δ, ϵ). Then the following two conditions are equivalent.*

(i) *$m : B \otimes B \longrightarrow B$ and $\eta : \mathbb{K} \longrightarrow B$ are coalgebra homomorphisms.*
(ii) *$\Delta : B \longrightarrow B \otimes B$ and $\epsilon : B \longrightarrow \mathbb{K}$ are algebra homomorphisms.*

Under these conditions, B is called a **bialgebra**.

Example 1.12. (i) Let M be a monoid. Then $\mathbb{K}[M]$ is a bialgebra.
(ii) Let $\mathfrak{g}$ be a Lie algebra. We have Lie algebra homomorphisms

$$\mathfrak{g} \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g}), \quad \mathfrak{g} \longrightarrow \mathbb{K}$$

which send x to $x \otimes 1 + 1 \otimes x$ and 0 respectively. They extend to algebra homomorphisms $\Delta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$ and $\epsilon : U(\mathfrak{g}) \longrightarrow \mathbb{K}$. In this way, $U(\mathfrak{g})$ becomes a bialgebra.

1.7. **Hopf algebras.** Let $(B, m, \eta, \Delta, \epsilon)$ be a bialgebra. Then $\mathrm{Hom}(B, B)$ equipped with convolution product is an algebra. Call B a *Hopf algebra* if the linear map $\mathrm{Id} : B \longrightarrow B$ is invertible with respect to the convolution product. The inverse is called the *antipode* $S : B \longrightarrow B$.

To an algebra (A, m, η) we attach its *opposite algebra* $(A, m\sigma, \epsilon)$, denoted by A^{op}. To a coalgebra (C, Δ, ϵ) we attach its *opposite coalgebra* $(C, \sigma\Delta, \epsilon)$, denoted by C^{cop}.

Example 1.13. (i) Let M be a monoid. The bialgebra $\mathbb{K}[M]$ is a Hopf algebra if and only if M is a group. The antipode is given by $S(e_x) = e_{x^{-1}}$.
(ii) Let $\mathfrak{g}$ be a Lie algebra. We have a Lie algebra homomorphism

$$\mathfrak{g} \longrightarrow U(\mathfrak{g})^{\mathrm{op}}, \quad x \mapsto -x,$$

which extends to an algebra homomorphism $S : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})^{\mathrm{op}}$, which is the antipode. Therefore, $U(\mathfrak{g})$ is always a Hopf algebra.

2. Quantum groups I

2.1. **Antipode and duality.** Let $(B, m, \eta, \Delta, \epsilon)$ be a bialgebra. Given two B-modules U and V, we can equip their tensor product $U \otimes V$ with a B-module structure by setting $a(u \otimes v) = \sum_i a_i u \otimes b_i v$ for $\Delta(a) = \sum_i a_i \otimes b_i$ and $u \in U,\ v \in V$.

The counit $\epsilon : B \longrightarrow \mathbb{K}$ equips $\mathbb{K}$ with a B-module structure.

For three B-modules U, V and W, the identity maps are B-module morphisms:

$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W), \quad \mathbb{K} \otimes U \cong U \cong U \otimes \mathbb{K}.$$

If $f : U \longrightarrow U'$ and $g : V \longrightarrow V'$ are B-module morphisms, then so is $f \otimes g$.

Proposition 2.1. *In a Hopf algebra $(H, m, \eta, \Delta, \epsilon, S)$, the antipode $S : H \longrightarrow H$ is an anti-homomorphism of algebras and an anti-homomorphism of coalgebras.*

Proof. We shall prove the first half, namely, $Sm = (S \otimes S)m\sigma$ in the convolution algebra $\mathrm{Hom}_{\mathbb{K}}(H \otimes H, H)$. Since m is a coalgebra homomorphism, we have

$$Sm * m = m(S \otimes \mathrm{Id})(m \otimes m)\Delta_{H \otimes H} = m(S \otimes \mathrm{Id})\Delta m = \eta\epsilon m = m * Sm.$$

So Sm is the convolution-inverse of m. On the other hand, for $x, y \in H$ with $\Delta(x) = \sum_i a_i \otimes b_i$ and $\Delta(y) = \sum_j c_j \otimes d_j$ we have $\Delta_{H \otimes H}(x \otimes y) = \sum_{i,j} a_i \otimes c_j \otimes b_i \otimes d_j$ and

$$\begin{aligned}(S \otimes S)m\sigma * m(x \otimes y) &= \sum_{i,j} S(c_j)S(a_i)b_i d_j = \sum_j S(c_j) \times \sum_i S(a_i)b_i \times d_j \\ &= \sum_j S(c_j) \times \epsilon(x)1 \times d_j = \epsilon(x) \sum_j S(c_j)d_j = \epsilon(x)\epsilon(y)1.\end{aligned}$$

By invertibility of m, we must have $Sm = (S \otimes S)m\sigma$. □

Definition 2.2. Let $(H, m, \eta, \Delta, \epsilon, S)$ be a Hopf algebra. Given two H-modules U and V, the space $\mathrm{Hom}_{\mathbb{K}}(U, V)$ of linear maps from U to V has a H-module structure: for $x \in H$ with coproduct $\Delta(x) = \sum_i a_i \otimes b_i$ and for $f : U \longrightarrow V$ a linear map, af is another linear map from U to V given by

$$\langle af, u\rangle = \sum_i a_i \langle f, (S(b_i)u\rangle \quad \text{for } u \in U.$$

It follows that the dual $U^* = \mathrm{Hom}_{\mathbb{K}}(U, \mathbb{K})$ of an H-module U is still an H-module. In particular, the natural linear map

$$V \otimes U^* \longrightarrow \mathrm{Hom}_{\mathbb{K}}(U, V)$$

is a H-module homomorphism.

Lemma 2.3. *Let H be a Hopf algebra and U be a finite-dimensional H-module. Then the evaluation map $e_U : U^* \otimes U \longrightarrow \mathbb{K}$ and the coevaluation map $c_U : \mathbb{K} \longrightarrow U \otimes U^*$ are H-module morphisms.*

Proof. Choose a basis $(u_1, u_2, \cdots, u_n)$ of U and let $(u_1^*, u_2^*, \cdots, u_n^*)$ be the dual basis of U^*. Then $c_U(1) = \sum_i u_i \otimes u_i^*$. For $h \in H$ with $\Delta(h) = \sum_i a_i \otimes b_i$ we have

$$\begin{aligned}hc_U(1) &= h \sum_s u_s \otimes u_s^* = \sum_{i,s} a_i u_s \otimes b_i u_s^* = \sum_{i,s,t} a_i u_s \otimes \langle b_i u_s^*, u_t\rangle u_t^* \\ &= \sum_{i,s,t} a_i u_s \otimes \langle u_s^*, S(b_i)u_t\rangle u_t^* = \sum_{i,t} a_i (\sum_s \langle u_s^*, S(b_i)u_t\rangle u_s) \otimes u_t^* \\ &= \sum_{i,t} a_i S(b_i)u_t \otimes u_t^* = \epsilon(h)c_U(1) = c_U(h1).\end{aligned}$$

So c_U is H-linear. For $f \in U^*$ and $u \in U$, we have

$$e_U(h(f \otimes u)) = e_U(\sum_i a_i f \otimes b_i u) = \sum_i \langle a_i f, b_i u\rangle = \sum_i f(S(a_i)b_i u)$$
$$= f(\sum_i S(a_i)b_i u) = f(\epsilon(h)u) = \epsilon(h)\langle f, u\rangle = he_U(f \otimes u).$$

So e_U is H-linear. □

Let U be a finite-dimensional H-module. Then the double dual module U^{**} is the pullback of the H-module U along the algebra homomorphism S^2.

Assume that there exists an invertible element $h_0 \in H^\times$ such that $S^2(h) = h_0 h h_0^{-1}$ for $h \in H$. Then we have an H-module isomorphism

$$U^{**} \longrightarrow U \quad u \mapsto h_0^{-1} u.$$

As a consequence, the following are H-module morphisms:

$$\tilde{e}_U : U \otimes U^* \longrightarrow \mathbb{K}, \quad u \otimes f \mapsto \langle f, h_0 u\rangle,$$
$$\tilde{c}_U : \mathbb{K} \longrightarrow U^* \otimes U, \quad 1 \mapsto \sum_i u_i^* \otimes h_0^{-1} u_i.$$

2.2. **Braided bialgebras.** A bialgebra B is *braided* (or *quasi-triangular*) if there exists an invertible element $R \in (H \otimes H)^\times$ satisfying

$$\Delta^{\text{cop}}(x) = R\Delta(x)R^{-1} \quad \text{for } x \in H,$$
$$(\Delta \otimes \text{Id})(R) = R_{13}R_{12}, \quad (\text{Id} \otimes \Delta)(R) = R_{13}R_{23}$$

where for $R = \sum_i a_i \otimes b_i$ we set

$$R_{12} = \sum_i a_i \otimes b_i \otimes 1, \quad R_{13} = \sum_i a_i \otimes 1 \otimes b_i, \quad R_{23} = \sum_i 1 \otimes a_i \otimes b_i.$$

Given two B-modules U and V, the following defines a B-module morphism

$$c_{U,V} : U \otimes V \longrightarrow V \otimes U, \quad c_{U,V}(u \otimes v) = \sum_i b_i v \otimes a_i u.$$

Namely, $c_{U,V} = \sigma_{U,V} R|_{U \otimes V}$. It is invertible with inverse given by $c_{U,V}^{-1} = R^{-1}|_{U,V}\sigma_{V,U}$. These isomorphisms are functorial in the sense that if $f : U \longrightarrow U'$ and $g : V \longrightarrow V'$ are B-module morphisms then

$$c_{U',V'}(f \otimes g) = (g \otimes f)c_{U,V}.$$

Furthermore, they satisfy the two relations

$$c_{U \otimes V,W} = (c_{U,W} \otimes \text{Id}_V)(\text{Id}_U \otimes c_{V,W}), \quad c_{U,V \otimes W} = (\text{Id}_V \otimes c_{U,W})(c_{U,V} \otimes \text{Id}_W).$$

The category of B-modules is therefore braided monoidal.

