Master class 2016-2017 in geometry, topology and physics

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Introduction to Quantum Topology I

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Exercise sheet 4

Exercise 1.
Let Z be a n-dimensional TQFT (over $\Bbbk$) and M be a closed oriented n-manifold. Then the tuple $(M, \emptyset, \emptyset, \mathrm{id}_\emptyset)$ is a n-cobordism and the $\Bbbk$-linear map

$$Z_0^{-1} Z(M, \emptyset, \emptyset, \mathrm{id}_\emptyset) Z_0 \colon \Bbbk \to \Bbbk$$

is the multiplication by a scalar $Z(M) \in \Bbbk$. Prove that $Z(M)$ is a topological invariant of M.

Exercise 2.
Let Z be a n-dimensional TQFT (over $\Bbbk$) and C be a closed oriented $(n-1)$-manifold. Prove that the $\Bbbk$-vector space $Z(C)$ is finite-dimensional.

Exercise 3.
Prove that any 1-dimensional TQFT (over $\Bbbk$) is entirely determined (up to equivalence) by an non-negative integer (the dimension of the $\Bbbk$-vector space associated by the TQFT to the point).

Exercise 4. (Mednykh's identity)

a. Let $(A, m, u, \Delta, \varepsilon)$ be a semisimple commutative Frobenius $\mathbb{C}$-algebra and let $e_1, \ldots, e_n$ be its primitive idempotents. Consider the 2-dimensional TQFT Z_A associated to A. Prove that for any closed connected oriented surface Σ of genus g,

$$Z_A(\Sigma) = \sum_{i=1}^{n} \varepsilon(e_i)^{1-g}.$$

b. Let G be a finite group. Denote by $\mathrm{Irr}(G)$ the set of isomorphic classes of irreducible complex representations of G. Prove that for any closed connected orientable surface Σ,

$$\big|\mathrm{Hom}(\pi_1(\Sigma), G)\big| = |G|^{1-\chi(\Sigma)} \sum_{V \in \mathrm{Irr}(G)} \dim(V)^{\chi(\Sigma)}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ.
Hint : consider the state sum triple $(\mathbb{C}[G], a, \mu)$ of Exercise 1 in Sheet 1.