Master class 2016-2017 in geometry, topology and physics

——

Introduction to Quantum Topology I

——

Exercise sheet 9

In what follows, $\Bbbk$ is a field.

Exercise 1.
Let G be a finite group and $D(G)$ be the Drinfeld double of the group algebra $\Bbbk[G]$.
a. Prove that a left $D(G)$-module is left G-module M endowed with a direct sum decomposition

$$M = \bigoplus_{g\in G} M_g$$

such that

$$gM_h \subset M_{ghg^{-1}}$$

for all $g, h \in G$.
b. Let $M = \bigoplus_{g\in G} M_g$ and $N = \bigoplus_{g\in G} N_g$ be two left $D(G)$-modules. Prove that the braiding

$$c_{M,N} \colon M \otimes N \to N \otimes M$$

induced by the R-matrix of $D(G)$ is computed by

$$c_{M,N}(m \otimes n) = n \otimes gm$$

for all $g \in G$, $m \in M$, and $n \in N_g$.

Exercise 2. (Yetter-Drinfeld modules)
Let $A = (A, \mu, \eta, \Delta, \varepsilon, S)$ be a finite-dimensional Hopf $\Bbbk$-algebra. A *Yetter-Drinfeld A-module* is a $\Bbbk$-vector space endowed with $\Bbbk$-linear maps $r\colon A \otimes M \to M$ and $\rho\colon M \to M \otimes A$ such that

- (M, r) is a left A-module, that is,
$$r(\mathrm{id}_A \otimes r) = r(\mu \otimes \mathrm{id}_M) \quad \text{and} \quad r(\eta \otimes \mathrm{id}_M) = \mathrm{id}_M;$$
- (M, ρ) is a right A-comodule, that is,
$$(\rho \otimes \mathrm{id}_A)\rho = (\mathrm{id}_M \otimes \Delta)\rho \quad \text{and} \quad (\mathrm{id}_M \otimes \varepsilon)\rho = \mathrm{id}_M;$$
- the following diagram commutes:

$$\begin{array}{ccccccc}
A\otimes M & \xrightarrow{\Delta\otimes\rho} & A\otimes A\otimes M\otimes A & \xrightarrow{\mathrm{id}_A\otimes\tau_{A,M}\otimes\mathrm{id}_A} & A\otimes M\otimes A\otimes A & \xrightarrow{r\otimes\mu} & M\otimes A \\
\Big\downarrow{\scriptstyle \Delta\otimes\mathrm{id}_A} & & & & & & \Big\uparrow{\scriptstyle \mathrm{id}_M\otimes\mu} \\
A\otimes A\otimes M & \xrightarrow{\mathrm{id}_A\otimes r} & A\otimes M & \xrightarrow{\tau_{A,M}} & M\otimes A & \xrightarrow{\rho\otimes\mathrm{id}_A} & M\otimes A\otimes A.
\end{array}$$

Prove that there is a bijective correspondence between:

(i) Yetter-Drinfeld A-modules,
(ii) left modules over the Drinfeld double $D(A)$ of A.