2.3. **The quantum group $U_q(\mathfrak{sl}_2)$.** Let $\mathbb{K}$ be a field. Fix $q \in \mathbb{K}^\times$ which is not a root of unity. For $t \in \mathbb{Z}$ and $n \in \mathbb{N}$ define the q-numbers in $\mathbb{K}^\times$

$$(t)_q := \frac{q^{2t}-1}{q^2-1}, \quad (n)_q! := \prod_{m=1}^{n} (m)_q, \quad \binom{t}{n}_q := \prod_{m=1}^{n} \frac{(t-m+1)_q}{(m)_q}.$$

The *quantum group* $U_q(\mathfrak{sl}_2)$ is the algebra generated by E, F, K, K^{-1} subject to relations

$$KK^{-1} = K^{-1}K = 1, \quad KE = q^2 EK, \quad KF = q^{-2}FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

It is a Hopf algebra whose coproduct is determined by

$$\Delta(K) = K \otimes K, \quad \Delta(E) = K \otimes E + E \otimes 1, \quad \Delta(F) = 1 \otimes F + F \otimes K^{-1}.$$

Its antipode is the algebra anti-automorphism $S : U_q(\mathfrak{sl}_2) \longrightarrow U_q(\mathfrak{sl}_2)$ determined by

$$S(K) = K^{-1}, \quad S(E) = -K^{-1}E, \quad S(F) = -FK.$$

Proposition 2.4. *We have $S^2(x) = K^{-1}xK$ for $x \in U_q(\mathfrak{sl}_2)$. As a vector space, $U_q(\mathfrak{sl}_2)$ is spanned by the monomials $F^m K^n E^p$ where $m, p \in \mathbb{N}$ and $n \in \mathbb{Z}$.*

The Hopf algebra $U_q(\mathfrak{sl}_2)$ is $2\mathbb{Z}$-graded by setting the degrees of the generators to be

$$\deg E = 2, \quad \deg F = -2, \quad \deg K = \deg K^{-1} = 0.$$

2.4. **The quasi R-matrix.** This is the following power series in z with coefficients in the tensor product algebra $U_q(\mathfrak{sl}_2)^{\otimes 2}$:

$$\overline{\mathcal{R}}(z) := \sum_{n=0}^{+\infty} z^n c_n E^n \otimes F^n \quad \text{where } c_n := \frac{(q^{-1}-q)^n}{(n)_q!} \in \mathbb{K}^\times. \tag{2.1}$$

It is invertible in the algebra $U_q(\mathfrak{sl}_2)^{\otimes 2}[[z]]$. The constant term of $\overline{\mathcal{R}}(z)^{-1}$ is still 1 and its coefficient of z^n is proportional to $E^n \otimes F^n$ for $n \in \mathbb{N}$.

Lemma 2.5. *Given two elements x and y in an algebra A such that $xy = q^2 yx$, we have the following q-binomial formula for $n \in \mathbb{N}$:*

$$(x+y)^n = \sum_{s=0}^{n} \binom{n}{s}_q y^s x^{n-s} \in A.$$

Proposition 2.6. *The quasi R-matrix satisfies the following equations in the algebras $U_q(\mathfrak{sl}_2)^{\otimes 2}[[z]]$ and $U_q(\mathfrak{sl}_2)^{\otimes 3}[[z]]$:*

$$\overline{\mathcal{R}}(z)(1 \otimes E + zE \otimes K^{-1}) = (1 \otimes E + zE \otimes K)\overline{\mathcal{R}}(z), \tag{2.2}$$

$$\overline{\mathcal{R}}(z)(F \otimes 1 + zK \otimes F) = (F \otimes 1 + zK^{-1} \otimes F)\overline{\mathcal{R}}(z), \tag{2.3}$$

$$(\Delta \otimes \mathrm{Id})\overline{\mathcal{R}}(z) = \overline{\mathcal{R}}_{13}(z)\overline{\mathcal{R}}_{23}(zK_{(1)}), \tag{2.4}$$

$$(\mathrm{Id} \otimes \Delta)\overline{\mathcal{R}}(z) = \overline{\mathcal{R}}_{13}(z)\overline{\mathcal{R}}_{12}(zK_{(3)}^{-1}). \tag{2.5}$$

Here $\overline{\mathcal{R}}_{13}(z) := \sum_{n=0}^{+\infty} z^n c_n (E^n \otimes 1 \otimes F^n)$ and

$$\overline{\mathcal{R}}_{23}(zK_{(1)}) := \sum_{n=0}^{+\infty} z^n c_n (K^n \otimes E^n \otimes F^n), \quad \overline{\mathcal{R}}_{12}(zK_{(3)}^{-1}) := \sum_{n=0}^{+\infty} z^n c_n (E^n \otimes F^n \otimes K^{-n}).$$

Proof. By definition $c_n = \frac{q^{-1}-q}{(n)_q} c_{n-1}$ for $n > 0$ and $c_0 = 1$. We have

$$
\begin{aligned}
[1\otimes E, \overline{\mathcal{R}}(z)] &= [1\otimes E, \sum_{n\geq 0} z^n c_n E^n \otimes F^n] = \sum_{n\geq 1} z^n c_n E^n \otimes [E, F^n] \\
&= \sum_{n\geq 1} z^n c_n E^n \otimes \sum_{s=1}^{n} F^{s-1} \frac{K-K^{-1}}{q-q^{-1}} F^{n-s} \\
&= \sum_{n\geq 1} z^n c_n \sum_{s=1}^{n} \frac{q^{2(s-1)}}{q-q^{-1}} E^n \otimes (KF^{n-1} - F^{n-1}K^{-1}) \\
&= \sum_{n\geq 1} z^n c_n \frac{(n)_q}{q^{-1}-q} E^n \otimes (F^{n-1}K^{-1} - KF^{n-1}) \\
&= \sum_{n\geq 1} z^n c_{n-1} E^n \otimes (F^{n-1}K^{-1} - KF^{n-1}) \\
&= \overline{\mathcal{R}}(z)(zE\otimes K^{-1}) - (zE\otimes K)\overline{\mathcal{R}}(z), \\
[\overline{\mathcal{R}}(z), F\otimes 1] &= \sum_{n\geq 0} z^n c_n [E^n, F]\otimes F^n = \sum_{n\geq 1} z^n c_n \sum_{s=1}^{n} E^{s-1}\frac{K-K^{-1}}{q-q^{-1}} E^{n-s}\otimes F^n \\
&= \sum_{n\geq 1} z^n c_n \frac{(n)_q}{q^{-1}-q} (K^{-1}E^{n-1} - E^{n-1}K)\otimes F^n \\
&= \sum_{n\geq 1} z^n c_{n-1} (K^{-1}E^{n-1} - E^{n-1}K)\otimes F^n \\
&= (zK^{-1}\otimes F)\overline{\mathcal{R}}(z) - \overline{\mathcal{R}}(z)(zK\otimes F), \\
(\Delta\otimes \mathrm{Id})(\overline{\mathcal{R}}(z)) &= \sum_{n\geq 0} z^n c_n (K\otimes E + E\otimes 1)^n \otimes F^n \\
&= \sum_{n\geq 0} z^n c_n \sum_{s=0}^{n} \frac{(n)_q!}{(s)_q!(n-s)_q!} (E^s\otimes 1)(K^{n-s}\otimes E^{n-s})\otimes F^n \\
&= (\sum_{n\geq 0} z^n c_n E^n \otimes 1\otimes F^n)(\sum_{n\geq 0} z^n c_n K^n \otimes E^n \otimes F^n) \\
&= \overline{\mathcal{R}}_{13}(z)\overline{\mathcal{R}}_{23}(zK_{(1)}), \\
(\mathrm{Id}\otimes\Delta)(\overline{\mathcal{R}}(z)) &= \sum_{n\geq 0} z^n c_n E^n \otimes (F\otimes K^{-1} + 1\otimes F)^n \\
&= \sum_{n\geq 0} z^n c_n \sum_{s=0}^{n} \frac{(n)_q!}{(s)_q!(n-s)_q!} E^n \otimes (1\otimes F^s)(F^{n-s}\otimes K^{s-n}) \\
&= (\sum_{n\geq 0} z^n c_n E^n \otimes 1\otimes F^n)(\sum_{n\geq 0} z^n c_n E^n \otimes F^n \otimes K^{-n}) \\
&= \overline{\mathcal{R}}_{13}(z)\overline{\mathcal{R}}_{12}(zK^{-1}_{(3)}).
\end{aligned}
$$

□

2.5. **Ore extension.** Let A be an algebra and $\varphi : A \longrightarrow A$ be an algebra automorphism. The vector space $A[t, t^{-1}]$ of Laurent polynomials with coefficients in A is an algebra with multiplication

$$(at^m) * (bt^n) := a\varphi^m(b)t^{m+n} \quad \text{for } a, b \in A \text{ and } m, n \in \mathbb{Z}.$$

The resulting algebra is called *Ore extension* of A by φ, and denoted by $A[t, t^{-1}; \varphi]$. It contains A and the Laurent polynomial algebra $\mathbb{K}[t, t^{-1}]$ as subalgebras.

Suppose $f : A \longrightarrow B$ is an algebra homomorphism. If there exits an invertible element $\psi \in B^\times$ such that $\psi f(a) = f(\varphi(a))\psi$ for all $a \in A$, then we can extend f uniquely to an algebra homomorphism $\tilde{f} : A[t, t^{-1}; \varphi] \longrightarrow B$ which sends t to ψ.

Example 2.7. We have an algebra automorphism φ of the tensor algebra $U_q(\mathfrak{sl}_2)^{\otimes 2}$ such that for $x, y \in U_q(\mathfrak{sl}_2)$ homogeneous of degrees $2r, 2s$:

$$\varphi(x \otimes y) = q^{-2rs} x K^{-s} \otimes y K^{-r}.$$

Let $\mathcal{A}_2$ denote the Ore extension of $U_q(\mathfrak{sl}_2)^{\otimes 2}$ by φ.

