Master class 2016-2017 in geometry, topology and physics

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Introduction to Quantum Topology I

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Exercise sheet 10

In what follows, $\Bbbk$ is a field.

Exercise 1. (Hopf modules)
Let $A = (A, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf $\Bbbk$-algebra. A *left Hopf A-module* is a $\Bbbk$-vector space M endowed with $\Bbbk$-linear maps $r\colon A \otimes M \to M$ and $\rho\colon M \to A \otimes M$ such that (M, r) is a left A-module, (M, ρ) is a left A-comodule, and the following diagram commutes:

$$\begin{array}{ccccc} A \otimes M & \xrightarrow{\;r\;} & M & \xrightarrow{\;\rho\;} & A \otimes M \\ {\scriptstyle \Delta\otimes\rho}\downarrow & & & & \uparrow{\scriptstyle \mu\otimes r} \\ A \otimes A \otimes A \otimes M & & \xrightarrow{\;\mathrm{id}_A \otimes \tau_{A,A} \otimes \mathrm{id}_M\;} & & A \otimes A \otimes A \otimes M. \end{array}$$

a. Let V be a $\Bbbk$-vector space. Prove that $A \otimes V$ is a left Hopf A-module with action $r = \mu \otimes \mathrm{id}_V$ and coaction $\rho = \Delta \otimes \mathrm{id}_V$.

b. Let M be a left Hopf A-module. Consider the $\Bbbk$-vector space

$$M^{\mathrm{coinv}} = \{m \in M \mid \rho(m) = 1_A \otimes m\}$$

and the left Hopf A-module $A \otimes M^{\mathrm{coinv}}$. Prove that the action of M induces an isomorphism

$$A \otimes M^{\mathrm{coinv}} \simeq M$$

of Hopf A-modules (i.e., an isomorphism of A-modules and of A-comodules).
Hint: prove that the map

$$P = r(S \otimes \mathrm{id}_M)\rho\colon M \to M$$

is a projector with image M^{coinv} verifying $Pr = \varepsilon \otimes P$ and $r(\mathrm{id}_A \otimes P)\rho = \mathrm{id}_M$, and then consider the map $(\mathrm{id}_A \otimes P)\rho\colon M \to A \otimes M^{\mathrm{coinv}}$.

Exercise 2. (Bijectivity of the antipode)
The aim of this exercise is to prove that the antipode of a finite-dimensional Hopf $\Bbbk$-algebra is bijective. Let $A = (A, \mu, \eta, \Delta, \varepsilon, S)$ be a finite-dimensional Hopf $\Bbbk$-algebra.

a. Prove that A^* is a left Hopf A-module with action

$$r = (e \otimes \mathrm{id}_{A^*})(\mathrm{id}_{A^*} \otimes \mu \otimes \mathrm{id}_{A^*})(\mathrm{id}_{A^*} \otimes S \otimes b)\tau_{A,A^*}\colon A \otimes A^* \to A^*$$

and coaction

$$\rho = (e \otimes \mathrm{id}_A \otimes \mathrm{id}_{A^*})(\mathrm{id}_{A^*} \otimes \Delta \otimes \mathrm{id}_{A^*})(\mathrm{id}_{A^*} \otimes b)\colon A^* \to A \otimes A^*,$$

where $e\colon A^* \otimes A \to \Bbbk$ is the evaluation pairing with inverse $b\colon \Bbbk \to A \otimes A^*$.

b. By considering the isomorphism $A \otimes (A^*)^{\mathrm{coinv}} \simeq A^*$ of Hopf A-modules induced by the action of A^* (see Exercise 1.b), prove that S is injective.

c. Conclude.