Master class 2016-2017 in geometry, topology and physics

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Introduction to Quantum Topology I

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Exercise sheet 8

Exercise 1. (The restricted dual)
We denote the dual of a $\Bbbk$-vector space V by $V^* = \mathrm{Hom}_\Bbbk(V, \Bbbk)$ and the transpose of a linear map $f\colon V \to W$ by $f^*\colon W^* \to V^*$. Let A be an $\Bbbk$-algebra with product $\mu\colon A \otimes A \to A$ and unit $\eta\colon \Bbbk \to A$. We view $A^* \otimes A^*$ as a subset of $(A \otimes A)^*$ as follows: for any $f, g \in A^*$, we define $f \otimes g \in (A \otimes A)^*$ by setting

$$(f \otimes g)(a \otimes b) = f(a)g(b)$$

for all $a, b \in A$. The *restricted dual* of the algebra A is the subspace of the dual A^* defined by

$$A^\circ = (\mu^*)^{-1}(A^* \otimes A^*),$$

where $\mu^*\colon A^* \to (A \otimes A)^*$ is the transpose of μ.

a. Prove that a form $f \in A^*$ belongs to A° if and only if there is an ideal I of A such that $\dim(A/I) < \infty$ and $f(I) = 0$.

b. Prove that A° is the largest subspace of A^* such that $\mu^*(A^\circ) \subset A^\circ \otimes A^\circ$.

c. Prove that A° is a coalgebra with coproduct and counit

$$\Delta_{A^\circ} = \mu^*\colon A^\circ \to A^\circ \otimes A^\circ \quad \text{and} \quad \varepsilon_{A^\circ} = \eta^*\colon A^\circ \subset A^* \to \Bbbk^* = \Bbbk.$$

d. Assume that A is a Hopf algebra with coproduct $\Delta\colon A \to A \otimes A$, counit $\varepsilon\colon A \to \Bbbk$, and antipode $S\colon A \to A$. Prove that

$$\Delta^*(A^\circ \otimes A^\circ) \subset A^\circ, \qquad \varepsilon^*(\Bbbk^*) \subset A^\circ, \qquad S^*(A^\circ) \subset A^\circ.$$

Deduce that the restricted dual A° of A is a Hopf algebra with product, unit, and antipode defined by

$$\mu_{A^\circ} = \Delta^*\colon A^\circ \otimes A^\circ \to A^\circ, \quad \eta_{A^\circ} = \varepsilon^*\colon \Bbbk = \Bbbk^* \to A^\circ, \quad S_{A^\circ} = S^*\colon A^\circ \to A^\circ.$$

e. Let A be a finite-dimensional Hopf algebra. Then $A^\circ = A^*$, so that A^* is a Hopf algebra. Liklewise $A^{**} = (A^*)^*$ is a Hopf algebra. Prove that $A^{**} \simeq A$ as Hopf algebras.

Exercise 2.
Let G be a finite group, so that the group algebra $\Bbbk[G]$ and the algebra $F(G)$ of $\Bbbk$-valued functions on G are finite-dimensional. Prove that

$$F(G)^* \simeq \Bbbk[G] \quad \text{and} \quad \Bbbk[G]^* \simeq F(G)$$

as Hopf algebras.

Exercise 3. (Grouplike elements)
Let A be a Hopf algebra. An element $g \in A$ is *grouplike* if $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$.

a. Prove that the set $G(A)$ of grouplike elements of A is a group (under multiplication).

b. Prove that the grouplike elements of A are linearly independent.

c. Prove that the grouplike elements of the restricted dual A° are the algebra morphisms $A \to \Bbbk$.

Exercise 4.
Let A be a Hopf algebra with coproduct Δ and counit ε. Consider the kernel I of ε.

a. Prove that $A = \Bbbk \oplus I$ as vector spaces.

b. Prove that for all $a \in I$,

$$\Delta(a) = a \otimes 1 + 1 \otimes a \mod I \otimes I.$$