The universal R-matrix of $U_q(\mathfrak{sl}_2)$ is the following power series in z with coefficients in the Ore extension $\mathcal{A}_2$ of $U_q(\mathfrak{sl}_2)^{\otimes 2}$ by the algebra automorphism φ:

$$\mathcal{R}(z) := \overline{\mathcal{R}}(z)t \in \mathcal{A}_2[[z]]. \tag{2.6}$$

3. Representations of the quantum group $U_q(\mathfrak{sl}_2)$

3.1. **Universal R-matrix of $U_q(\mathfrak{sl}_2)$.** Since the $\mathbb{Z}$-grading on $U_q(\mathfrak{sl}_2)$ is compatible with the Hopf algebra structure, we can define two algebra homomorphisms Δ_z and Δ_z^{cop} from $U_q(\mathfrak{sl}_2)$ to $U_q(\mathfrak{sl}_2)^{\otimes 2}[z, z^{-1}]$ as follows. For $x \in U_q(\mathfrak{sl}_2)$ homogeneous, write $\Delta(x) = \sum_i a_i \otimes b_i$ where all a_i, b_i are homogeneous. Then set

$$\Delta_z(x) := \sum_i a_i z^{\frac{\deg(a_i)}{2}} \otimes y_i, \quad \Delta_z^{\mathrm{cop}}(x) := \sum_i b_i z^{\frac{\deg(b_i)}{2}} \otimes a_i.$$

Proposition 3.1. *The universal R-matrix $\mathcal{R}(z)$ satisfies the equation:*

$$\mathcal{R}(z)\Delta_z(x) = \Delta_z^{\mathrm{cop}}(x)\mathcal{R}(z) \in \mathcal{A}_2((z)) \quad \text{for } x \in U_q(\mathfrak{sl}_2)$$

Proof.

$$\begin{aligned}
\mathcal{R}(z)\Delta_z(E) &= \overline{\mathcal{R}}(z)t(K \otimes E + zE \otimes 1) = \overline{\mathcal{R}}(z)(1 \otimes E + zE \otimes K^{-1})t \\
&= (1 \otimes E + zE \otimes K)\overline{\mathcal{R}}(z)t = \Delta_z^{\mathrm{cop}}(E)\mathcal{R}(z), \\
\mathcal{R}(z)\Delta_z(F) &= \overline{\mathcal{R}}(z)t(1 \otimes F + z^{-1}F \otimes K^{-1}) = \overline{\mathcal{R}}(z)(K \otimes F + z^{-1}F \otimes 1)t \\
&= (K^{-1} \otimes F + z^{-1}F \otimes 1)\overline{\mathcal{R}}(z)t = \Delta_z^{\mathrm{cop}}(F)\mathcal{R}(z).
\end{aligned}$$

□

3.2. **Category $\mathscr{F}$.** From now on assume that there exists a fixed square root $q^{\frac{1}{2}} \in \mathbb{K}$ of q. Recall that the quantum group $U_q(\mathfrak{sl}_2)$ is graded with respect to the conjugate action of the invertible element K.

Given a $U_q(\mathfrak{sl}_2)$-module V, for $n \in \mathbb{Z}$ let V_n denote the eigenspace of K of eigenvalue q^n. It follows that

$$EV_n \subset V_{n+2}, \quad FV_n \subset V_{n-2}, \quad KV_n = V_n.$$

Call the module V of type 1 if it is a direct sum of the V_n for $n \in \mathbb{Z}$.

Lemma 3.2. *Given V and W two $U_q(\mathfrak{sl}_2)$-modules of type 1, define the linear isomorphism $\Psi_{V,W}$ by*

$$\Psi_{V,W}|_{V_m \otimes W_n} := q^{-\frac{mn}{2}} \mathrm{Id}_{V_m \otimes W_n} \quad \text{for } m, n \in \mathbb{Z}.$$

Then the $U_q(\mathfrak{sl}_2)^{\otimes 2}$-module structure on $V \otimes W$ is extended to a $\mathcal{A}_2$-module structure such that t acts as $\Psi_{V,W}$.

Proof. Let $x, y \in U_q(\mathfrak{sl}_2)$ be of degrees $2r$ and $2s$ respectively. Let $v \in V_m$ and $w \in W_n$. We need to check that

$$\Psi_{V,W}(xv \otimes yw) = q^{-2rs}(xK^{-s} \otimes yK^{-r})\Psi_{V,W}(v \otimes w).$$

Since $xv \in V_{m+2r}$ and $yw \in W_{n+2s}$, the left-hand side is $q^{-\frac{(m+2r)(n+2s)}{2}} xv \otimes yw$. The right-hand side is

$$q^{-2rs-\frac{mn}{2}} xK^{-s}v \otimes yK^{-r}w = q^{-2rs-\frac{mn}{2}-sm-rn} xv \otimes yw.$$

□

Let $\mathscr{F}$ denote the category of finite-dimensional $U_q(\mathfrak{sl}_2)$-modules of type 1. It is closed under dual, quotients, submodules and tensor product.

Example 3.3. Fix $n \in \mathbb{N}$. Choose a basis $(v_0, v_1, \cdots, v_n)$ of the vector space $\mathbb{K}^{n+1}$. The following assignments define a $U_q(\mathfrak{sl}_2)$-module structure on $\mathbb{K}^{n+1}$, denoted by $L(n)$:

$$Kv_i = q^{n-2i} v_i, \quad Fv_i = v_{i+1}, \quad Ev_i = \frac{(q^{n-2i+2} - q^{-n})(q^{2i} - 1)}{(q - q^{-1})(q^2 - 1)} v_{i-1}.$$

It is an simple module in category $\mathscr{F}$ and self dual: $L(n)^* \cong L(n)$.

Theorem 3.4. *Let V and W be $U_q(\mathfrak{sl}_2)$-modules in category $\mathscr{F}$. Then we have a module isomorphism*

$$c_{V,W} := \sigma_{V,W} \overline{\mathcal{R}}(1)|_{V \otimes W} \Psi_{V,W} : V \otimes W \longrightarrow W \otimes V.$$

Furthermore, if U is another $U_q(\mathfrak{sl}_2)$-module in category $\mathscr{F}$, then

$$c_{U \otimes V, W} = (c_{U,W} \otimes \mathrm{Id}_V)(\mathrm{Id}_U \otimes c_{V,W}), \quad c_{U, V \otimes W} = (\mathrm{Id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{Id}_W).$$

Proof. The first statement follows from Proposition 3.1, the second from Eqs.(2.4)–(2.5) and the following commutation relations in $\mathrm{End}(U \otimes V \otimes W)$:

$$\overline{\mathcal{R}}_{12}(K_{(3)}^{-1})(\Psi_{U,W})_{13} = (\Psi_{U,W})_{13} \overline{\mathcal{R}}_{12}(1), \quad \overline{\mathcal{R}}_{23}(K_{(3)})(\Psi_{U,W})_{13} = (\Psi_{U,W})_{13} \overline{\mathcal{R}}_{23}(1).$$

□

Example 3.5. Set $V = W = L(1)$. Then $\overline{\mathcal{R}}(1)|_{V\otimes W} = 1 + (q^{-1} - q)E \otimes F$ and

$$c_{V,W}(v_0 \otimes v_0) = q^{-\frac{1}{2}} v_0 \otimes v_0, \quad c_{V,W}(v_1 \otimes v_1) = q^{-\frac{1}{2}} v_1 \otimes v_1,$$
$$c_{V,W}(v_0 \otimes v_1) = q^{\frac{1}{2}} v_1 \otimes v_0, \quad c_{V,W}(v_1 \otimes v_0) = q^{\frac{1}{2}} v_0 \otimes v_1 + (q^{-\frac{1}{2}} - q^{\frac{3}{2}}) v_1 \otimes v_0.$$

3.3. **Cyclicity is equivalent to simplicity.** Call a $U_q(\mathfrak{sl}_2)$-module V *highest weight* if there exists a nonzero vector $v \in V$ such that

$$V = U_q(\mathfrak{sl}_2)v, \quad Kv \in \mathbb{K}v, \quad Ev = 0.$$

Lemma 3.6. *Let V be a highest weight $U_q(\mathfrak{sl}_2)$-module. Then there exists a unique nonzero scalar $\lambda \in V$ such that: λ is an eigenvalue of K acting on V; all the eigenvalues of K are of the form λq^{-2s} with $s \in \mathbb{N}$.*

Call λ the highest weight of V. Any nonzero vector of eigenvalue λ is called a *highest weight vector.*

Proof. Write $Kv = \lambda v$. Then $\lambda \in \mathbb{K}^\times$ because K is invertible. Since $U_q(\mathfrak{sl}_2)$ is spanned by the monomials $F^m K^p E^n$ with $m, n \in \mathbb{N}$ and $p \in \mathbb{Z}$, we have that $U_q(\mathfrak{sl}_2)v$ is spanned by the $F^m v$. If $F^m v$ is nonzero, then it is an eigenvector of eigenvalue λq^{-2m}. □

Proposition 3.7. *Let V be $U_q(\mathfrak{sl}_2)$-module in category $\mathscr{F}$ of highest weight $\lambda \in \mathbb{K}^\times$. Then there exists a unique $n \in \mathbb{N}$ such that $\lambda = q^n$ and $V \cong L(n)$.*

Proof. Since V is of type 1, there exists $k \in \mathbb{Z}$ such that $\lambda = q^k$. Choose a highest weight vector w_0 and set $w_i := F^i w_0$. If all the w_i are nonzero, then the q^{k-2i} for $i \in \mathbb{N}$ form an infinite sequence of eigenvalues of K acting on the finite-dimensional space V, a contradiction. Let $n \in \mathbb{N}$ be such that $w_n \neq 0$ and $w_{n+1} = 0$. Then

$$Ew_{n+1} = EF^{n+1}w_0 = \frac{(q^{k-2n} - q^{-k})(q^{2n+2} - 1)}{(q - q^{-1})(q^2 - 1)} w_n = 0.$$

It follows that $q^{k-2n} - q^{-k} = 0$ and so $n = k$. One shows directly that the assignments $v_i \mapsto w_i$ for $0 \le i \le n$ define a module isomorphism from $L(n)$ to V. □

Corollary 3.8. *The simple objects in category $\mathscr{F}$ are precisely the $L(n)$ for $n \in \mathbb{N}$.*

3.4. **Semi-simplicity of category $\mathscr{F}$.** Let V be a module over an algebra. A submodule W of V is called a *direct factor* if there exists another submodule W' such that $V \cong W \oplus W'$.

