Master class 2016-2017 in geometry, topology and physics

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Introduction to Quantum Topology I

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Exercise sheet 7

Exercise.
Let $V = \mathbb{C}^2$ with canonical basis $\mathcal{B} = (b_1, b_2)$. Consider the following basis of $V \otimes V$:

$$\mathcal{B} \otimes \mathcal{B} = (b_1 \otimes b_1, b_1 \otimes b_2, b_2 \otimes b_1, b_2 \otimes b_2).$$

Pick a parameter $t \in \mathbb{C}$ and define $\mathbb{C}$-linear morphisms $R\colon V \otimes V \to V \otimes V$ and $h\colon V \to V$ by

$$\mathrm{Mat}_{\mathcal{B}\otimes\mathcal{B}}(R) = \begin{pmatrix} t^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & t^{-\frac{1}{2}} - t^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & -t^{\frac{1}{2}} \end{pmatrix} \quad \text{and} \quad \mathrm{Mat}_{\mathcal{B}}(h) = \begin{pmatrix} t^{\frac{1}{2}} & 0 \\ 0 & -t^{\frac{1}{2}} \end{pmatrix}.$$

a. Prove that R and h satisfy

$$(h \otimes h)R = R(h \otimes h),$$
$$\mathrm{tr}_2((\mathrm{id}_V \otimes h)R^{\pm 1}) = \mathrm{id}_V,$$
$$(R^{-1})^{\circlearrowleft}\big((\mathrm{id}_V \otimes h)R(h^{-1} \otimes \mathrm{id}_V)\big)^{\circlearrowright} = \mathrm{id}_{V\otimes V^*},$$
$$(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R) = (R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V).$$

b. Consider the isotopy invariant $F = F_{(V,R,h)}$ of oriented tangles.
Prove that $F(L) = 0$ for all oriented link.

c. Prove that for an oriented $(1,1)$-tangle T, there is $c_T \in \mathbb{C}$ such that

$$F(T) = c_T \,\mathrm{id}_V.$$

d. For an oriented link L, set

$$\Delta_L(t) = c_T$$

where T is any oriented $(1,1)$-tangle whose closure is L. Prove that $\Delta_L(t)$ is a well-defined isotopy invariant of L satisfying the skein relation:

$$\Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})\Delta_{L_0}(t).$$