Master class 2016-2017 in geometry, topology and physics

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Introduction to Quantum Topology I

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Exercise sheet 3

Exercise 1.
Consider the 2-dimensional TQFT $Z = Z_{(V,a,\mu)}$ associated to a state sum triple (V, a, μ). Let C be a closed oriented 1-manifold and t be a triangulation of C. Prove that the vector space $Z(C)$ is isomorphic to the image of the projector $p^C_{t,t}$.

Exercise 2.
Let G be a finite group. Consider the 2-dimensional TQFT $Z = Z_{(\mathbb{C}[G],a,\mu)}$ where

$$a = \frac{1}{|G|^2} \sum_{\substack{g,h,k \in G \\ ghk=1}} g \otimes h \otimes k \qquad \text{and} \qquad \mu(g \otimes h) = |G|\, \delta_{gh,1}.$$

Compute the dimension of the $\mathbb{C}$-vector space $Z(S^1)$.

Exercise 3.
Let (V, a, μ) be a state sum triple. The goal of this exercise is to determine the Frobenius algebra associated to the 2-dimensional TQFT $Z_{(V,a,\mu)}$. We say that two state sum triples are *equivalent* if they define isomorphic TQFTs.

a. Let $(\overline{V}, \overline{a}, \overline{\mu})$ be the state sum triple of Exercise 2 of Sheet 2.
Prove that the state sum triples $(\overline{V}, \overline{a}, \overline{\mu})$ et (V, a, μ) are équivalent.

b. Let $(\widetilde{V}, \widetilde{a}, \widetilde{\mu})$ be the state sum triple of Exercise 3 of Sheet 2.
Prove that the state sum triples $(\widetilde{V}, \widetilde{a}, \widetilde{\mu})$ et (V, a, μ) are équivalent.

c. Assume that $V = \{(\mathrm{id}_V \otimes \phi)(a) \,|\, \phi \in (V \otimes V)^*\}$ and μ is non-degenerate.
Prove that the Frobenius algebra associated to the 2-dimensional TQFT $Z_{(V,a,\mu)}$ is isomorphic to the center of the $\mathbb{k}$-algebra (V, m), where

$$m = (\mu \otimes \mathrm{id}_V \otimes \mu)(\mathrm{id}_V \otimes a \otimes \mathrm{id}_V) \colon V \otimes V \to V.$$

(See Exercise 4 of Sheet 2.)

d. Conclude by noticing that the state sum triple $(\widetilde{\overline{V}}, \widetilde{\overline{a}}, \widetilde{\overline{\mu}})$ satisfies the assumptions of question c.