Lemma 3.9. *Let V be a module in category $\mathscr{F}$ which contains a submodule W such that $W \cong L(0)$ and $V/W \cong L(n)$ for certain $n \in \mathbb{N}$. Then W is a direct factor.*

Proof. Suppose $n > 0$. Then the eigenvalues of K are of the form q^s where $-n \le s \le n$ and the eigenspace V_n of eigenvalue q^n is one-dimensional. Choose a nonzero vector $v \in V_n$. Then $Ev = 0$ and the submodule W' of V generated by v is a highest weight module isomorphic to $L(n)$. Since $L(0)$ and $L(n)$ are non-isomorphic, $W + W'$ is a direct sum and $V = W \oplus W'$.

Suppose $n = 0$. Then $E = F = 0$. Any subspace of V is a submodule. □

Lemma 3.10. *Let V be a module in category $\mathscr{F}$ which contains a submodule W such that $V/W \cong L(0)$ and $W \cong L(n)$ for certain $n \in \mathbb{N}$. Then W is a direct factor.*

Lemma 3.11. *Let $\pi : V \longrightarrow L(0)$ be a surjective module morphism in category $\mathscr{F}$. Then there exists a submodule V_0 of V such that $V_0 \cong L(0)$ and $\pi(V_0) = L(0)$.*

Proof. We prove by induction on $\dim V > 0$. For $\dim V = 1$ this is trivial. Assume $\dim V > 1$. Then V is not simple and so it contains a simple submodule $W \cong L(n)$. If $\pi(W) = L(0)$, then necessarily $n = 0$ and we are done.

There remains the case $\pi(W) = \{0\}$. Then π induces another surjective module morphism $\pi' : V/W \longrightarrow L(0)$. By induction hypothesis, there exists a submodule T of V/W isomorphic to $L(0)$ and $\pi'(T) = L(0)$. Write $T = V'/W$ with V' a submodule of V. Then $V'/W \cong L(0)$ and $W \cong L(n)$. So there exists a submodule V'' of V' such that $V' = V'' \oplus W$. Clearly $V'' \cong L(0)$ is a submodule of V. Since $\pi(V') = L(0)$ and $\pi(W) = \{0\}$, we must have $\pi(V'') = L(0)$. □

If H is a Hopf algebra and V is an H-module, set

$$V^H := \{v \in V \mid hv = \epsilon(h)v \text{ for } h \in H\}.$$

Lemma 3.12. *Let H be a Hopf algebra and V and W be two H-modules. Then*

$$\mathrm{Hom}_{\mathbb{K}}(V,W)^H = \mathrm{Hom}_H(V,W).$$

Theorem 3.13. *Let V be a $U_q(\mathfrak{sl}_2)$-module in category $\mathscr{F}$. Then all submodules of V are direct factors.*

Proof. Let W be a submodule. The injective module morphism $\iota : W \longrightarrow V$ induces a surjective module morphism

$$F : \mathrm{Hom}_{\mathbb{K}}(V,W) \longrightarrow \mathrm{Hom}_{\mathbb{K}}(W,W) \quad f \mapsto f\iota.$$

Notice that $\mathbb{K}\mathrm{Id}_W$ is a submodule of $\mathrm{Hom}_{\mathbb{K}}(W,W)$ isomorphic to $L(0)$. Its pre-image $T := F^{-1}(\mathbb{K}\mathrm{Id}_W)$ is a submodule of $\mathrm{Hom}_{\mathbb{K}}(V,W)$. Since T admits $L(0)$ as a quotient module, it contains a submodule T_0 isomorphic to $L(0)$ and $F(T_0) = \mathbb{K}\mathrm{Id}_W$. Namely, there exists a module morphism $f : V \longrightarrow W$ such that $f\iota = \mathrm{Id}_W$. One shows then $V = W \oplus \ker(f)$. □

Corollary 3.14. *Any module in category $\mathscr{F}$ is a finite direct sum of the $L(n)$ for $n \in \mathbb{N}$.*

4. Quantum groups II

4.1. **Fusion rule in category $\mathscr{F}$.** Recall the simple $U_q(\mathfrak{sl}_2)$-module $L(n)$ in category $\mathscr{F}$ for $n \in \mathbb{N}$. Let w_0^n be a highest weight vector. Set $w_i^n := \frac{1}{(i)_q!}F^i w_0^n$ for $i \in \mathbb{N}$ and $w_i^n = 0$ for $i < 0$. Then $w_i^n = 0$ for $i > n$ and $(w_0^n, w_1^n, \cdots, w_n^n)$ forms a basis of $L(n)$ with respect to which the $U_q(\mathfrak{sl}_2)$-action is given by

$$Kw_i^n = q^{n-2i}w_i^n, \quad Fw_i^n = (i+1)_q w_{i+1}^n, \quad Ew_i^n = q^{1-n}(n-i+1)_q w_{i-1}^n.$$

Up to nonzero scalar multiplication, w_0^n is the unique highest weight vector of $L(n)$.

Proposition 4.1. *For $m, n \in \mathbb{N}$ we have a decomposition of modules in category $\mathscr{F}$:*

$$L(m) \otimes L(n) \cong \bigoplus_{j=0}^{\min(m,n)} L(m+n-2j) = \bigoplus_{p=0}^{\min(m,n)} L(|m-n|+2p).$$

Proof. By comparing dimensions, it suffices to show that for each $0 \le j \le \min(m,n)$, there exists a vector in $L(m)\otimes L(n)$ of highest weight q^{m+n-2j}. In view of the K-action, such a vector is of the form $\sum_{i=0}^{j} \alpha_i w_i^m \otimes w_{j-i}^n$ where $\alpha_0, \alpha_1, \cdots, \alpha_j \in \mathbb{K}$ are chosen such that the vector is annihilated by E, namely,

$$\begin{aligned} 0 &= \sum_{i=0}^{j} \alpha_i (K w_i^m \otimes E w_{j-i}^n + E w_i^m \otimes w_{j-i}^n) \\ &= \sum_{i=0}^{j-1} \alpha_i q^{m-2i+1-n}(n-j+i+1)_q w_i^m \otimes w_{j-i-1}^n \\ &\quad + \sum_{i=1}^{j} \alpha_i q^{1-m}(m-i+1)_q w_{i-1}^m \otimes w_{j-i}^n \\ &= \sum_{i=0}^{j-1} (\alpha_i q^{m-2i+1-n}(n-j+i+1)_q + \alpha_{i+1} q^{1-m}(m-i)_q) w_i^m \otimes w_{j-i-1}^n. \end{aligned}$$

We are reduced to the recursion $\alpha_{i+1} = -\alpha_i q^{2m-2n-2i} \frac{(n-j+1+i)_q}{(m-i)_q}$ for $0 \le i < j$. $\square$

In the above proof, one has a solution $\alpha_i = (-1)^i q^{i(2m-2n-i+1)}(m-i)_q!(n-j+i)_q!$ for $0 \le i \le j$ to the recursion and a highest weight vector in $L(m)\otimes L(n)$:

$$w_j^{m,n} := \sum_{i=0}^{j} (-1)^i q^{i(2m-2n-i+1)}(m-i)_q!(n-j+i)_q! w_i^m \otimes w_{j-i}^n \in L(m)\otimes L(n).$$

Consider the module isomorphism $c_{L(m),L(n)} : L(m)\otimes L(n) \longrightarrow L(n)\otimes L(m)$ defined by $\sigma\overline{\mathcal{R}}(1)\Psi_{L(m),L(n)}$ in Theorem 3.4. It sends $w_j^{m,n}$ to $\lambda_j w_j^{n,m}$ for a unique $\lambda_j \in \mathbb{K}^\times$. Notice that $c_{L(m),L(n)}(w_j^{m,n})$ modulo the subspace $\mathrm{Vect}(w_i^n \otimes w_{j-i}^m :\ 0 < i \le j)$ is

$$q^{-\frac{(m-2j)n}{2}}(-1)^j q^{j(2m-2n-j+1)}(m-j)_q!(n)_q! w_0^n \otimes w_j^m = \lambda_j (n)_q!(m-j)_q! w_0^n \otimes w_j^m,$$
$$\Rightarrow \quad \lambda_j = q^{-\frac{mn}{2}}(-1)^j q^{j(2m-n-j+1)}.$$

Let P_j denote the composition $L(m)\otimes L(n) \longrightarrow L(m+n-2j) \longrightarrow L(n)\otimes L(n)$ sending $w_j^{m,n}$ to $w_j^{n,m}$. We obtain the following *spectral decomposition of R-matrix*

$$q^{\frac{mn}{2}} c_{L(m),L(n)} = \bigoplus_{j=0}^{\min(m,n)} (-1)^j q^{j(2m-n-j+1)} P_j.$$

In the particular case $m = n$, the P_j are projections.

4.2. **Divided power algebra of Lusztig.** Let $U_q(\mathfrak{sl}_2)$ denote the quantum group over the field $\mathbb{C}(q)$ with q an indeterminate. Let $\mathcal{A} := \mathbb{C}[q,q^{-1}]$ be the subalgebra of Laurent polynomials. For two integers $n, r \in \mathbb{Z}$ with $r \ge 0$, the q-numbers $(n)_q, (r)_q!$ and $\binom{n}{r}_q$ belong to $\mathcal{A}$, so they can be specialized to an arbitrary complex number λ. The resulting complex numbers are denoted by $(n)_\lambda, (r)_\lambda!$ and $\binom{n}{r}_\lambda$.

Definition 4.2. *Lusztig's divided power algebra*, denoted by $\mathbf{U}$, is the $\mathcal{A}$-subalgebra of $U_q(\mathfrak{sl}_2)$ generated by $K^{\pm 1}$ and the q-divided powers

$$E^{(r)} := \frac{1}{(r)_q!}E^r, \quad F^{(r)} := \frac{1}{(r)_q!}F^r \quad \text{for } r \in \mathbb{N}.$$

Lemma 4.3. *For $m, n \in \mathbb{Z}$ and $r \in \mathbb{N}$ we have in $\mathcal{A}$ the identity*

$$\binom{m+n}{r}_q = \sum_{s=0}^{r} \binom{m}{s}_q \binom{n}{r-s}_q q^{2s(n-r+s)}.$$

Proof. Both sides are polynomials in the q^m and q^n with coefficients in $\mathbb{C}(q)$. To prove the identity one can assume $m, n \geq 0$. let $\mathcal{B}$ be an algebra containing x and y such that $xy = q^2yx$ and the monomials y^ix^j for $i, j \in \mathbb{N}$ are linearly independent. Then the coefficient of z^r in the polynomial $(zx+y)^{m+n} \in \mathcal{B}[z]$ is precisely

$$\binom{m+n}{s}_q y^{m+n-s}x^r.$$

Decomposing $(zx+y)^{m+n} = (zx+y)^m(zx+y)^n$, we get the coefficient of z^r:

$$\sum_{s=0}^{r} \binom{m}{s}_q y^{m-s}x^s \binom{n}{r-s}_q y^{n-r+s}x^{r-s} = \sum_{s=0}^{r} \binom{m}{s}_q \binom{n}{r-s}_q q^{2s(n-r+s)}y^{m+n-r}x^r.$$

□

For $n \in \mathbb{Z}$ and $r \geq \mathbb{N}$, define the following element

$$\binom{K;n}{r}_q := \prod_{s=1}^{r} \frac{K^2q^{2n-2s+2}-1}{q^{2s}-1} \in U_q(\mathfrak{sl}_2).$$

Proposition 4.4. (i) *In the $\mathbb{C}(q)$-algebra $U_q(\mathfrak{sl}_2)$ we have for $p, r \in \mathbb{N}$ and $n \in \mathbb{Z}$:*

$$\binom{K;n}{r}_q E^{(p)} = E^{(p)} \binom{K;n+2p}{r}_q, \quad \binom{K;n}{r}_q F^{(p)} = E^{(p)} \binom{K;n-2p}{r}_q,$$

$$E^{(p)}F^{(r)} = \sum_{t\geq 0} q^t F^{(r-t)}K^{-t}\binom{K;2t-p-r}{t}_q E^{(p-t)}, \quad \binom{K;n}{r}_q \in \mathbf{U}.$$

(ii) *The $\mathcal{A}$-subalgebra $\mathbf{U}$ is a Hopf algebra. We have for $m, n \in \mathbb{Z}$ and $r, p \in \mathbb{N}$:*

$$\Delta(E^{(r)}) = \sum_{s=0}^{r} E^{(r-s)}K^s \otimes E^{(s)}, \quad \Delta(F^{(r)}) = \sum_{s=0}^{r} F^{(s)} \otimes K^{-s}F^{(r-s)},$$

$$\Delta(\binom{K;m+n}{r}_q) = \sum_{s=0}^{r} q^{2s(s-r)} \binom{K;m}{s}_q \otimes \binom{K;n}{r-s}_q K^{2s},$$

$$S(\binom{K;n}{r}_q) = (-1)^r K^{-2r} q^{r(2n-r+1)} \binom{K;-n+r-1}{r}_q.$$

(iii) *The $\mathcal{A}$-module $\mathbf{U}$ is generated by the $F^{(r)}\phi E^{(s)}$ where $r, s \in \mathbb{N}$ and ϕ is a monomial in the K^{-1} and $\binom{K;n}{p}_q$ for $n \in \mathbb{Z}$ and $p \in \mathbb{Z}$.*

It is understood that $E^{(r)} = 0 = F^{(r)}$ for $r < 0$.

Proof. (i) We shall only prove $\binom{K;n}{r}_q \in \mathbf{U}$, by induction on $r \geq 0$. For $r = 0$ this is trivial. Assume this is true for $0 \leq r < t$. In view of the recursion formula

$$\binom{K; n+1}{r}_q = q^{2r}\binom{K;n}{r}_q + \binom{K;n}{r-1}_q$$

we only need to show that $\binom{K;0}{t}_q \in \mathbf{U}$. By Proposition 4.4(i),

$$E^{(t)}F^{(t)} = q^t K^{-t}\binom{K;0}{t}_q + \sum_{r=0}^{t-1} q^r F^{(t-r)} K^{-r}\binom{K; 2r-2t}{r}_q E^{(t-r)}.$$

The summation at the right-hand side belongs to $\mathbf{U}$ by induction hypothesis. Together with $E^{(t)}F^{(t)}, K^{\pm 1}, q^{\pm 1} \in \mathbf{U}$ by definition, we obtain $\binom{K;0}{t}_q \in \mathbf{U}$.

(ii) The first and second formulas are almost clear. Let H denote the $\mathbb{C}(q)$-subalgebra of $U_q(\mathfrak{sl}_2)$ generated by the $K^{\pm 1}$. Then H is a Hopf subalgebra containing all the $\binom{K;n}{r}_q$. It follows that $\Delta(\binom{K;m+n}{r}_q) = \sum_{i,j\in\mathbb{Z}} c_{ij} K^i \otimes K^j$ with $c_{ij} \in \mathbb{C}(q)$. To prove the third formula, it suffice to show that for all $m_1, m_2 \in \mathbb{Z}$ we have

$$\sum_{i,j\in\mathbb{Z}} c_{ij} q^{m_1 i + m_2 j} = \sum_{s=0}^{r} \binom{m_1+m}{s}_q \binom{m_2+n}{r-s}_q q^{2s(m_2+n-r+s)}.$$

Let $\mathbb{C}(q)_{m_1}$ denote the one-dimensional H-module such that K acts as q^{m_1}, so that $\binom{K;n}{r}_q$ acts as $\binom{m_1+n}{r}_q$. On the same module $\mathbb{C}(q)_{m_1} \otimes_{\mathbb{C}(q)} \mathbb{C}(q)_{m_2} \cong \mathbb{C}(q)_{m_1+m_2}$, the action of $\binom{K;m+n}{r}_q$ is given on the one-hand by $\binom{m_1+m_2+m+n}{r}_q$, and on the other hand by $\sum_{ij\in\mathbb{Z}} c_{ij} q^{m_1 i + m_2 j}$. The desired formula follows from Lemma 4.3.

(iii) This follows from (i) and $E^{(r)}E^{(s)} = \binom{r+s}{r}_q E^{(r+s)}$ for $r, s \in \mathbb{N}$. □

Notice that the quasi R-matrix is a power series in z with coefficients in $\mathbf{U} \otimes_{\mathcal{A}} \mathbf{U}$:

$$\overline{\mathcal{R}}(z) = \sum_{r\geq 0} \frac{(q^{-1}-q)^r}{(r)_q!} E^r \otimes F^r z^r = \sum_{r\geq 0} (q^{-1}-q)^r (r)_q! E^{(r)} \otimes F^{(r)} z^r.$$

4.3. **Quantum groups at roots of unity.** From now on we fix $\ell > 1$ an odd integer and $\varepsilon \in \mathbb{C}$ a primitive ℓ-th root of unity. The ground field is $\mathbb{C}$.

Definition 4.5. Define the algebra $\mathbf{U}_\varepsilon$ to be the extension $\mathbf{U} \otimes_{\mathcal{A}} \mathbb{C}_\varepsilon$ where $\mathbb{C}_\varepsilon$ is $\mathbb{C}$ regarded as a $\mathcal{A}$-algebra with q acting as multiplication by ε. Let $\overline{\mathbf{U}}_\varepsilon$ denote the quotient algebra of $\mathbf{U}_\varepsilon$ by the two-sided ideal generated by $K^\ell - 1$.

By abuse of language, in the algebra $\mathbf{U}_\varepsilon$ or its quotient $\overline{\mathbf{U}}_\varepsilon$, let x denote $x \otimes_{\mathcal{A}} 1$ or its quotient for $x \in \{E^{(r)}, F^{(r)}, K\}$, and let $\binom{K;n}{r}_\varepsilon$ denote $\binom{K;n}{r}_q \otimes_{\mathcal{A}} 1$ or its quotient.

Proposition 4.4 tells that $\mathbf{U}$ is a Hopf algebra over $\mathcal{A}$. After evaluation $\mathbf{U}_\varepsilon$ is a complex Hopf algebra. Since K is grouplike, $\overline{\mathbf{U}}_\varepsilon$ is a Hopf algebra.

Lemma 4.6. *In the algebra* $\mathbf{U}_\varepsilon$*, the element* K^ℓ *is central and*

$$K^{2\ell} - 1 = E^\ell = F^\ell = 0.$$

Proof. In the $\mathcal{A}$-algebra $\mathbf{U}$ we have

$$(\ell)_q!\binom{K;0}{\ell}_q = \prod_{s=1}^{\ell}(K^2q^{2s-2}-1), \quad (\ell)_q!E^{(\ell)} = E^\ell, \quad (\ell)_q!F^{(\ell)} = F^\ell.$$

Evaluate q at ε. Since the ε^{2s-2} for $1 \le s \le \ell$ are precisely the roots of the polynomial $X^\ell - 1 \in \mathbb{C}[X]$ and since $(\ell)_\varepsilon! = 0$, we obtained the desired equations. □

Theorem 4.7. *The Hopf algebra $\overline{\mathbf{U}}_\varepsilon$ is quasi-triangular with universal R-matrix*

$$\mathcal{R}_\varepsilon := \left(\sum_{r=0}^{\ell-1}(\varepsilon^{-1}-\varepsilon)^r(r)_\varepsilon!E^{(r)}\otimes F^{(r)}\right) \times \left(\frac{1}{\ell}\sum_{i,j=0}^{\ell-1}\varepsilon^{2ij}K^i\otimes K^j\right).$$

Proof. Let $\overline{\mathcal{R}}_\varepsilon$ and ψ_ε denote the two factors at the right-hand side. The $\mathbb{C}(q)$-algebra automorphism Ψ of $U_q(\mathfrak{sl}_2)\otimes_{\mathbb{C}(q)}U_q(\mathfrak{sl}_2)$ in Example 2.7 restricts to an $\mathcal{A}$-algebra automorphism of $\mathbf{U}\otimes_{\mathcal{A}}\mathbf{U}$ still denoted by Ψ. Its evaluation at $q=\varepsilon$ induces an algebra automorphism $\mathbf{U}_\varepsilon\otimes\mathbf{U}_\varepsilon$ factorizes through the quotient map $\mathbf{U}_\varepsilon\otimes\mathbf{U}_\varepsilon \longrightarrow \overline{\mathbf{U}}_\varepsilon\otimes\overline{\mathbf{U}}_\varepsilon$. Let Ψ_ε denote the resulting algebra automorphism of $\overline{\mathbf{U}}_\varepsilon\otimes\overline{\mathbf{U}}_\varepsilon$.

The quasi R-matrix $\overline{\mathcal{R}}(z)$ has coefficients in $\mathbf{U}\otimes_{\mathcal{A}}\mathbf{U}$. By Proposition 2.6:

$$\overline{\mathcal{R}}(z)\Psi(\Delta_z(x)) = \Delta_z^{\mathrm{cop}}(x)\overline{\mathcal{R}}(z) \in (\mathbf{U}\otimes_{\mathcal{A}}\mathbf{U})((z)) \quad \text{for } x\in\mathbf{U}.$$

After evaluation at $q=\varepsilon$, the power series $\overline{\mathcal{R}}(z)$ truncates to a polynomial in z. Together with the polynomiality of $\Delta_z(x)$ and $\Delta_z^{\mathrm{cop}}(x)$ by definition, we can evaluate z at 1 to get the relation

$$\overline{\mathcal{R}}_\varepsilon\Psi_\varepsilon(\Delta(x)) = \Delta^{\mathrm{cop}}(x)\overline{\mathcal{R}}_\varepsilon \in \overline{\mathbf{U}}_\varepsilon\otimes\overline{\mathbf{U}}_\varepsilon \quad \text{for } x\in\overline{\mathbf{U}}_\varepsilon.$$

To show that $\overline{\mathcal{R}}_\varepsilon\psi_\varepsilon$ is a universal R-matrix, as in the proof of Theorem 3.4 we need to establish the following assertions:

(i) The element $\psi_\varepsilon\in\overline{\mathbf{U}}_\varepsilon^{\otimes 2}$ is invertible and $\Psi_\varepsilon(X) = \psi_\varepsilon X\psi_\varepsilon^{-1}$ for $X\in\overline{\mathbf{U}}_\varepsilon^{\otimes 2}$.

(ii) We have in $\overline{\mathbf{U}}_\varepsilon^{\otimes 3}$ the equations

$$(\Delta\otimes\mathrm{Id})(\psi_\varepsilon) = \psi_{\varepsilon,13}\psi_{\varepsilon,23}, \quad (\mathrm{Id}\otimes\Delta)(\psi_\varepsilon) = \psi_{\varepsilon,13}\psi_{\varepsilon,12},$$
$$\overline{\mathcal{R}}_{12}(K_{(3)}^{-1})\psi_{\varepsilon,13} = \psi_{\varepsilon,13}\overline{\mathcal{R}}_{12}, \quad \overline{\mathcal{R}}_{23}(K_{(3)})\psi_{\varepsilon,13} = \psi_{\varepsilon,13}\overline{\mathcal{R}}_{23}.$$

One checks directly that $\frac{1}{\ell}\sum_{i,j=0}^{\ell-1}\varepsilon^{-2ij}K^i\otimes K^j$ is the inverse of ψ_ε; here the relation $K^\ell = 1$ is necessary. Furthermore, if $x\in\overline{\mathbf{U}}_\varepsilon$ satisfies $Kx = \varepsilon^{2m}xK$ then

$$\begin{aligned}\psi_\varepsilon(x\otimes 1) &= \sum_{i,j=0}^{\ell-1}\varepsilon^{2ij}K^ix\otimes K^j = \sum_{i,j=0}^{\ell-1}\varepsilon^{2ij+2im}xK^i\otimes K^j \\ &= (x\otimes K^{-m})\sum_{i,j=0}^{\ell-1}\varepsilon^{2i(j+m)}K^i\otimes K^{j+m} = (x\otimes K^{-m})\psi_\varepsilon.\end{aligned}$$

It follows that $\Psi_\varepsilon(x\otimes 1) = \psi_\varepsilon(x\otimes 1)\psi_\varepsilon^{-1}$. Similarly, $\Psi_\varepsilon(1\otimes x) = \psi(1\otimes x)\psi^{-1}$. This proves (i). For (ii), the first equation follows from

$$\begin{aligned}
\psi_{\varepsilon,13}\psi_{\varepsilon,23} &= \frac{1}{\ell^2}\sum_{i,j,s,t=0}^{\ell} \varepsilon^{2ij+2st} K^i\otimes K^s\otimes K^{j+t} \\
&= \frac{1}{\ell^2}\sum_{i,s,p=0}^{\ell-1} K^i\otimes K^s\otimes K^p \sum_{j=0}^{\ell-1}\varepsilon^{2ij+2s(p-j)} \\
&= \frac{1}{\ell^2}\sum_{i,s,p=0}^{\ell-1} \varepsilon^{2sp} K^i\otimes K^s\otimes K^p \sum_{j=0}^{\ell-1}\varepsilon^{2j(i-s)} \\
&= \frac{1}{\ell}\sum_{s,p=0}^{\ell-1} \varepsilon^{2sp} K^s\otimes K^s\otimes K^p = (\Delta\otimes \mathrm{Id})(\psi_\varepsilon).
\end{aligned}$$

Here we used the identity $\sum_{j=0}^{\ell-1}\varepsilon^{2jm} = \ell\delta_{m0}$ for $1-\ell\le m\le \ell-1$. The remaining three equations are proved in the same way. □

Let $\overline{\mathbf{u}}_\varepsilon$ denote the subalgebra of $\overline{\mathbf{U}}_\varepsilon$ generated by E, F, K. Then it is a finite-dimensional quasi-triangular Hopf algebra since $\mathcal{R}_\varepsilon \in \overline{\mathbf{u}}_\varepsilon\otimes\overline{\mathbf{u}}_\varepsilon$.

Lemma 4.8. *Let $m_0, m_1, r_0, r_1\in\mathbb{Z}$ such that $0\le m_0, r_0 < \ell$ and $r_1\ge 0$. Then*

$$\binom{m_0+\ell m_1}{r_0+\ell r_1}_\varepsilon = \binom{m_0}{r_0}_\varepsilon\binom{m_1}{r_1}.$$

Proposition 4.9. *The algebra $\mathbf{U}_\varepsilon$ is generated by $K, E, F, E^{(\ell)}, F^{(\ell)}$.*

Proof. For $m\in\mathbb{N}$, let $m = m_0+\ell m_1$ be the euclidean division of m by ℓ, so that m_0, m_1 are positive integers and $0\le m_0<\ell$. In the $\mathcal{A}$-algebra $\mathbf{U}$ we have

$$\binom{m}{m_0}_q E^{(m)} = \frac{(m)_q!}{(m_0)_q!(\ell m_1)_q!}E^{(m)} = E^{(m_0)}E^{(\ell m_1)}.$$

Evaluating q at ε and noticing $\binom{m}{m_0}_\varepsilon = 1$ we get $E^{(m)} = E^{(m_0)}E^{(\ell m_1)}\in\mathbf{U}_\varepsilon$.

Next, in the $\mathcal{A}$-algebra $\mathbf{U}$ we have $E^{m_0} = (m_0)_q!E^{(m_0)}$. Evaluating q at ε and noticing that $(m_0)_\varepsilon!\in\mathbb{C}^\times$ we get $E^{(m_0)} = \frac{1}{(m_0)_\varepsilon!}E^{m_0}\in\mathbf{U}_\varepsilon$.

In the $\mathcal{A}$-algebra $\mathbf{U}$ we have for $n\ge 0$:

$$\binom{n\ell+\ell}{\ell}_q E^{(n\ell+\ell)} = E^{(n\ell)}E^{(\ell)}.$$

Evaluating q at ε and noticing $\binom{n\ell+\ell}{\ell}_\varepsilon = n+1$ we get $E^{(n\ell)}E^{(\ell)} = (n+1)E^{(n\ell+\ell)}\in\mathbf{U}_\varepsilon$. It follows by induction on $m_1\ge 0$ that $E^{(\ell m_1)} = \frac{1}{m_1!}(E^{(\ell)})^{m_1}\in\mathbf{U}_\varepsilon$.

In summary, we have

$$E^{(m)} = \frac{1}{(m_0)_\varepsilon! m_1!}E^{m_0}(E^{(\ell)})^{m_1}\in\mathbf{U}_\varepsilon.$$

Similarly, $F^{(m)}$ can be expressed in terms of F and $F^{(\ell)}$. □

5. Representations of the quantum group $\mathbf{U}_\varepsilon$

5.1. **Negligible modules.** The category of finite-dimensional $\mathbf{U}_\varepsilon$-modules is monoidal. It has left duality and right duality (because $S(x) = K^{-1}xK$ for $x \in \mathbf{U}_\varepsilon$)

$$e_V : \begin{cases} V^* \otimes V \longrightarrow \mathbb{C} \\ f \otimes v \mapsto f(v), \end{cases} \qquad c_V : \begin{cases} \mathbb{C} \longrightarrow V \otimes V^* \\ 1 \mapsto \sum_i v_i \otimes v_i^*, \end{cases}$$

$$\tilde{e}_V : \begin{cases} V \otimes V^* \longrightarrow \mathbb{C} \\ f \otimes v \mapsto f(K^{-1}v), \end{cases} \qquad \tilde{c}_V : \begin{cases} \mathbb{C} \longrightarrow V^* \otimes V \\ 1 \mapsto \sum_i v_i^* \otimes Kv_i. \end{cases}$$

$$\varphi_V : V \longrightarrow V^{**}, \quad v \mapsto K^{-1}v.$$

Here (v_i) is a basis of V and (v_i^*) is the dual basis of $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$.

Definition 5.1. Let V be a finite-dimensional $\mathbf{U}_\varepsilon$-module.

(i) The *quantum trace* of a module morphism $f : V \longrightarrow V$ is $\mathrm{qtr}(f) := \mathrm{tr}_V(Kf)$. The *quantum dimension of* V is $\mathrm{qdim}(V) := \mathrm{qtr}(\mathrm{Id}) = \mathrm{tr}_V(K)$.
(ii) Call V *negligible* if $\mathrm{qtr}(f) = 0$ for all module morphisms $f : V \longrightarrow V$.

By Krull–Schmidt decomposition, such a module is a direct sum of indecomposable submodules. It is negligible if and only if each indecomposable submodule is negligible.

Proposition 5.2. *Let U, V and W be finite-dimensional $\mathbf{U}_\varepsilon$-modules.*

(i) *If W is indecomposable, then W is negligible if and only if* $\mathrm{qdim}W = 0$.
(ii) *If W is negligible, then so is $V \otimes W$.*
(iii) *If W is negligible and U and V are simple non-negligible, then any module morphism of the form $U \longrightarrow W \longrightarrow V$ is zero.*

Proof. (i) By Fitting Lemma, each module morphism $f : W \longrightarrow W$ is either an automorphism or nilpotent. Let $\lambda \in \mathbb{C}$ be an eigenvalue of f. Then $f - \lambda \mathrm{Id}_W$ is nilpotent. So is $K(f - \lambda\mathrm{Id}_W)$ and it is traceless. It follows that

$$\mathrm{qtr}(f) = \mathrm{tr}(\lambda K + K(f - \lambda \mathrm{Id}_W)) = \lambda \mathrm{tr}(K) = \lambda \times \mathrm{qdim}(W).$$

(ii) Recall the module morphisms $\tilde{c}_V : \mathbb{C} \longrightarrow V^* \otimes V$ and $e_V : V^* \otimes V \longrightarrow \mathbb{C}$. Given a module morphism $f : V \otimes W \longrightarrow V \otimes W$, we obtain another module morphism

$$\tilde{f} = (e_V \otimes \mathrm{Id}_W)(\mathrm{Id}_{V^*} \otimes f)(\tilde{c}_V \otimes \mathrm{Id}_W) : W \longrightarrow W.$$

One verifies directly that $\mathrm{tr}_V(K\tilde{f}) = \mathrm{tr}_{V\otimes W}(Kf)$ and so $\mathrm{qtr}(f) = \mathrm{qtr}(\tilde{f}) = 0$.

(iii) By Schur Lemma such a module morphism, if nonzero, is an isomorphism. So V is negligible as an indecomposable submodule of W. □

5.2. **Category $\mathscr{F}_\varepsilon$.** Let V be a $\mathbf{U}_\varepsilon$-module. For $n \in \mathbb{Z}$, define V_n to be the subspace

$$V_n := \{v \in V \mid Kv = \varepsilon^n v, \quad \binom{K; m}{r}_\varepsilon v = \binom{n+m}{r}_\varepsilon v \quad \text{for } m \in \mathbb{Z} \text{ and } r \in \mathbb{N}\}.$$

If $V_n \neq \{0\}$, then n is called a weight of V, and V_n the weight space of weight n.

Lemma 5.3. *Let V and W be $\mathbf{U}_\varepsilon$-modules. Let $m, n \in \mathbb{Z}$ and $p \in \mathbb{N}$.*

(i) *The sum of weight spaces of V is direct.*

(ii) *We have* $E^{(p)}V_n \subset V_{n+2p}$ *and* $F^{(p)}V_n \subset V_{n-2p}$.
(iii) *We have* $V_m \otimes W_n \subset (V \otimes W)_{m+n}$ *for* $m, n \in \mathbb{Z}$.
(iv) *If* V *is a finite sum of weight spaces, then so is* V^* *and* $(V_n)^* = (V^*)_{-n}$.

Proof. (i) follows from Lemma 4.8, and the rest from Proposition 4.4(i)–(ii). □

Let $\mathscr{F}_\varepsilon$ denote the full subcategory of finite-dimensional $\mathbf{U}_\varepsilon$-modules which are direct sums of weight spaces. The module structure factorizes through the quotient map $\mathbf{U}_\varepsilon \longrightarrow \overline{\mathbf{U}}_\varepsilon$. So $\mathscr{F}_\varepsilon$ is braided as a full subcategory of finite-dimensional modules over the quasi-triangular Hopf algebra $\overline{\mathbf{U}}_\varepsilon$. Category $\mathscr{F}_\varepsilon$ is closed under submodule, quotient, tensor product and dual.

Lemma 5.4. *Let* V *be a* $\mathbf{U}_\varepsilon$*-module in category* $\mathscr{F}_\varepsilon$ *and* $n \in \mathbb{Z}$ *such that* $\dim V_n = 1$. *Then* V *admits a indecomposable direct factor containing* V_n. *Moreover, such a direct factor is unique up to isomorphism.*

Proof. By Krull-Schmidt decomposition V is a direct sum of indecomposable submodules $T^1 \oplus T^2 \oplus \cdots \oplus T^r$. Since $\dim V_n = 1$, there exists a unique $1 \leq i \leq r$ such that $(T^i)_n = V_n$ and $(T^j)_n = \{0\}$ for $j \neq i$. If $V = S^1 \oplus S^2 \oplus \oplus S^t$ is another Krull-Schmidt decomposition, then there exists another $1 \leq k \leq t$ such that $(S^k)_n = V_n$ and $(S^l)_n = \{0\}$ for $l \neq k$. It follows from uniqueness that $S^i \cong S^k$. □

Example 5.5. For $n \in \mathbb{N}$ we define the Weyl module $W(n)$ over $\mathbf{U}_\varepsilon$ in two ways.

(i) Recall the irreducible $U_q(\mathfrak{sl}_2)$-module $L(n)$ of highest weight q^n, defined over the field $\mathbb{C}(q)$. Fix w_0 a highest weight vector. Set $w_r := F^{(r)}w_0$ for $r \in \mathbb{N}$. Then $(w_0, w_1, \cdots, w_n)$ forms a $\mathbb{C}(q)$-basis of $L(n)$ with $U_q(\mathfrak{sl}_2)$-action:
$$K^{\pm 1}w_r = q^{\pm(n-2r)}w_r, \quad \binom{K;m}{p}_q w_r = \binom{n+m}{p}_q w_r,$$
$$F^{(p)}w_r = \binom{p+r}{p}_q w_{r+p}, \quad E^{(p)}w_r = q^{p-pn}\binom{n+p-r}{p}_q w_{r-p}.$$
Here $w_p := 0$ if $p < 0$ or $p > n$. It follows that $\oplus_{r=0}^n \mathcal{A}w_r$ is a $\mathbf{U}$-submodule of $L(n)$, denoted by $L_{\mathcal{A}}(n)$. Define the *Weyl module* $W(n)$ to be
$$W(n) := L_{\mathcal{A}}(n) \otimes_{\mathcal{A}} \mathbb{C}_\varepsilon = L_{\mathcal{A}}(n)/(q-\varepsilon)L_{\mathcal{A}}(n).$$
We have $W(n)_{n-2r} = \mathbb{C}w_r$ for $0 \leq r \leq n$. The Weyl module $W(n)$ is indecomosable of quantum dimension $(n+1)_\varepsilon$, and it is negligible if and only if ℓ divides $n+1$. If $n \leq \ell - 1$, then $W(n)$ is simple and self dual.

(ii) The Weyl module $W(n)$ is the $\mathbf{U}_\varepsilon$-module generated by w_0 subject to the following relations for $p \in \mathbb{N}$ and $m \in \mathbb{Z}$:
$$E^{(p+1)}w_0 = F^{(n+p+1)}w_0 = 0, \quad Kw_0 = \varepsilon^n w_0, \quad \binom{K;m}{p}_\varepsilon w_0 = \binom{n+m}{p}_\varepsilon w_0.$$

Example 5.6. For $n \in \mathbb{N}$, the tensor power $W(1)^{\otimes n}$ is a module in category $\mathscr{F}_\varepsilon$ and $(W(1)^{\otimes n})_n = \mathbb{C}w_0^{\otimes n}$. Up to isomorphism it contains a unique indecomposable direct factor containing $w_0^{\otimes n}$, denoted by $T(n)$ and called a *tilting module*.

For $m, n \in \mathbb{N}$, by uniqueness $T(m+n)$ is a direct factor of $T(m) \otimes T(n)$.

5.3. **Semi-simplification.** Call a $\mathbf{U}_\varepsilon$-module V of highest weight $n \in \mathbb{Z}$ if it is generated by a nonzero vector $v \in V_n$ such that $E^{(r)}v = 0$ for all $r > 0$. For example, the Weyl module $W(m)$ is of highest weight $m \in \mathbb{N}$.

Proposition 5.7. *If V is a finite-dimensional $\mathbf{U}_\varepsilon$-module of highest weight $n \in \mathbb{Z}$, then $n \geq 0$ and $V_{-n} \neq \{0\}$. Moreover, V is a quotient of the Weyl module $W(n)$.*

Proof. Choose a nonzero vector $v \in V_n$. We need to prove that $n \in \mathbb{N}$, $F^{(n)}v \neq 0$ and $F^{(s)}v = 0$ for $s > n$. Let $n = n_0 + \ell n_1$ and $s = s_0 + \ell s_1$ be the euclidean divisions of n and s by ℓ, so that $0 \leq n_0, s_0 \leq \ell - 1$. Then $s > n$ implies $s_1 \geq n_1$.

Since V is finite-dimensional, $F^{(r)}V_p \subset V_{p-2r}$ and $F^{(m\ell)} = \frac{1}{m!}(F^{(\ell)})^m$ for $m \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $F^{(m\ell)}v \neq 0$ and $F^{(r\ell)}v = 0$ for $r > m$. Apply the following relation to v:

$$E^{(\ell)}F^{(m\ell+\ell)} = \sum_{t\geq 0} \varepsilon^t F^{(m\ell+\ell-t)}K^{-t}\binom{K; 2t - m\ell - 2\ell}{t}_\varepsilon E^{(\ell-t)}.$$

We get $\binom{n-m\ell}{\ell}_\varepsilon = 0 = n_1 - m$. It follows that $n = n_0 + \ell n_1 = n_0 + \ell m \geq 0$.

Since $F^{(n_1\ell)}v \neq 0$ and $F^{(n_1\ell+\ell)}v = 0$, there exists $0 \leq p \leq \ell - 1$ such that $F^{(p+n_1\ell)}v \neq 0$ and $F^{(p+1+n_1\ell)}v = 0$. Apply the following relation to v:

$$EF^{(p+1+n_1\ell)} = F^{(p+1+n_1\ell)}E + \varepsilon F^{(p+n_1\ell)}K^{-1}\binom{K; -p - n_1\ell}{1}_\varepsilon.$$

We get $(n - p - n_1\ell)_\varepsilon = 0$, namely, ℓ divides $n - p - n_1\ell$. This forces $p = n_1$. As a consequence, $F^{(n)}v \neq 0$ and $F^{(n+1)}v = 0$.

If $s_1 > n_1$, then

$$F^{(s)}v = F^{(s_0)}F^{(s_1\ell)}v = 0.$$

If $s_1 = n_1$, then $s > n$ implies $s_0 > n_0$ and

$$\binom{s_0}{n_0+1}_\varepsilon F^{(s)}v = \binom{s}{n+1}_\varepsilon F^{(s)}v = F^{(s-n-1)}F^{(n+1)}v = 0.$$

Since $1 \leq n_0 + 1 \leq s_0 < \ell$, we have $\binom{s_0}{n_0+1}_\varepsilon \neq 0$ and $F^{(s)}v = 0$. □

Lowest weight modules are defined by replacing E with F everywhere. In particular, if V is finite-dimensional of lowest weight n, then $n \leq 0$ and $V_{-n} \neq \{0\}$.

Corollary 5.8. *Let V be a nonzero $\mathbf{U}_\varepsilon$-module in category $\mathscr{F}_\varepsilon$. Then there exists a unique $m_1 \in \mathbb{N}$ such that $V_{\pm m_1} \neq \{0\}$ and $V_m \neq \{0\}$ only if $-m_1 \leq m \leq m_1$. Any surjective module morphism $V \longrightarrow W(m_1)$ splits.*

Proof. Since V is a finite direct sum of the V_m, there exist two integers $m_0 \leq m_1$ such that $V_{m_0} \neq \{0\} \neq V_{m_1}$ and $V_m \neq \{0\}$ only if $m_0 \leq m \leq m_1$. Choose a nonzero vector $v_1 \in V_{m_1}$. Then $V_1 := \mathbf{U}_\varepsilon v_1$ is a submodule of highest weight m_1. Proposition 5.7 forces $m_1 \geq 0$ and $V_{-m_1} \neq \{0\}$. This implies $m_0 \leq -m_1$. On the other hand, any nonzero vector of V_{m_0} generates a submodule of lowest weight m_0, so $m_0 \leq 0$ and $V_{-m_0} \neq \{0\}$. This implies $-m_0 \leq m_1$. As a consequence, $m_0 + m_1 = 0$.

Let $f : V \longrightarrow W(m_1)$ be a surjective module morphism. Choose a pre-image $v \in V_{m_1}$ of $w_0 \in W(m_1)$. We have $E^{(p)}v \in V_{m_1+2p} = \{0\}$ and $F^{(m_1+p)}v \in V_{-m_1-2p} = \{0\}$ for

$p > 0$. So all the defining relations of $W(m_1)$ are satisfied and we have a module morphism $W(m_1) \longrightarrow V$ sending w_0 to v. This gives the desired splitting. □

Theorem 5.9. *Let V be a $\mathbf{U}_\varepsilon$-module in category $\mathscr{F}_\varepsilon$. Suppose that V is the sum of the V_m for $1-\ell \leq m \leq \ell-1$. Then V is isomorphic to a direct sum of the simple Weyl modules $W(n)$ for $0 \leq n \leq \ell-1$.*

Proof. We proceed by induction on $\dim V$. The case $\dim V = 0$ is trivial. Assume $\dim V > 0$ so that V is nonzero. Choose $m_1 \in \mathbb{N}$ as in Corollary 5.8. Then $V_{m_1} \neq \{0\}$ implies $m_1 \leq \ell-1$. Choose a nonzero vector $v_1 \in V_{m_1}$. Then the submodule $V_1 = \mathbf{U}_\varepsilon v_1$, being finite-dimensional of highest weight m_1, is isomorphic to the simple self dual Weyl module $W(m_1)$. This gives an injective module morphism $W(m_1) \longrightarrow V$. Taking duals gives a surjective module morphism $V^* \longrightarrow W(m_1)$. By Corollary 5.8 such a surjective module morphism splits, meaning that V_1 is a direct factor of V. Apply the induction hypothesis to V/V_1 and conclude. □

Corollary 5.10. *If $0 \leq n \leq \ell-2$, then $T(n) \cong W(n)$ is simple non-negligible. If $n \geq \ell-1$, then $T(n)$ is negligible. In all cases, $T(n)$ is self dual.*

Proof. For $n \leq \ell-1$, the tilting module $T(n)$ as an indecomposable submodule of the semi-simple module $W(1)^{\otimes n}$ is isomorphic to the simple Weyl module $W(n)$. It is negligible if and only if $n = \ell-1$.

Assume $T(n)$ negligible. Then $W(1) \otimes T(n) = T(1) \otimes T(n)$ is negligible. By Proposition 5.2(ii), its direct factor $T(n+1)$ is negligible. □

Definition 5.11.
- (i) Category $\mathscr{M}_\varepsilon$ is the full subcategory of category $\mathscr{F}_\varepsilon$ with the additional condition that V is the sum of the V_m for $2-\ell \leq m \leq \ell-2$.
- (ii) Category $\mathscr{T}_\varepsilon$ is the full subcategory of $\mathscr{F}_\varepsilon$ with the additional condition that V is isomorphic to a direct sum of tilting modules.

By Theorem 5.9 category $\mathscr{M}_\varepsilon$ is semi-simple with finitely many simple objects: $W(n)$ for $0 \leq n \leq \ell-2$. We have $\mathscr{M}_\varepsilon \subset \mathscr{T}_\varepsilon \subset \mathscr{F}_\varepsilon$. Category $\mathscr{T}_\varepsilon$ is not abelian. Category $\mathscr{M}_\varepsilon$ is not closed under tensor product.

By definition and uniqueness of Krull–Schmidt decomposition, a module V in category $\mathscr{T}_\varepsilon$ has a unique splitting $V = \mathbf{M}(V) \oplus \mathbf{Z}(V)$ where:

- (i) the submodule $\mathbf{M}(V)$ is a direct sum of simple non-negligible tilting modules;
- (ii) the submodule $\mathbf{Z}(V)$ is a direct sum of negligible tilting modules.

A morphism $f : V \longrightarrow W$ in category $\mathscr{T}_\varepsilon$ is encoded in a square matrix $\begin{pmatrix} f_{\mathbf{MM}} & f_{\mathbf{MZ}} \\ f_{\mathbf{ZM}} & f_{\mathbf{ZZ}} \end{pmatrix}$ whose entries $f_{XY} : Y(V) \longrightarrow X(W)$ for $X, Y \in \{\mathbf{Z}, \mathbf{M}\}$ are module morphisms.

Proposition 5.12. *The assignments $V \mapsto \mathbf{M}(V)$ and $f \mapsto f_{\mathbf{MM}}$ define a functor $\mathbf{M} : \mathscr{T}_\varepsilon \longrightarrow \mathscr{M}_\varepsilon$ whose restriction to the subcategory $\mathscr{M}_\varepsilon$ is the identity functor. We have $\mathbf{M}(V) = \{0\}$ if and only if V is negligible.*

Proof. Let $f : U \longrightarrow V$ and $g : V \longrightarrow W$ be morphisms in category $\mathscr{T}_\varepsilon$. We need to show that $g_{\mathbf{MZ}} f_{\mathbf{ZM}} = 0$. This is a morphism of the form $\mathbf{M}(U) \longrightarrow \mathbf{Z}(V) \longrightarrow \mathbf{M}(W)$. By construction $\mathbf{M}(U)$ and $\mathbf{M}(W)$ are direct sums of simple non-negligible modules and $\mathbf{Z}(V)$ is negligible. Apply Proposition 5.2(iii). □

5.4. **Fusion rule and quantum $6j$-symbol.** We record the following deep result in representation theory of quantum groups at roots of unity. Its proof requires an alternative characterization of tilting modules in terms of Weyl module filtrations.

Theorem 5.13 (Andersen, Paradowski)**.** *Category $\mathscr{T}_\varepsilon$ is closed under tensor product.*

Since negligible modules are stable under tensor product by an arbitrary module, the monoidal structure on category $\mathscr{T}_\varepsilon$ induces a monoidal structure $\underline{\otimes}$ on category $\mathscr{M}_\varepsilon$ defined in objects U, V and morphisms f, g as follows:

$$U \underline{\otimes} V := \mathbf{M}(U \otimes V), \quad f \underline{\otimes} g := \mathbf{M}(f \otimes g).$$

Theorem 5.14 (Reshetikhin–Turaev)**.** *Let $0 \leq m, n \leq \ell - 2$. In the semi-simple monoidal category $\mathscr{M}_\varepsilon$ we have*

$$T(m) \underline{\otimes} T(n) \cong \bigoplus_{p=0}^{\min(m,n,\ell-2-m,\ell-2-n)} T(|n-m| + 2p).$$

Call a triple (m, n, k) of integers *admissible* if $0 \leq m, n, k \leq \ell - 2$ and $T(k)$ is a direct factor of $T(m) \otimes T(n)$. For such a triple, setting $j = \frac{m+n-k}{2}$ we have a nonzero $\mathbf{U}_\varepsilon$-module morphism $Y^k_{m,n} : T(k) \longrightarrow T(m) \otimes T(n)$:

$$w^k_0 \mapsto w^{m,n}_j = \sum_{i=0}^{j} (-1)^i \varepsilon^{i(2m-2n-i+1)} (m-i)_\varepsilon! (n-j+i)_\varepsilon! w^m_i \otimes w^n_{j-i}.$$

Fix (a, d, c, f) four integers between 0 and $\ell - 2$. For all d such that (d, c, f) and (a, b, d) are admissible, we have a module morphism in category $\mathscr{T}_\varepsilon$:

$$(Y^d_{ab} \otimes 1_c) Y^f_{dc} : T(f) \longrightarrow T(d) \otimes T(c) \longrightarrow T(a) \otimes T(b) \otimes T(c).$$

Here 1_c denotes the identity map of $T(c)$. Their images by the functor $\mathbf{M}$ form a basis of the space of morphisms in category $\mathscr{M}_\varepsilon$ from $T(f)$ to $T(a) \underline{\otimes} T(b) \underline{\otimes} T(c)$. Likewise, for all e such that (a, e, f) and (b, c, e) are admissible, we have a module morphism

$$(1_a \otimes Y^e_{bc}) Y^f_{ae} : T(f) \longrightarrow T(a) \otimes T(e) \longrightarrow T(a) \otimes T(b) \otimes T(c).$$

Their images by the functor $\mathbf{M}$ form a basis of the same morphism space. We obtain therefore the so-called *quantum $6j$-symbol* $\begin{Bmatrix} a & b & d \\ c & f & e \end{Bmatrix}_\varepsilon$ such that

$$(1_a \underline{\otimes} Y^e_{bc}) Y^f_{ae} = \sum_d \begin{Bmatrix} a & b & d \\ c & f & e \end{Bmatrix}_\varepsilon (Y^d_{ab} \underline{\otimes} 1_c) Y^f_{dc}.$$